topology and supersymmetry

Simon Trebst
University of Cologne

Copenhagen
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collaborators

Jan Attig
PhD student

Michael Lawler
Cornell / Binghamton

Krishanu Roychowdury
Stockholm

arXiv:1809.08248
and
Phys. Rev. B **96**, 085145 (2017)
Editors' suggestion.

topological mechanics

topological mechanics

C.L. Kane & T.C. Lubensky, Nat. Phys. **10** (2014)

example #1: “floppy modes” in isostatic lattices

Maxwell relation

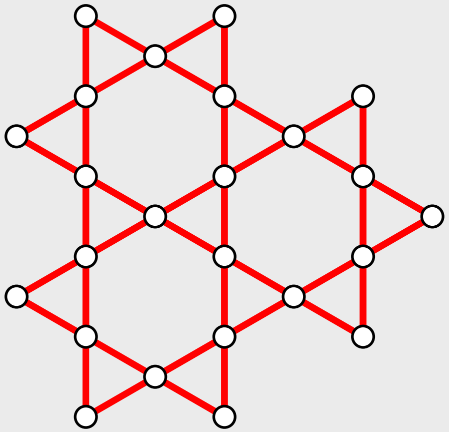
$$\nu \equiv N_0 - N_{\mathrm{ss}} = d \cdot n_{\mathrm{s}} - n_{\mathrm{b}}$$

isostatic lattices

$$\nu = 0$$

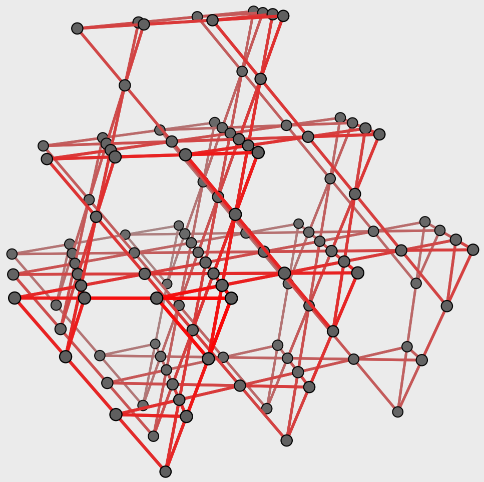
coordination number

$$z = 2 \cdot d$$



kagome lattice

$d = 2 \quad z = 4$



pyrochlore lattice

$d = 3 \quad z = 6$

topological mechanics

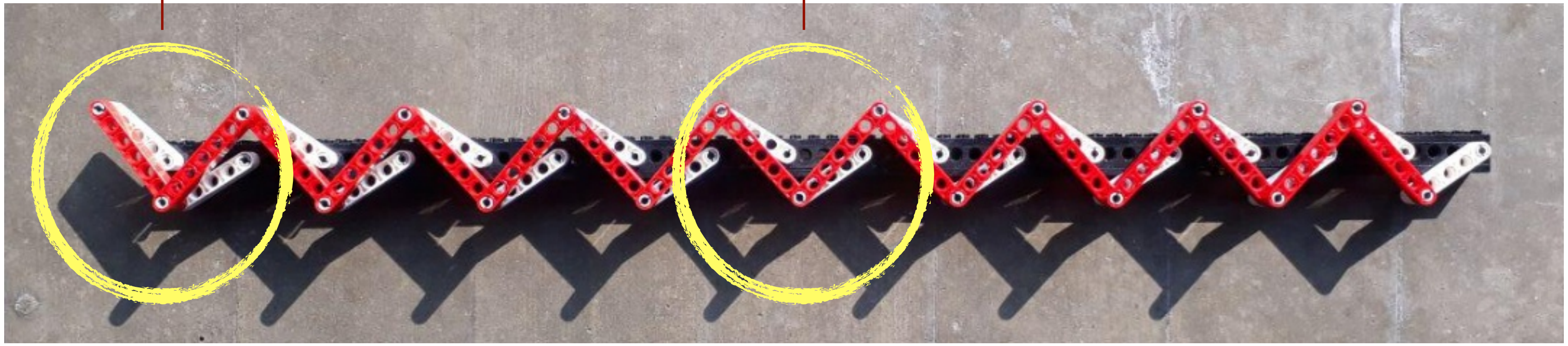
C.L. Kane & T.C. Lubensky, Nat. Phys. **10** (2014)

example #1: “floppy modes” in isostatic lattices

Maxwell relation

$$\nu \equiv N_0 - N_{\mathrm{ss}} = d \cdot n_{\mathrm{s}} - n_{\mathrm{b}}$$

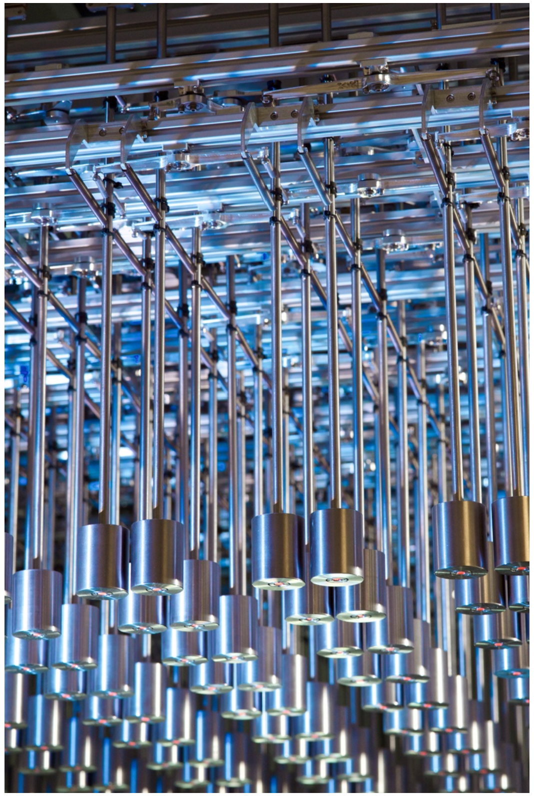
“floppy mode” **“rigid mode”**



“mechanical” SSH chain

topological mechanics

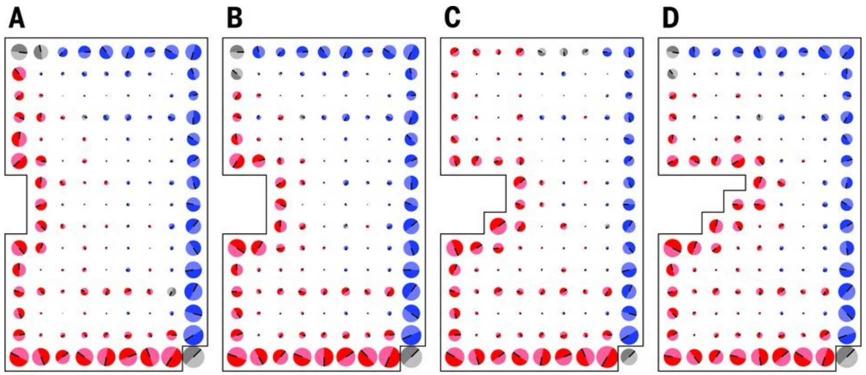
example #2: topological insulator from classical pendula



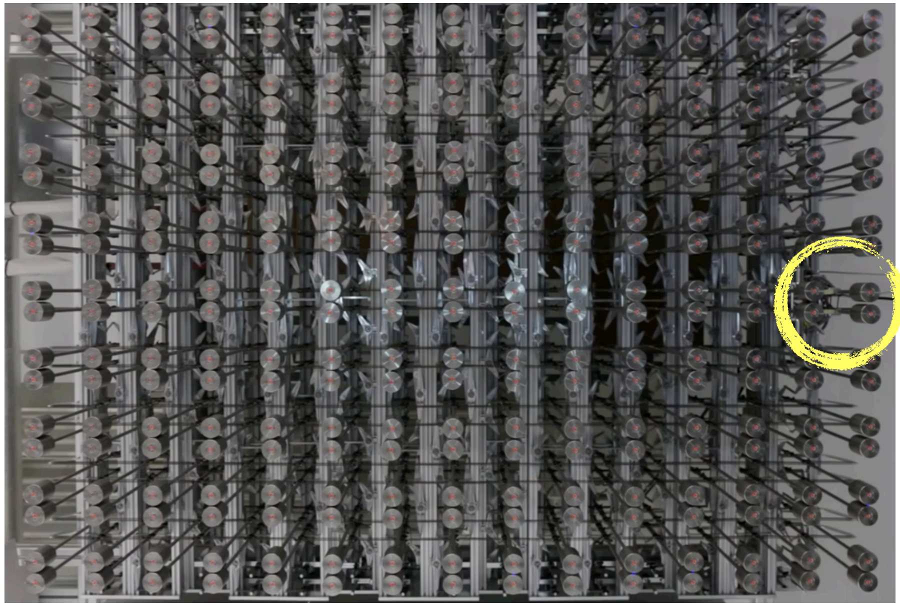
R. Süsstrunk and S. D. Huber, Science **349**, 47 (2015)
S. D. Huber, Nature Phys. **12**, 621 (2016)

topological mechanics

example #2: topological insulator from classical pendula



floppy modes constitute
boundary mode



R. Süsstrunk and S. D. Huber, Science **349**, 47 (2015)
S. D. Huber, Nature Phys. **12**, 621 (2016)

correspondence principles

topological mechanics

electronic system ⟷ mechanical system

(matrix) correspondence

Schrödinger equation

$$i\frac{\partial}{\partial t}\begin{pmatrix}\sqrt{\mathbf{D}}^T x\\ i\dot{x}\end{pmatrix} = \begin{pmatrix}0 & \sqrt{\mathbf{D}}^T\\ \sqrt{\mathbf{D}} & 0\end{pmatrix}\begin{pmatrix}\sqrt{\mathbf{D}}^T x\\ i\dot{x}\end{pmatrix}$$

symmetry class BDI

← square root

Newton's equation

$$\ddot{x} = -\mathbf{D}x$$

dynamical matrix

correspondence principles

topological mechanics

electronic system ⟷ mechanical system

(matrix) correspondence

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supersymmetry

fermions ⟷ bosons

SUSY

supersymmetry

basic ingredients of SUSY

non-hermitian
SUSY charge operator

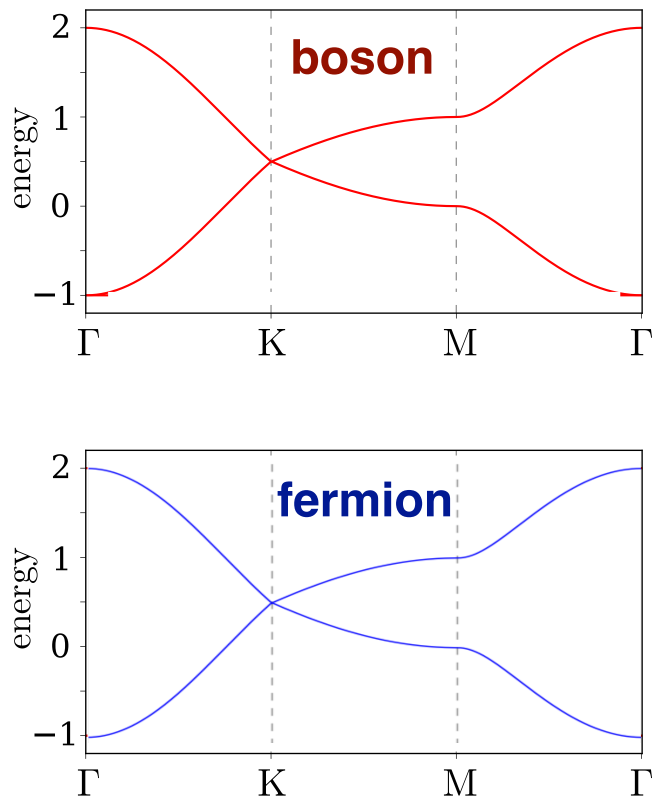
boson

$$\mathcal{Q} = c_i^\dagger \mathbf{R}_{ij} b_j$$

fermion arbitrary matrix

supersymmetric **Hamiltonian**

$$\mathcal{H}_{\text{SUSY}} = \{\mathcal{Q}, \mathcal{Q}^\dagger\}$$



isospectral
quadratic
Hamiltonians

SUSY & topological mechanics

topological mechanics – phase space coordinates (p, q)
as bosonic degrees of freedom

real fermions
= Majorana fermions

natural SUSY partners

real bosons $[\hat{q}_i, \hat{p}_j] = i\delta_{i,j}$

real bosons

SUSY charge $$\mathcal{Q} = \gamma_i^B \mathbf{1}_{ij} \hat{p}_j + \gamma_i^A \mathbf{A}_{ij} \hat{q}_j$$

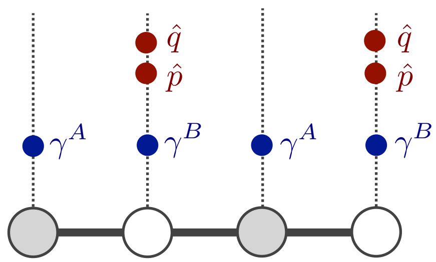
real fermions

SUSY & topological mechanics

SUSY charge $\mathcal{Q} = \gamma_i^B \mathbf{1}_{ij} \hat{p}_j + \gamma_i^A \mathbf{A}_{ij} \hat{q}_j$

encodes block-diagonal form

$$\mathbf{R} = \begin{pmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{A} \end{pmatrix}$$



$$H_{\mathrm{SUSY}} = \{\mathcal{Q}, \mathcal{Q}^\dagger\}$$

$$\mathcal{H}_{\mathrm{fermion}} = -i\gamma_j^A \, \mathbf{A}_{jk} \, \gamma_k^B + \mathrm{h.c.}$$

Majoranas hopping on two sublattices AB

$$\mathcal{H}_{\mathrm{boson}} = \hat{p}_i \hat{p}_i + \hat{q}_i (\mathbf{A}^T \mathbf{A})_{ij} \hat{q}_j$$

dynamical matrix

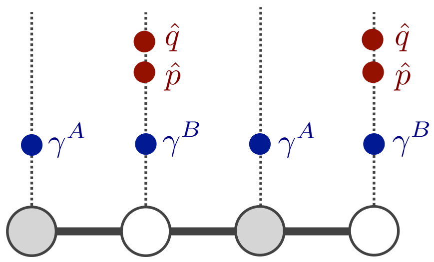
bosons on one sublattice (B)

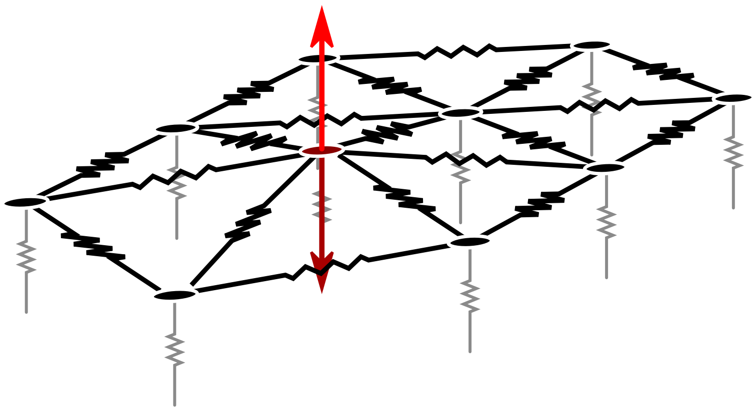
$\mathbf{R}$ is the **rigidity matrix** of the mechanical system.

It allows to directly connect mechanical systems to Majorana analogues, and vice versa.

SUSY & topological mechanics

From real bosons to classical **balls and springs**.





$$\mathcal{H}_{\text{boson}} = \hat{p}_i \hat{p}_i + \hat{q}_i (\mathbf{A}^T \mathbf{A})_{ij} \hat{q}_j$$

classical limit

$$\mathcal{H} = \sum_i \frac{p_i^2}{2m} + \sum_{ij} \frac{k_{ij}}{2} (q_i - q_j)^2 + \sum_i \frac{\kappa_i}{2} q_i^2$$

$$\sim \sum_i p_i^2 + \sum_{ij} q_i \mathbf{D}_{ij} q_j$$

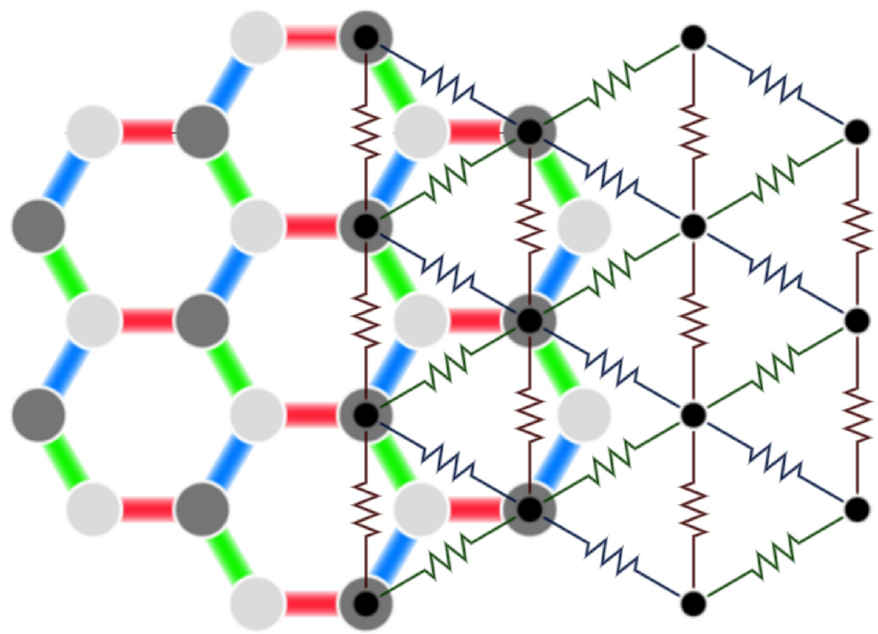
$$k_{ij} = -2 \sum_{a \in A} \mathbf{A}^T_{ia} \mathbf{A}_{aj} \qquad \kappa_i = 2 \sum_{a \in A} \mathbf{A}^2_{ia} - \sum_{b \in B} k_{ib}$$

Kitaev model

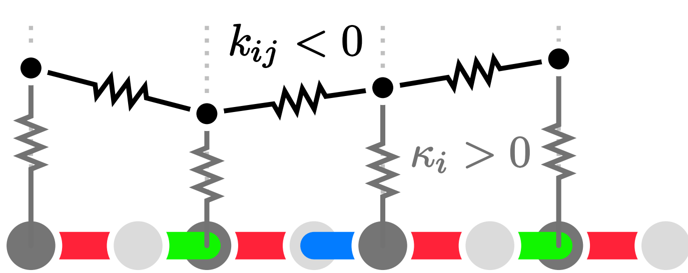
mechanical analogue

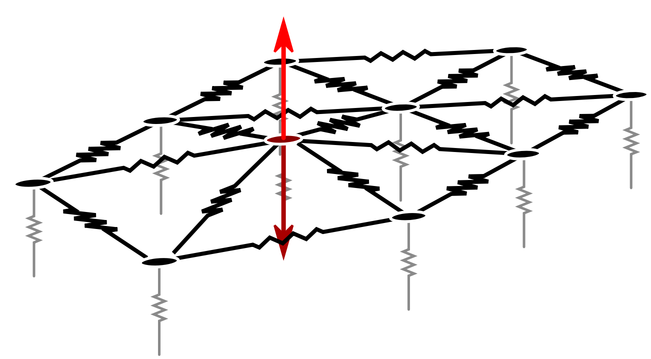
balls & springs Kitaev model

Majorana fermions
on **honeycomb** lattice

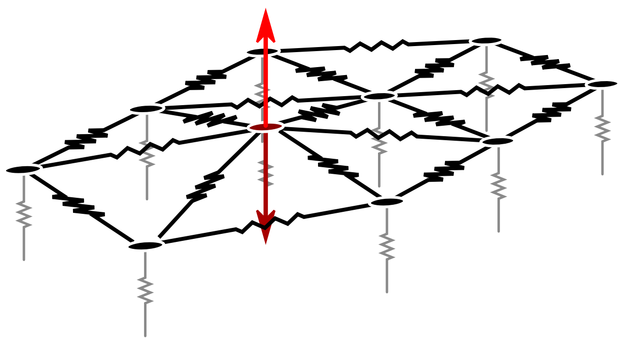


balls & springs
on **triangular** lattice



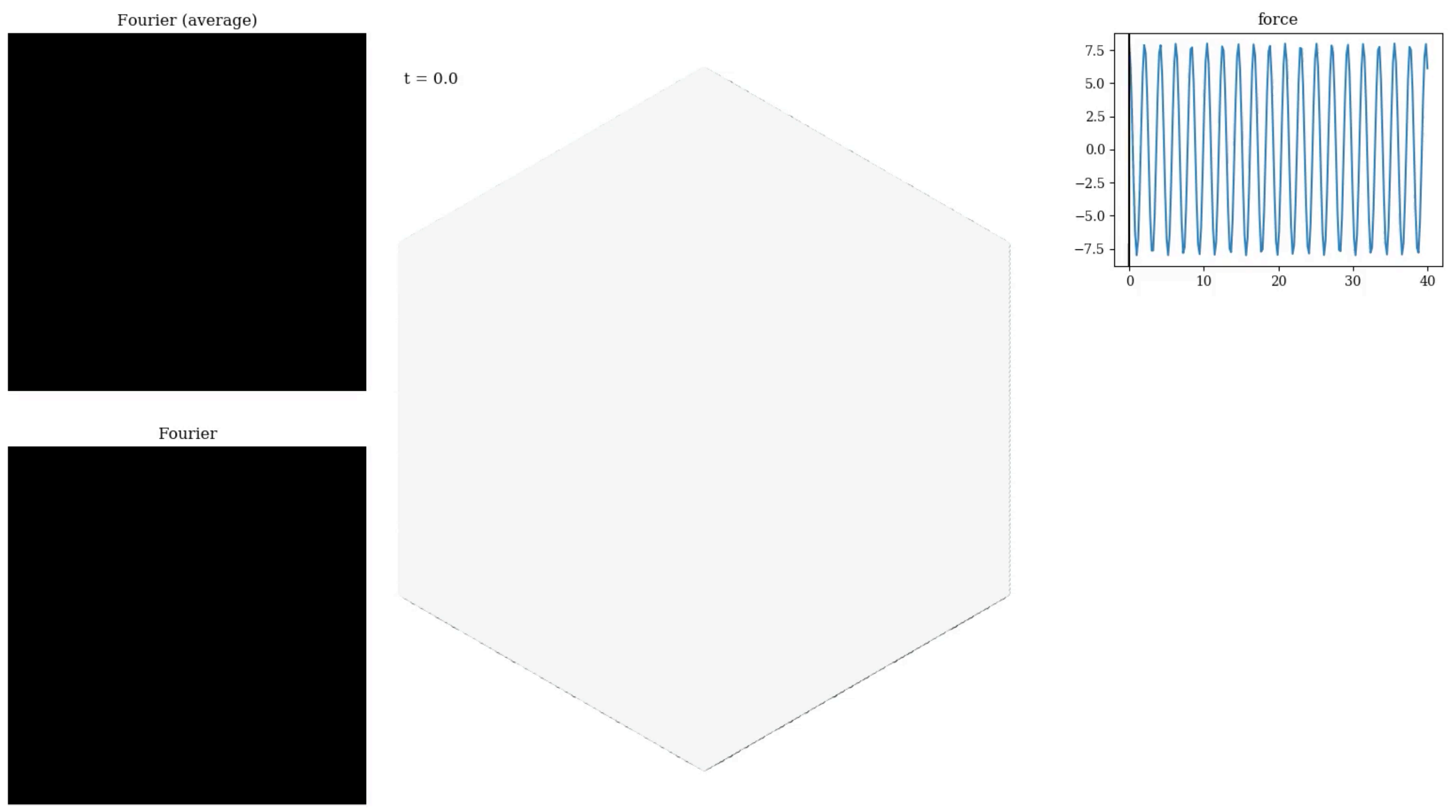


balls & springs Kitaev model

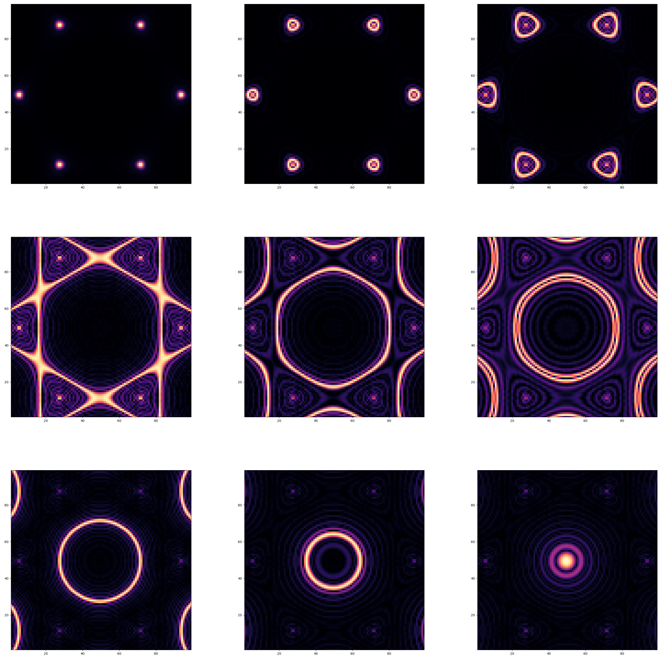


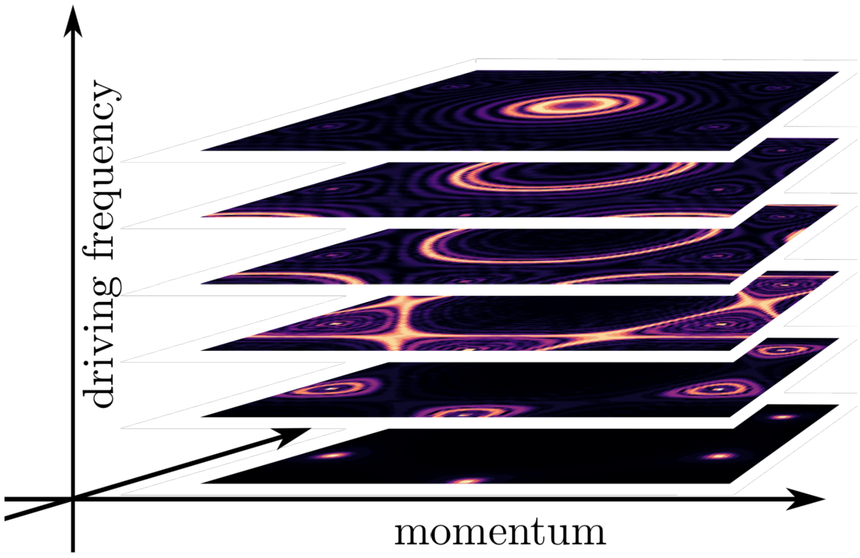
simulate balls & springs model by integrating
classical equations of motion

periodic drive of a single sites

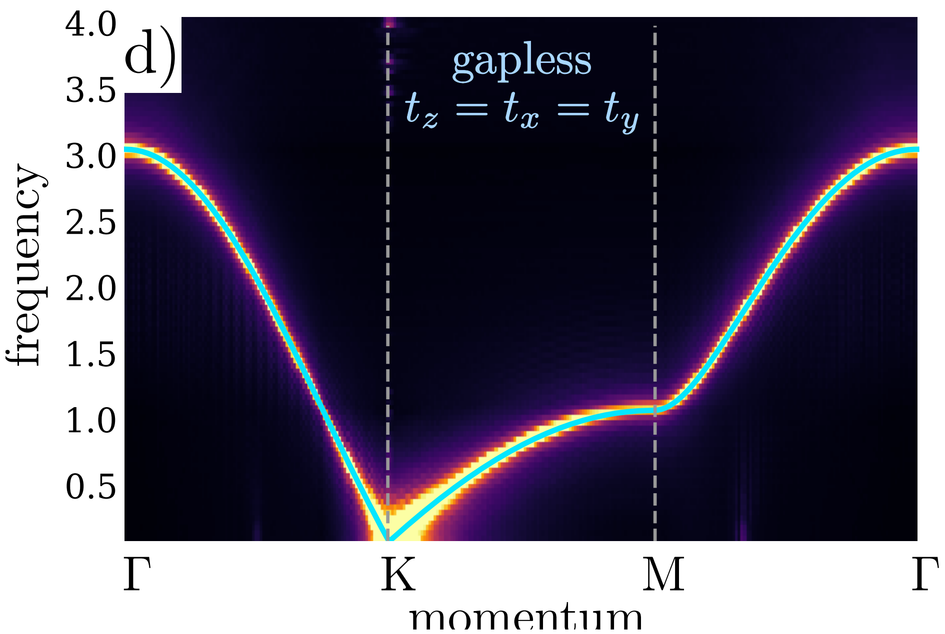


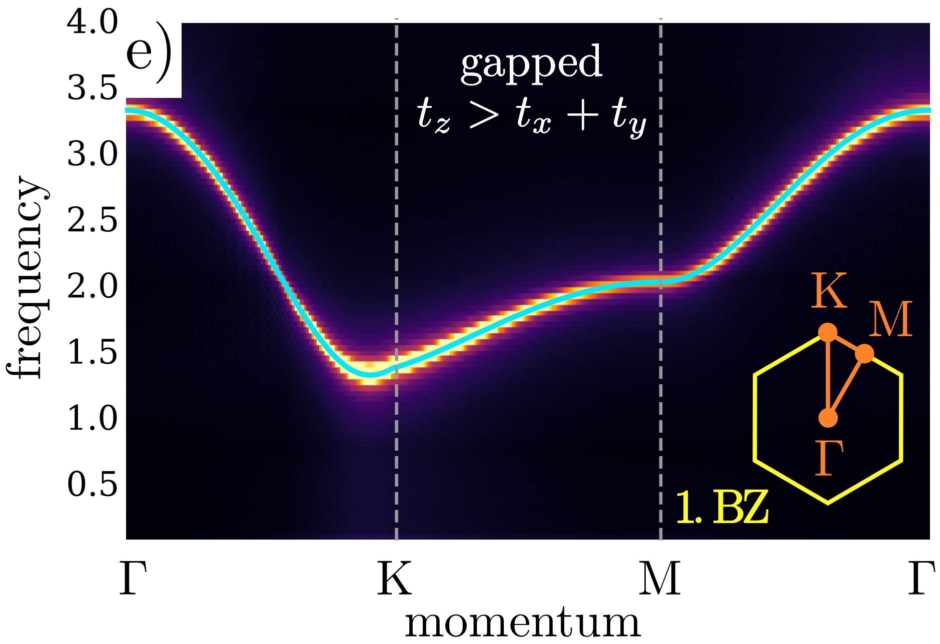
balls & springs Kitaev model



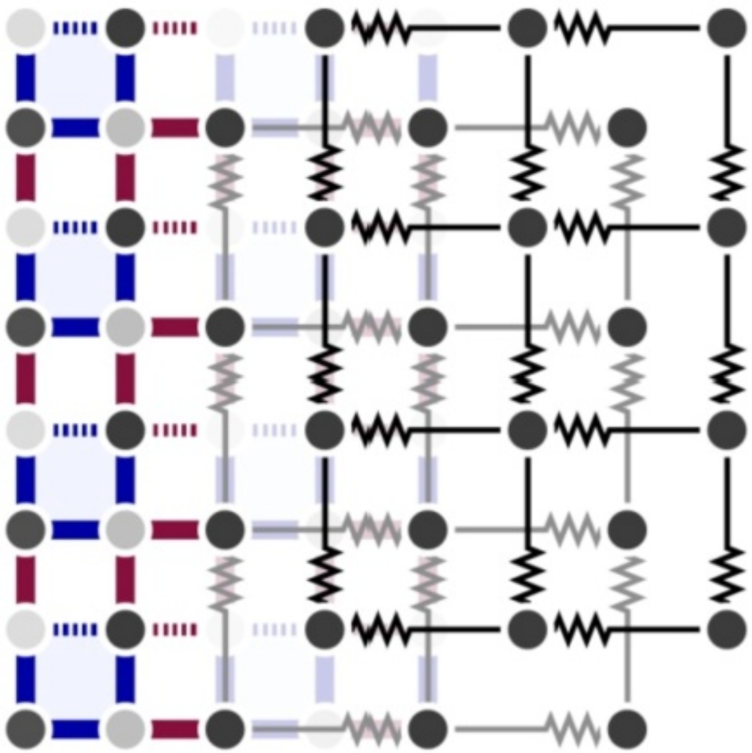


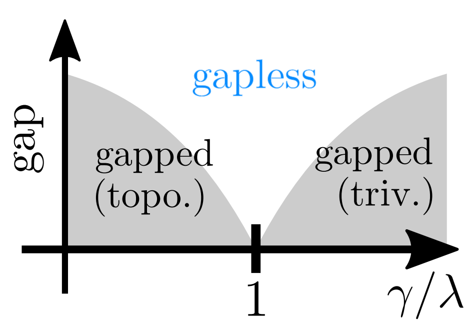
balls & springs Kitaev model

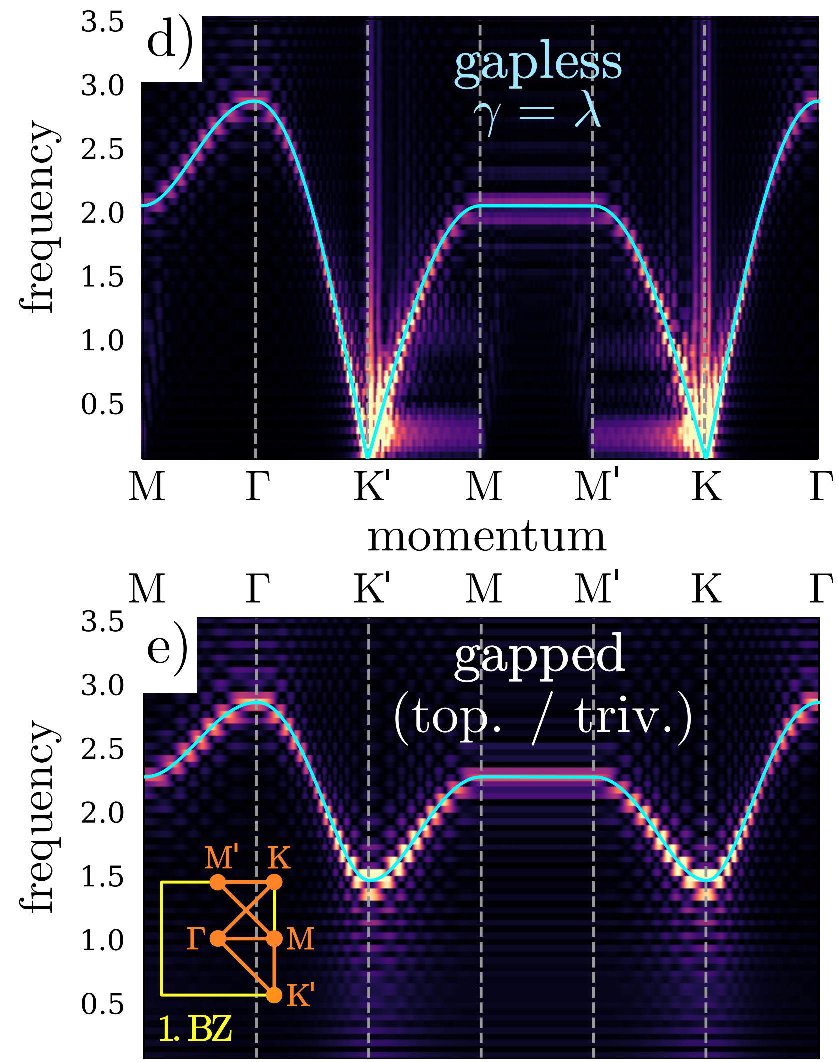




mechanical 2nd order TI







topological invariants

topological invariants

This SUSY construction allows to explore
topological properties of bosonic systems
by connecting the symplectic bosonic eigenfunctions
with a **fermionic Berry phase** of its SUSY partner.

fermionic Berry curvature

$$\mathcal{A} = \langle u_m(\mathbf{k}) | i\nabla_k | u_n(\mathbf{k}) \rangle$$

fermionic eigenstates

via **rigidity matrix**

$$|u_m(\mathbf{k})\rangle = \frac{\mathbf{R}(\mathbf{k})}{\sqrt{|\omega_m(\mathbf{k})|}} |v_m(\mathbf{k})\rangle \equiv \tilde{\mathbf{R}}(\mathbf{k}) |v_m(\mathbf{k})\rangle$$

bosonic eigenstates

bosonic Berry curvature

$$\begin{aligned} \mathcal{A}_{\mathrm{SUSY}} &= \langle v_m(\mathbf{k}) | i \tilde{\mathbf{R}}^\dagger \nabla_k \left(\tilde{\mathbf{R}} | v_n(\mathbf{k}) \rangle \right) \\ &= \langle v_m(\mathbf{k}) | i\sigma_2 \left(\nabla_k + \underbrace{\sigma_2 \tilde{\mathbf{R}}^\dagger \nabla_k \tilde{\mathbf{R}}} \right) | v_n(\mathbf{k}) \rangle \end{aligned}$$

additional covariant derivative

route to classify bosonic systems

This SUSY construction allows to explore
topological properties of bosonic systems
by connecting the symplectic bosonic eigenfunctions
with a **fermionic Berry phase** of its SUSY partner.

SUSY Berry curvature

$$\begin{aligned} \mathcal{A}_{\mathrm{SUSY}} &= \langle v_m(\mathbf{k})| i\tilde{\mathbf{R}}^\dagger \nabla_k \left(\tilde{\mathbf{R}} |v_n(\mathbf{k})\rangle \right) \\ &= \langle v_m(\mathbf{k})| i\sigma_2 \left(\nabla_k + \underbrace{\sigma_2 \tilde{\mathbf{R}}^\dagger \nabla_k \tilde{\mathbf{R}}}_{\text{additional covariant derivative}} \right) |v_n(\mathbf{k})\rangle \end{aligned}$$

Bosonic systems that are trivial with regard to conventional definition of Berry phase
can be non-trivial with regard to SUSY Berry phase!

spin spirals

Dirac magnons

spin liquids

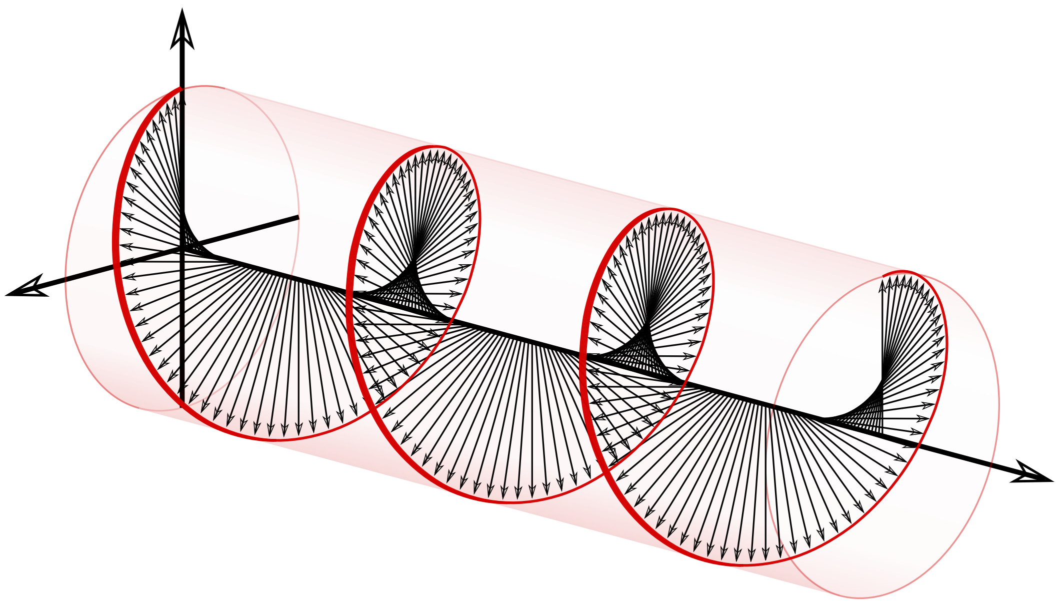
…

spin spirals

Coplanar spirals typically arise in the presence of **competing interactions**

Elementary ingredient for

- multiferroics
- spin textures/multi-q states
  - skyrmion lattices
  - Z_2 vortex lattices
- spiral spin liquids



Description in terms of a single wavevector

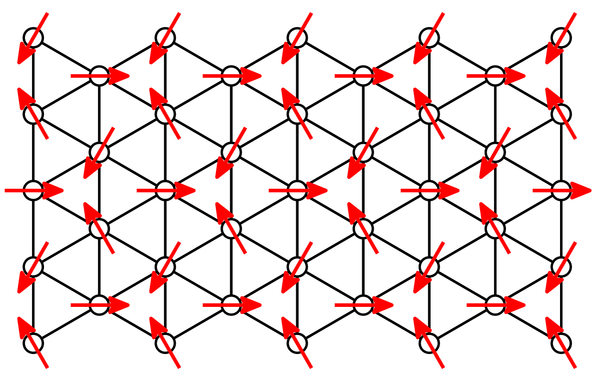
$$\vec{S}(\vec{r}) = \mathrm{Re}\left(\left(\vec{S}_1 + i\vec{S}_2\right) e^{i\vec{q}\vec{r}}\right)$$

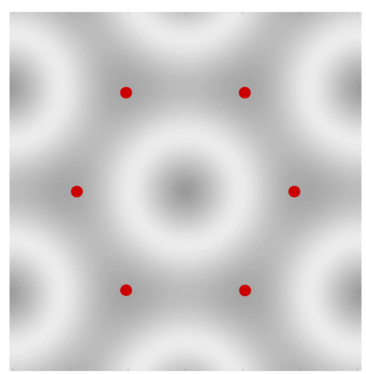
spin spirals

Coplanar spirals typically arise in the presence of **competing interactions**

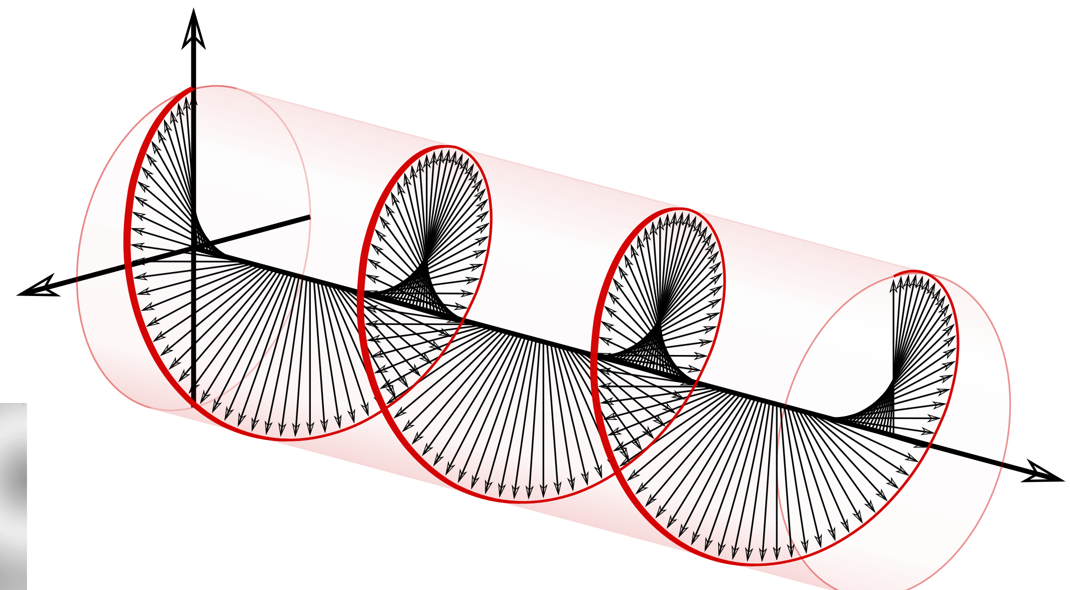
Familiar example

- **120º order** of Heisenberg AFM on triangular lattice





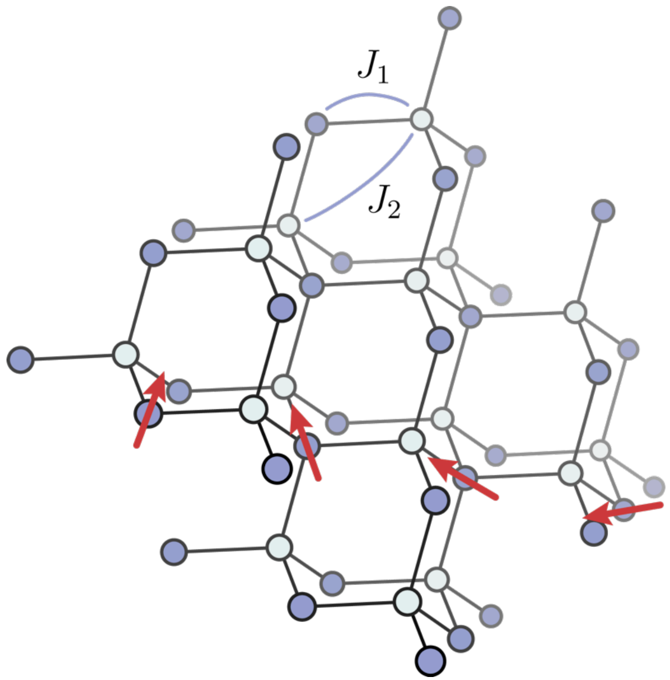
$$\vec{q} = \left(\pm \frac{2\pi}{3}, \frac{2\pi}{\sqrt{3}} \right)$$



$$\vec{S}(\vec{r}) = \mathrm{Re}\left(\left(\vec{S}_1 + i\vec{S}_2 \right) e^{i\vec{q}\vec{r}} \right)$$

spin spiral materials

Frustrated diamond lattice antiferromagnets

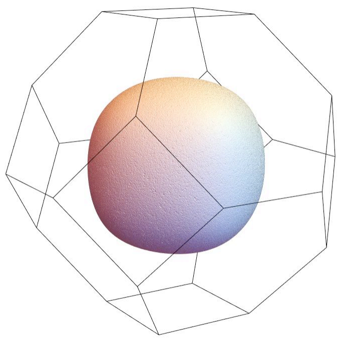


A-site spinels

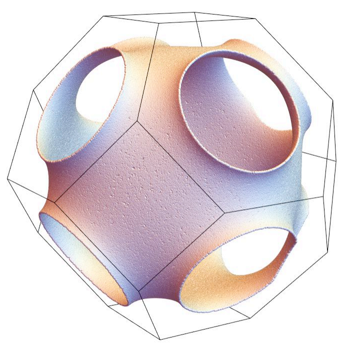
$MnSc_2S_4$	S=5/2
$FeSc_2S_4$	S=2
$CoAl_2O_4$	S=3/2
$NiRh_2O_4$	S=1

$$\mathcal{H} = J_1 \sum_{\langle i,j \rangle} \vec{S}_i \vec{S}_j + J_2 \sum_{\langle\langle i,j \rangle\rangle} \vec{S}_i \vec{S}_j$$

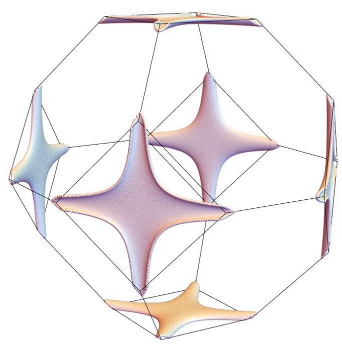
degenerate coplanar spirals form
spin spiral surfaces in *k*-space



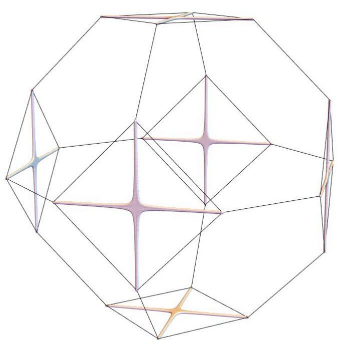
$J_2/J_1 = 0.2$



$J_2/J_1 = 0.4$



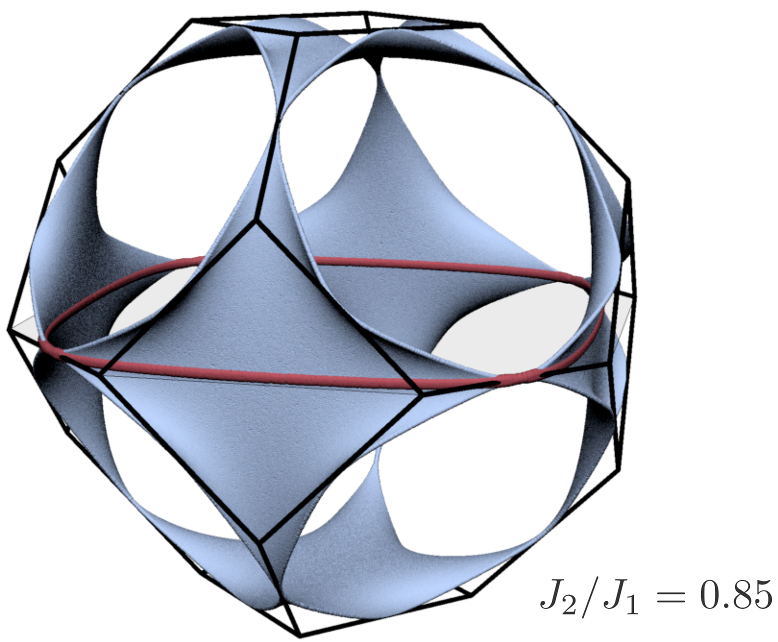
$J_2/J_1 = 3$



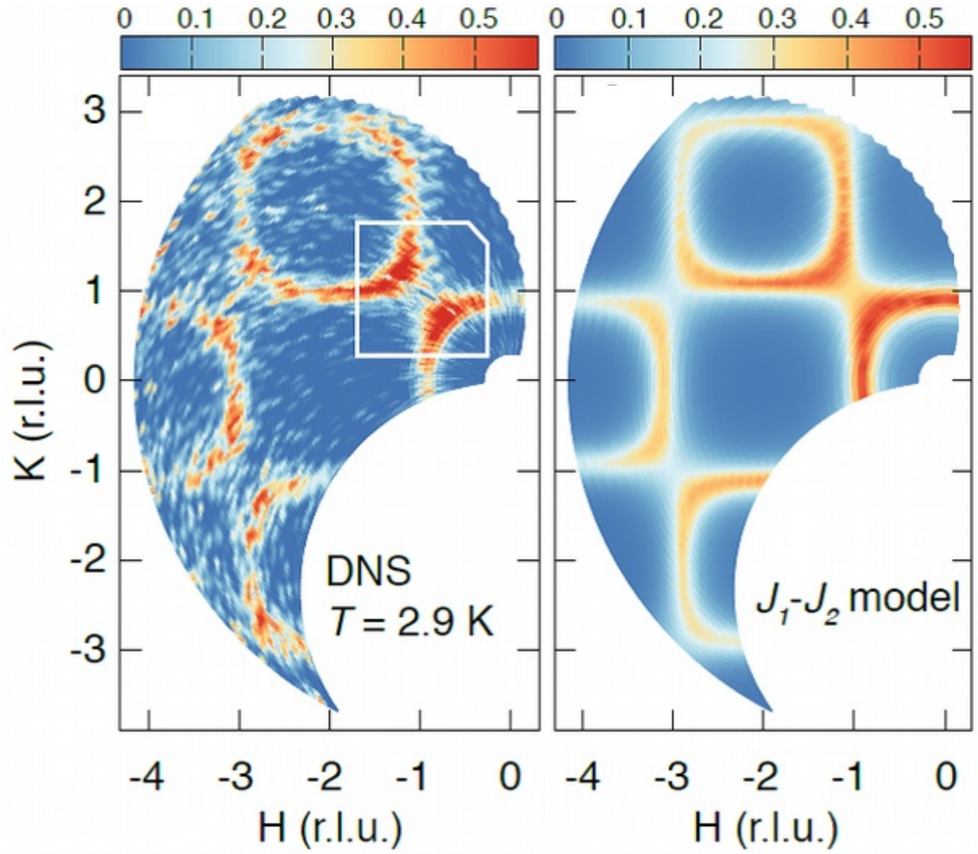
$J_2/J_1 = 100$

spin spiral materials

Experimental observation of spin spiral surface in inelastic neutron scattering of $MnSc_2S_4$.



Nature Phys. **3**, 487 (2007)

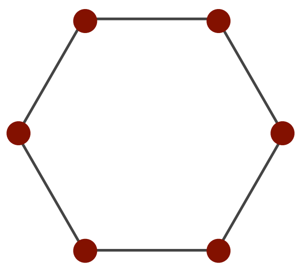


Nature Phys. **13**, 157 (2017)

spin spiral manifolds

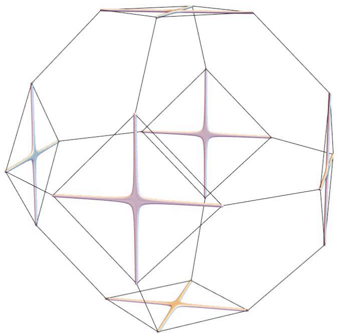
Spiral manifolds are extremely reminiscent of **Fermi surfaces**

triangular lattice



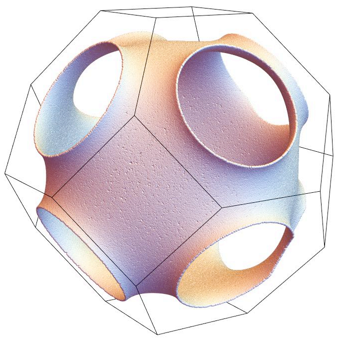
Dirac points

FCC lattice



nodal lines

diamond lattice



Fermi surface

But:
Spiral manifolds describe ground state of classical spin system, while Fermi surfaces are features in the middle of the energy spectrum of an electronic quantum system.

spin spiral manifolds

Luttinger-Tisza

spin spirals in a nutshell

$$\mathcal{H} = \sum_{\langle i,j \rangle} J_{ij}\, \vec{S}_i \vec{S}_j$$

Fourier transform
of spin model

$$= \sum_{\vec{k}} \sum_{A,B} S^A_{\vec{k}}\, \mathbf{M}_{A,B}(\vec{k})\, S^B_{-\vec{k}}$$

diagonalize
matrix

$$\mathbf{M}_{A,B}(\vec{k}) = \sum_{\vec{r}^A_{B,j}} J_{\vec{r}_j}\, e^{-i\vec{k}\cdot\vec{r}_j}$$

find **minimal**
eigenvalues

$$\lambda_j(\vec{k})$$

free fermions in a nutshell

$$\mathcal{H} = \sum_{\langle i,j \rangle} t_{ij}\, c_i^\dagger c_j$$

Fourier transform
of spin model

$$= \sum_{\vec{k}} \sum_{A,B} c^\dagger_{A,\vec{k}}\, \mathbf{H}_{A,B}(\vec{k})\, c_{B,\vec{k}}$$

diagonalize
matrix

$$\mathbf{H}_{A,B}(\vec{k}) = \sum_{\vec{r}^A_{B,j}} t_{\vec{r}_j}\, e^{-i\vec{k}\cdot\vec{r}_j}$$

find **zero**
eigenvalues

$$\epsilon_j(\vec{k})$$

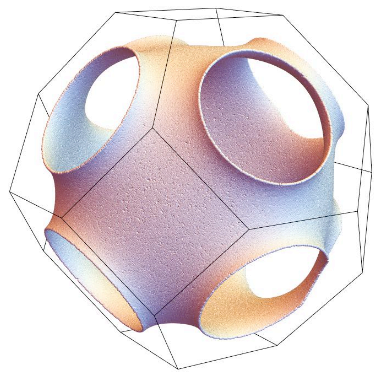
spin spiral manifolds

spin spirals in a nutshell

$\mathbf{M}_{A,B}(\vec{k})$

with **minimal** eigenvalues

$\lambda_j(\vec{k})$



make ansatz

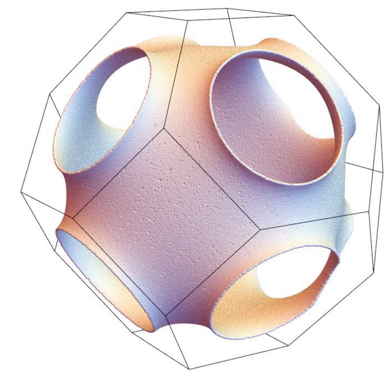
$$\mathbf{M}(\vec{k}) = \mathbf{H}(\vec{k})^2 - E_0 \cdot \mathbf{1}$$

free fermions in a nutshell

$\mathbf{H}_{A,B}(\vec{k})$

with **zero** eigenvalues

$\epsilon_j(\vec{k})$



$\mathbf{H}(\vec{k})^2$ has eigenvalues $\epsilon_j(\vec{k})^2$

zero eigenvalues of $\mathbf{H}(\vec{k})$ are minimal eigenvalues of $\mathbf{H}(\vec{k})^2$

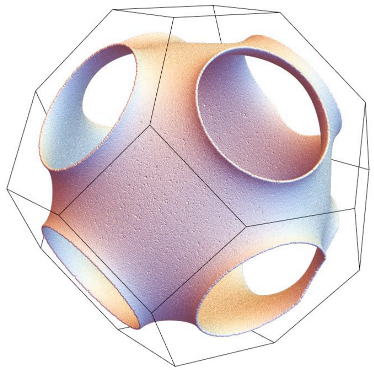
matrix correspondence

spin spirals in a nutshell

$\mathbf{M}_{A,B}(\vec{k})$

with **minimal** eigenvalues

$\lambda_j(\vec{k})$



$\mathbf{H}(\vec{k})^2$ ←

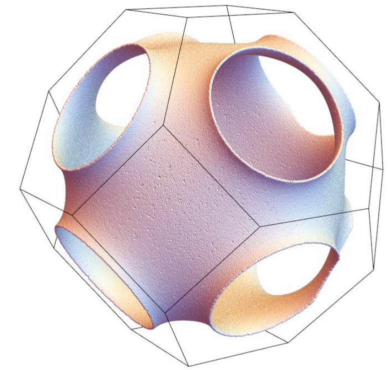
→ $\sqrt{\mathbf{M}(\vec{k})}$

free fermions in a nutshell

$\mathbf{H}_{A,B}(\vec{k})$

with **zero** eigenvalues

$\epsilon_j(\vec{k})$



$$\mathbf{M}(\vec{k}) = \mathbf{H}(\vec{k})^2 - E_0 \cdot \mathbf{1}$$

mapping of a classical to quantum system
(of same spatial dimensionality)
via a 1:1 matrix correspondence

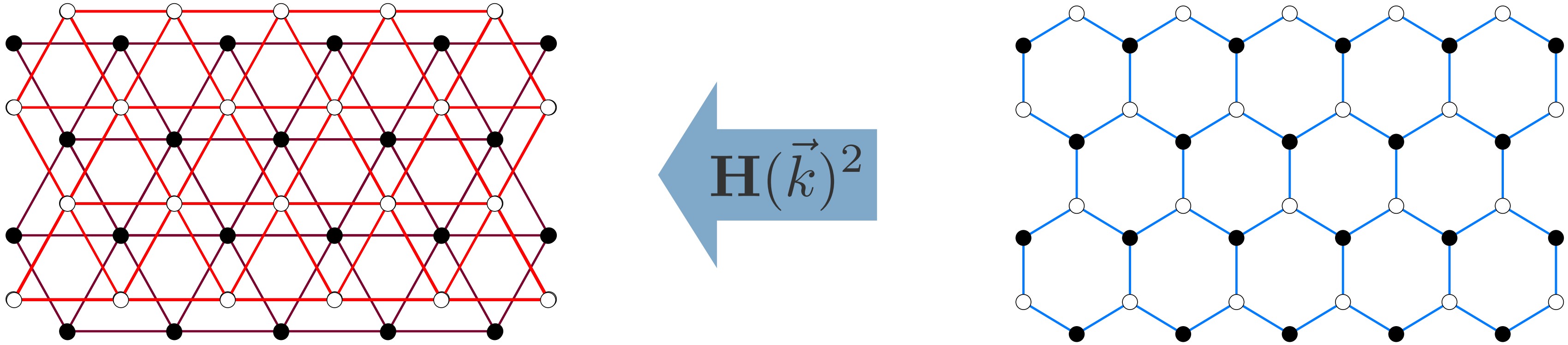
→ reminiscent of “topological mechanics”

lattice construction

$$\mathbf{M}(\vec{k}) = \mathbf{H}(\vec{k})^2 - E_0 \cdot \mathbf{1}$$

What does **"squaring"** of quantum system mean?

Explicit **lattice construction**.



coplanar spirals on triangular lattice

$$\vec{q} = \left(\pm\frac{2\pi}{3}, \frac{2\pi}{\sqrt{3}} \right)$$

free fermions on honeycomb lattice

$$\vec{q} = \left(\pm\frac{2\pi}{3}, \frac{2\pi}{\sqrt{3}} \right)$$

lattice construction

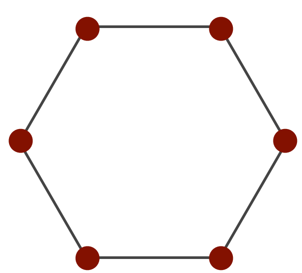
$$\mathbf{M}(\vec{k}) = \mathbf{H}(\vec{k})^2 - E_0 \cdot \mathbf{1}$$

What does **“squaring”** of quantum system mean?

Explicit **lattice construction**.

spin spirals
triangular lattice

120° order

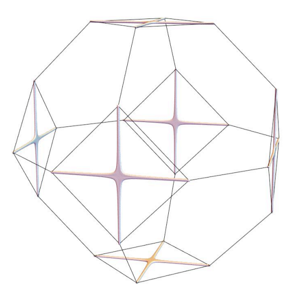


Dirac points

free fermions
honeycomb lattice

spin spirals
FCC lattice

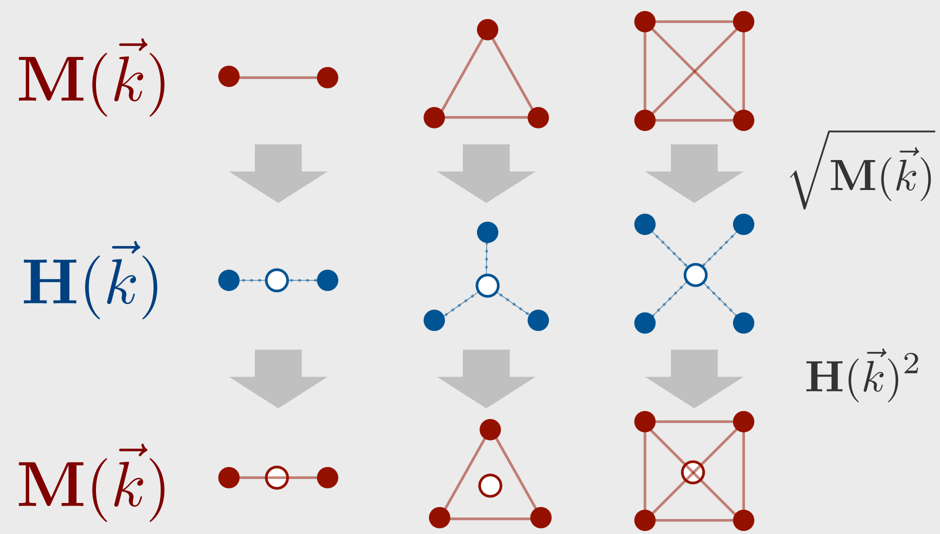
degenerate spirals



nodal lines

free fermions
diamond lattice

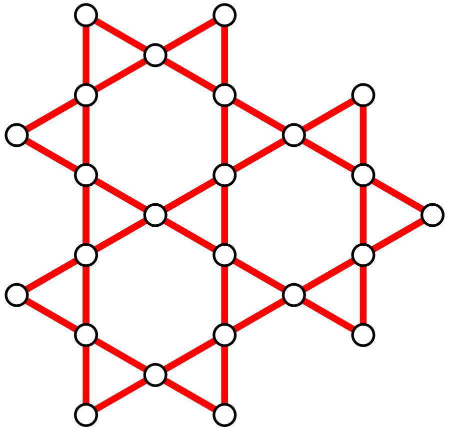
general lattice construction

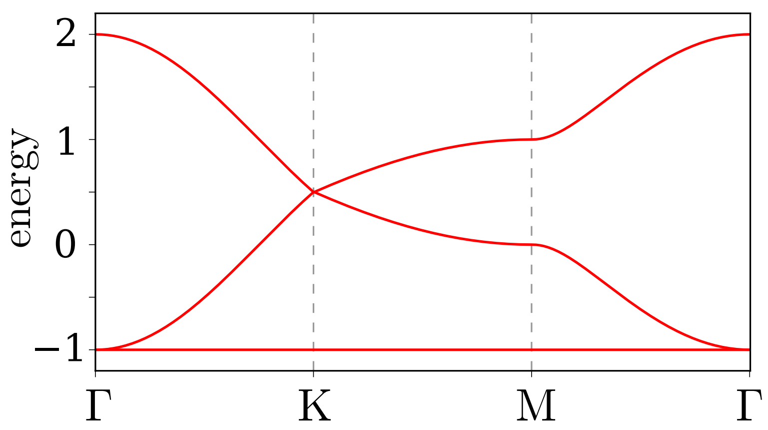


lattice construction – examples

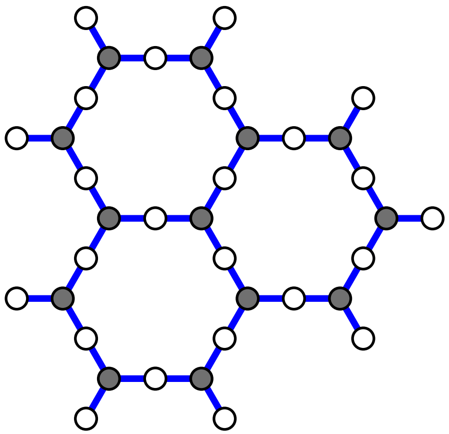
Spectra of the **kagome** and **extended honeycomb** lattice.

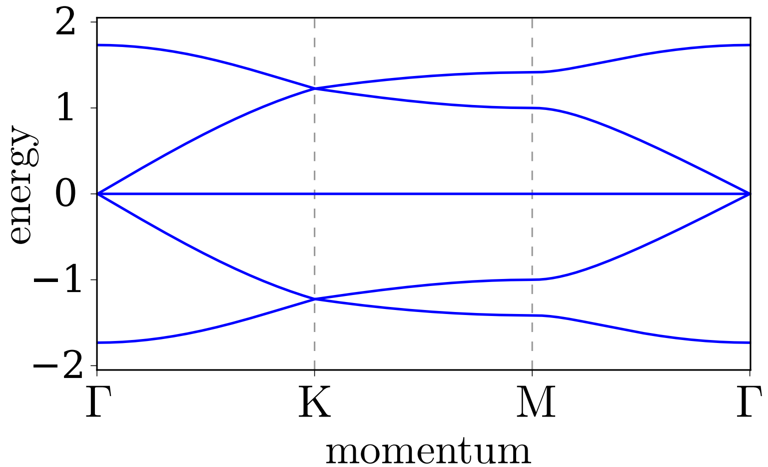
spins





fermions

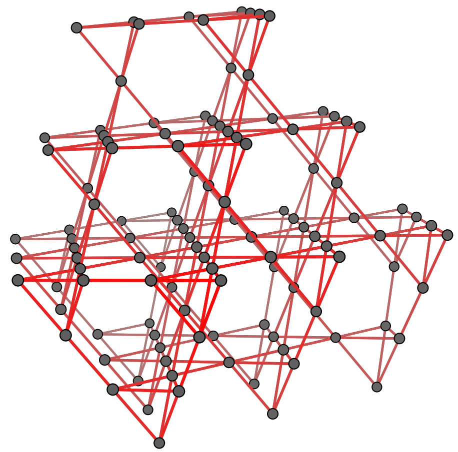


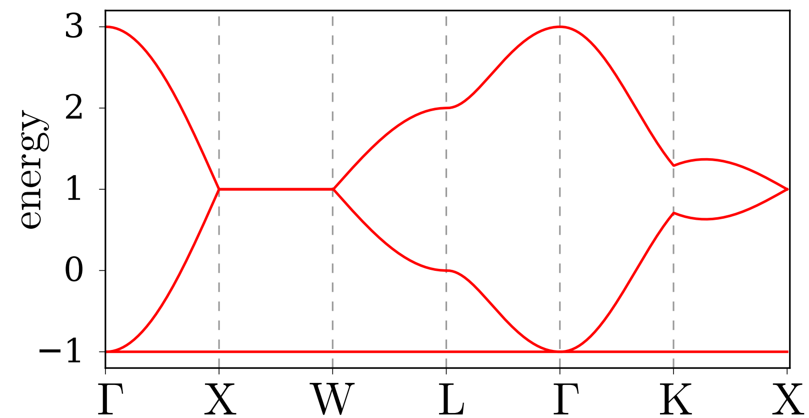


lattice construction – examples

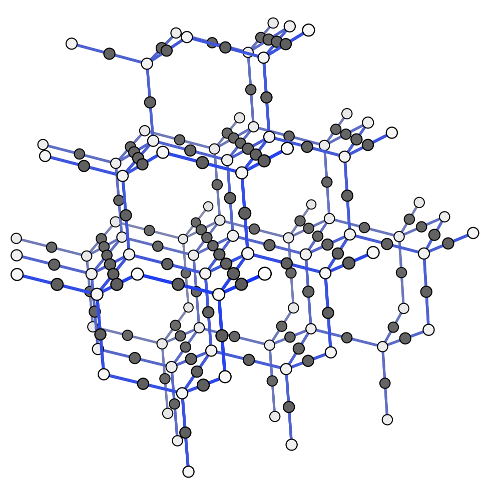
Spectra of the **pyrochlore** and **extended diamond** lattice.

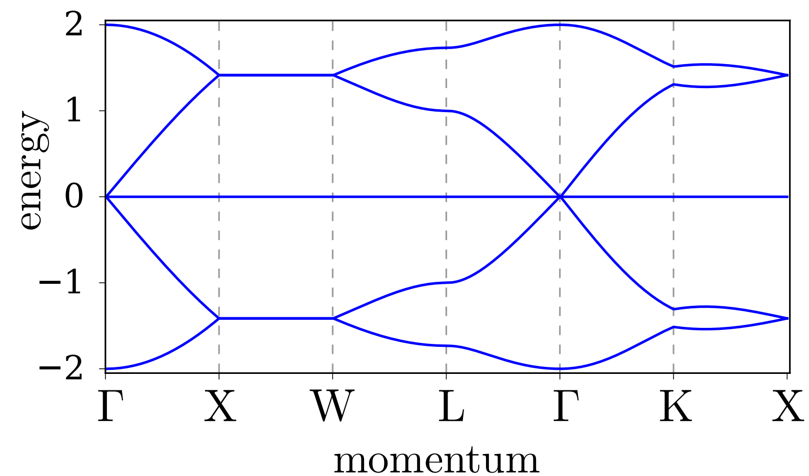
spins



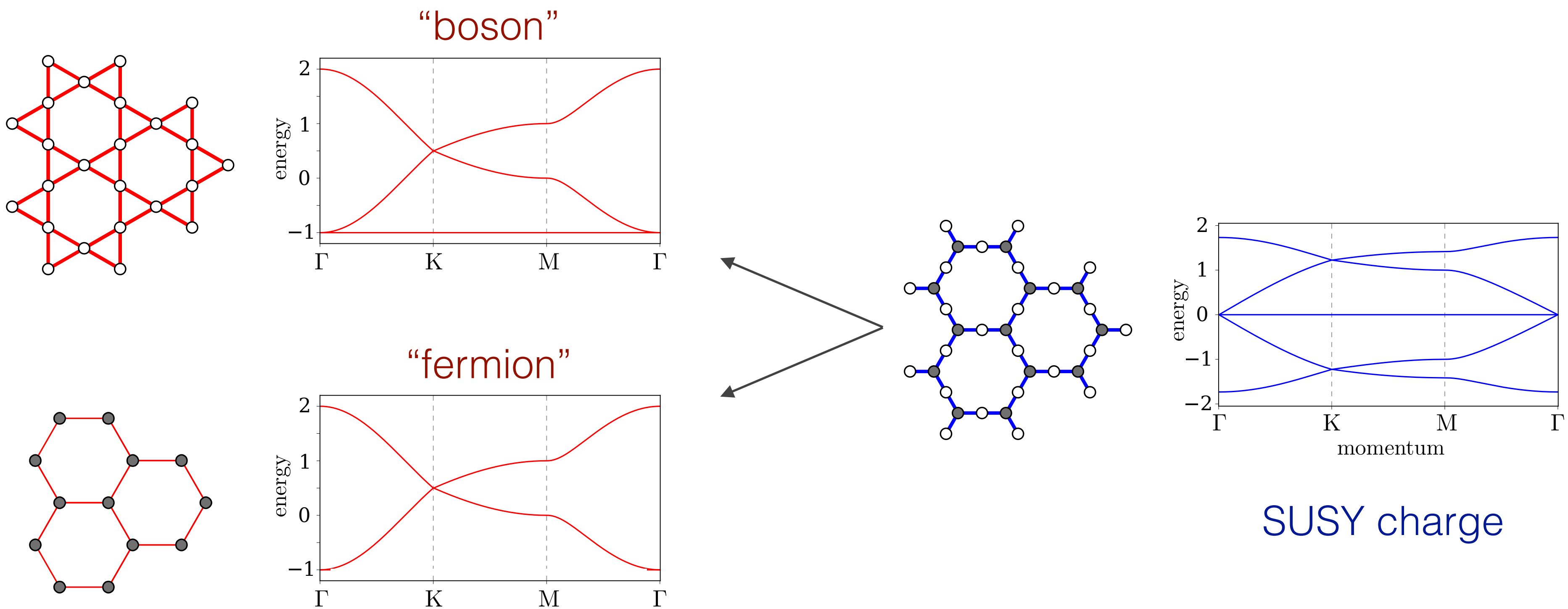


fermions





SUSY formulation



$$\mathcal{H}^2 = \begin{pmatrix} \mathbf{Q}^\dagger \mathbf{Q} & 0 \\ 0 & \mathbf{Q}\mathbf{Q}^\dagger \end{pmatrix}$$

square root

$$\mathcal{H} = \begin{pmatrix} 0 & \mathbf{Q}^\dagger \\ \mathbf{Q} & 0 \end{pmatrix}$$

Summary

topological mechanics from supersymmetry

$$H_{\mathrm{SUSY}} = \{\mathcal{Q}, \mathcal{Q}^\dagger\}$$

$$\mathcal{H}_{\mathrm{fermion}} = -i\gamma_j^A \, \mathbf{A}_{jk} \, \gamma_k^B + \mathrm{h.c.}$$

Majorana system (AB lattice)

$$\mathcal{H}_{\mathrm{boson}} = \hat{p}_i \hat{p}_i + \hat{q}_i (\mathbf{A}^T \mathbf{A})_{ij} \hat{q}_j$$

mechanical system (B sublattice)

novel topological invariant for boson systems

$$\mathcal{A}_{\mathrm{SUSY}} = \langle v_m(\mathbf{k}) | i\sigma_2 \left(\nabla_k + \underbrace{\sigma_2 \tilde{\mathbf{R}}^\dagger \nabla_k \tilde{\mathbf{R}}}_{\text{additional covariant derivative}} \right) | v_n(\mathbf{k}) \rangle$$

many other SUSY pairs – spin spirals, Dirac magnons, …

arXiv:1809.08248 and Phys. Rev. B **96**, 085145 (2017), Editors' suggestion.

Thanks!

@SimonTrebst