Interacting anyons in topological quantum liquids

Things golden

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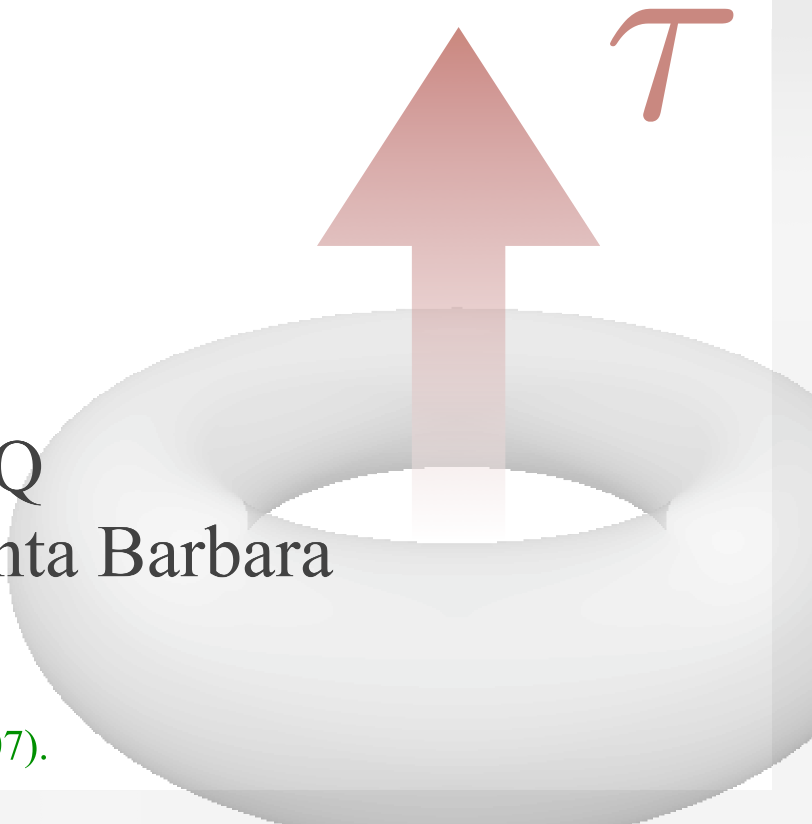

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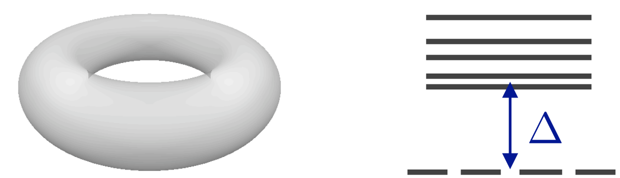
Outline

- **Topological quantum liquids**
  - Fractional quantum Hall liquids
  - Fibonacci anyons
- **Collective states of interacting anyons**
  - The golden chain
  - Topological stability & local perturbations
  - Variations of the chain: More things golden
- **Outlook**

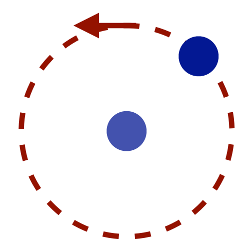
Topological quantum liquids

- Gapped spectrum
- **No** broken symmetry

- Degenerate ground state on torus

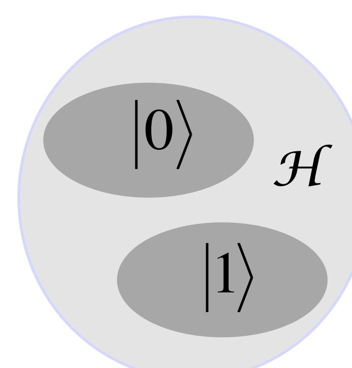


- Fractional statistics of excitations

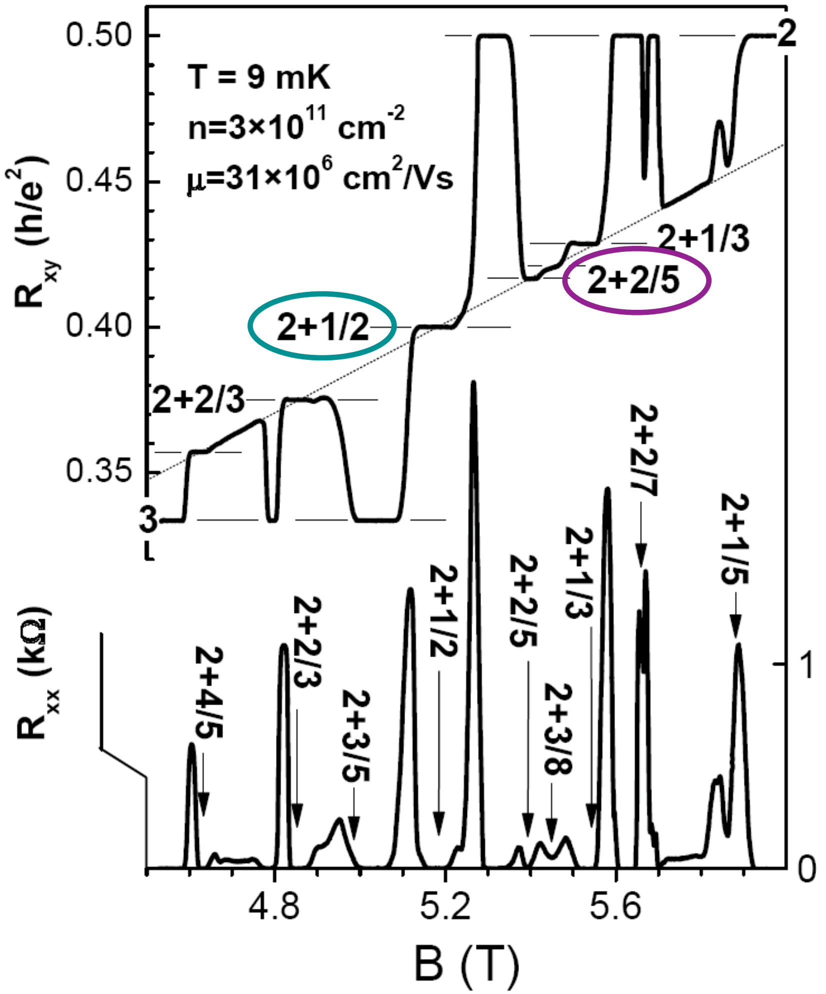


$e^{i\theta}$

- Hilbert space split into topological sectors
  No **local** matrix element mixes the sectors



Fractional quantum Hall liquids



J.S. Xia *et al.*, PRL (2004)

Moore-Read "Pfaffian" state

Moore & Read, Nucl. Physics B (1994)

Charge e/4 quasiparticles

Ising anyons

$SU(2)_2$

Nayak & Wilczek (1996)

Read-Rezayi "parafermion" state

Read & Rezayi, PRB (1999)

Charge e/5 quasiparticles

Fibonacci anyons

$SU(2)_3$

Slingerland & Bais (2001)

Anyonic statistics

Abelian anyons

$$\psi(x_2, x_1) = e^{i\pi\theta} \cdot \psi(x_1, x_2)$$

fractional phase

Non-Abelian anyons

matrix

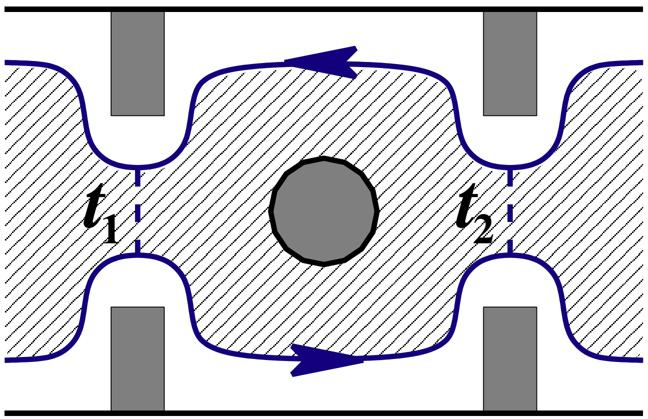
$$\psi(x_1 \leftrightarrow x_3) = M \cdot \psi(x_1, \ldots, x_n)$$

$$\psi(x_2 \leftrightarrow x_3) = N \cdot \psi(x_1, \ldots, x_n)$$

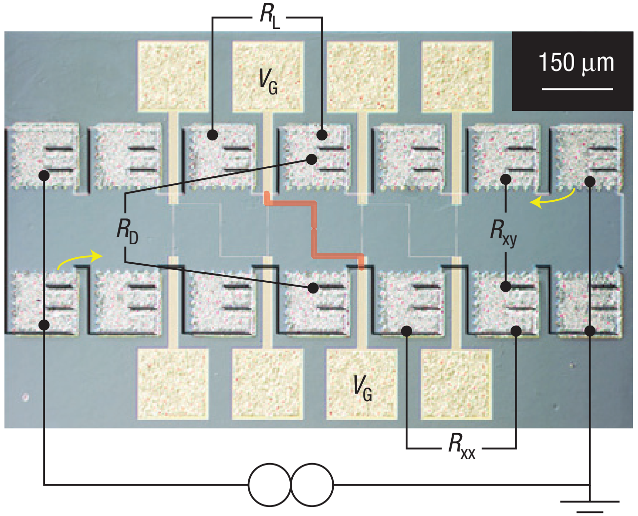
In general *M* and *N* do not commute!

Probing anyonic statistics

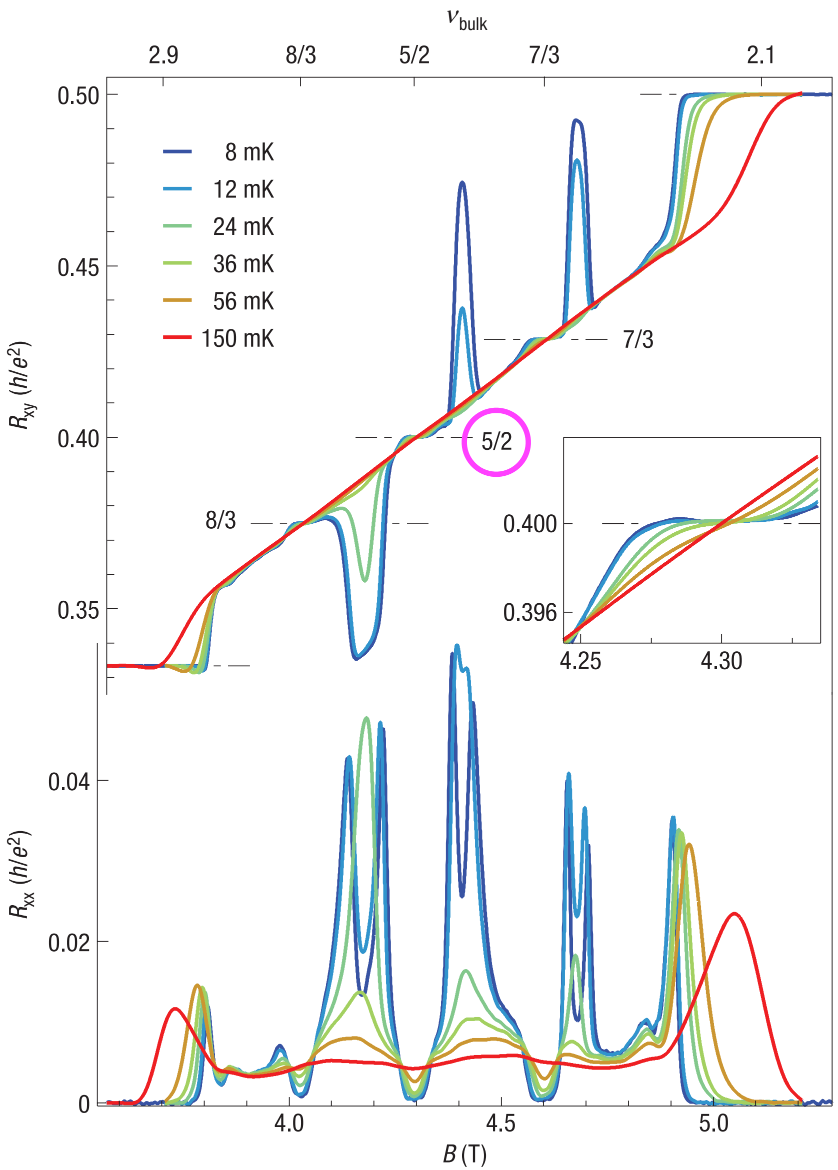
Interference experiments



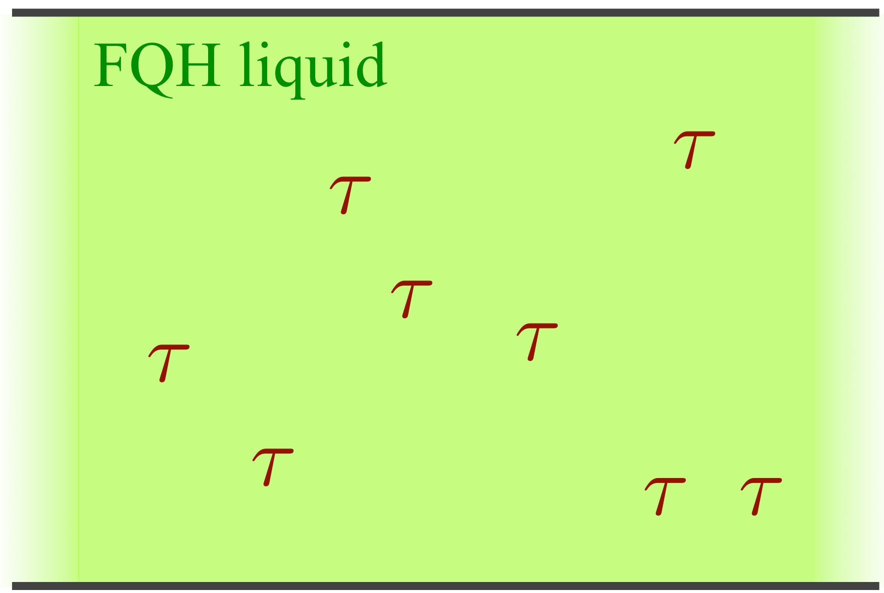
Bonderson, Kitaev, Shtengel, PRL (2006)
Stern & Halperin, PRL (2006)



J.B. Miller *et al.*, Nature Physics (2007)



Fibonacci anyons



How can we model interactions between anyons?

What is the **collective state**
of a set of **interacting anyons**?

Fibonacci anyons

Fusion rules for SU(2)$_3$

$$1 \times 1 = 1$$
$$1 \times \tau = \tau$$
$$\tau \times \tau = 1 + \tau$$

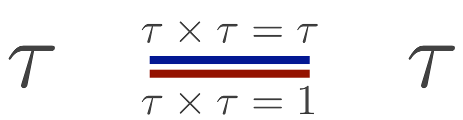
SU(2) spins

$$\frac{1}{2} \times \frac{1}{2} = 0 + 1$$

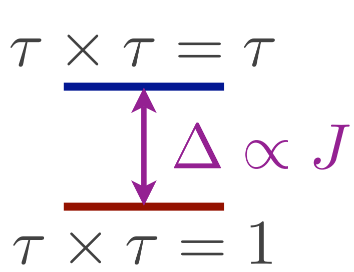
triplet

singlet

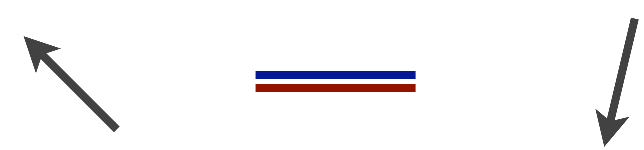
Weak interaction



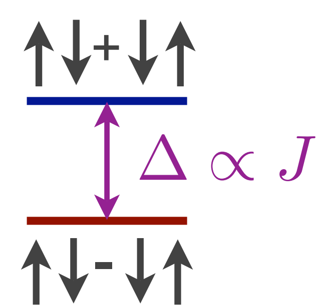
Strong interaction



Weak interaction



Strong interaction



The Heisenberg model

Fusion rules for SU(2)₃

$$1 \times 1 = 1$$
$$1 \times \tau = \tau$$
$$\tau \times \tau = 1 + \tau$$

SU(2) spins

$$\frac{1}{2} \times \frac{1}{2} = 0 + 1$$

triplet

singlet

Heisenberg Hamiltonian for (Fibonacci) anyons

$$H = J \sum_{\langle ij \rangle} \prod_{ij}^{1}$$

"antiferromagnet" favors 1 $(J > 0)$

"ferromagnet" favors τ $(J < 0)$

Heisenberg Hamiltonian

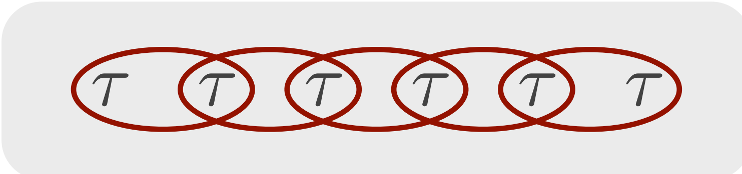
$$H = J \sum_{\langle ij \rangle} \vec{S}_i \cdot \vec{S}_j$$

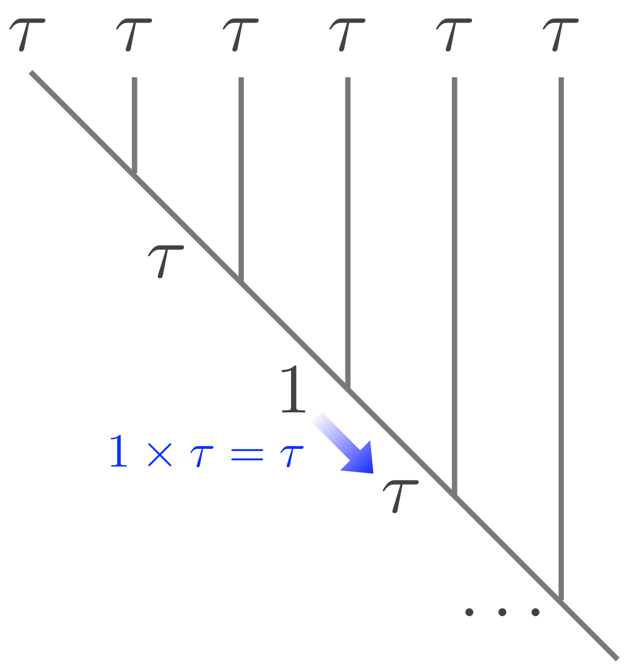
$$= J \sum_{\langle ij \rangle} \vec{J}_{ij}^{\,2} - \vec{S}_i^{\,2} - \vec{S}_j^{\,2}$$

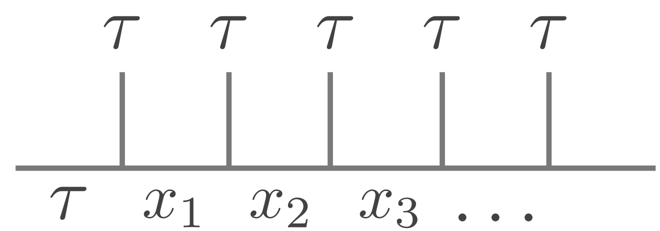
$$\vec{J}_{ij} = \vec{S}_i + \vec{S}_j$$

$$H = J \sum_{\langle ij \rangle} \prod_{ij}^{0}$$

The golden chain







Hilbert space: $|x_1, x_2, x_3, \ldots\rangle$

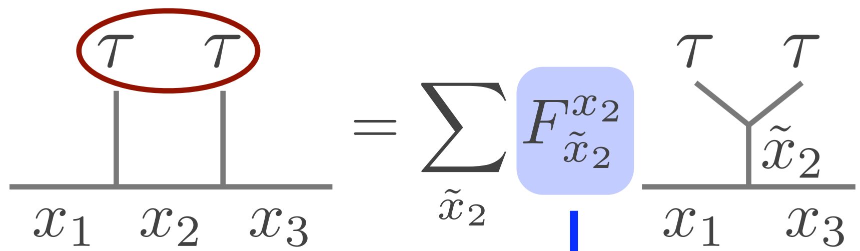
$$\dim_L = F_{L+1} \propto \phi^L$$

$$\phi = \frac{1+\sqrt{5}}{2} = 1.618\ldots$$

Hilbert space has **no natural decomposition** as tensor product of single-site states.

The golden chain

We want to construct a **local** Hamiltonian $H = \sum_i H_i$.

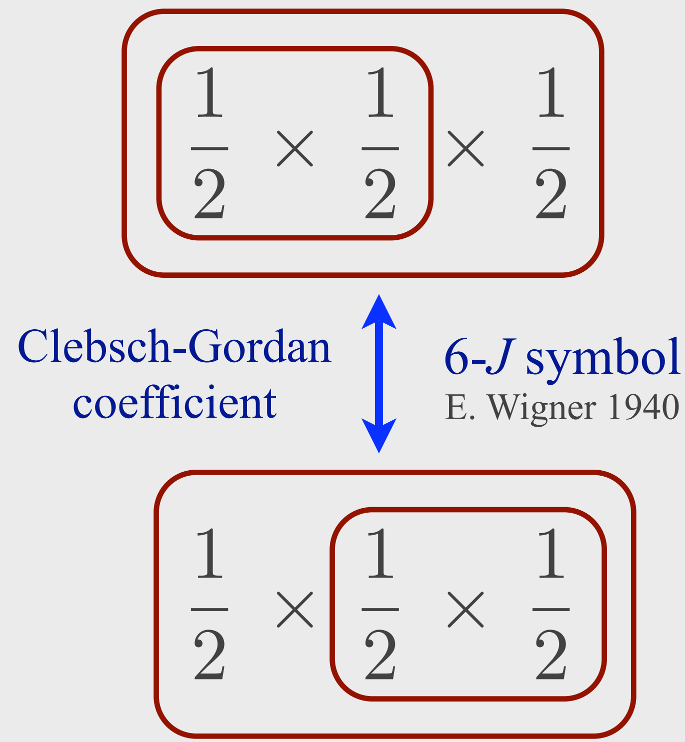


F-matrix

$$F = \begin{pmatrix} \phi^{-1} & \phi^{-1/2} \\ \phi^{-1/2} & -\phi^{-1} \end{pmatrix}$$

Local Hamiltonian: $H_i = F_i \, \Pi^1_i \, F_i$

SU(2) spins



The golden chain

Local Hamiltonian: $H_i = F_i \, \Pi_i^1 \, F_i$

$$H_i = -\begin{pmatrix} \phi^{-2} & \phi^{-3/2} \\ \phi^{-3/2} & \phi^{-1} \end{pmatrix}$$

Explicit form:

$$\begin{aligned} H_i \quad = \quad & -\mathcal{P}_{1\tau 1} - \phi^{-2}\mathcal{P}_{\tau 1 \tau} - \phi^{-1}\mathcal{P}_{\tau\tau\tau} \\ & -\phi^{-3/2}\left(\left|\tau 1 \tau\right\rangle \left\langle \tau\tau\tau\right| + \text{h.c.}\right) \end{aligned}$$

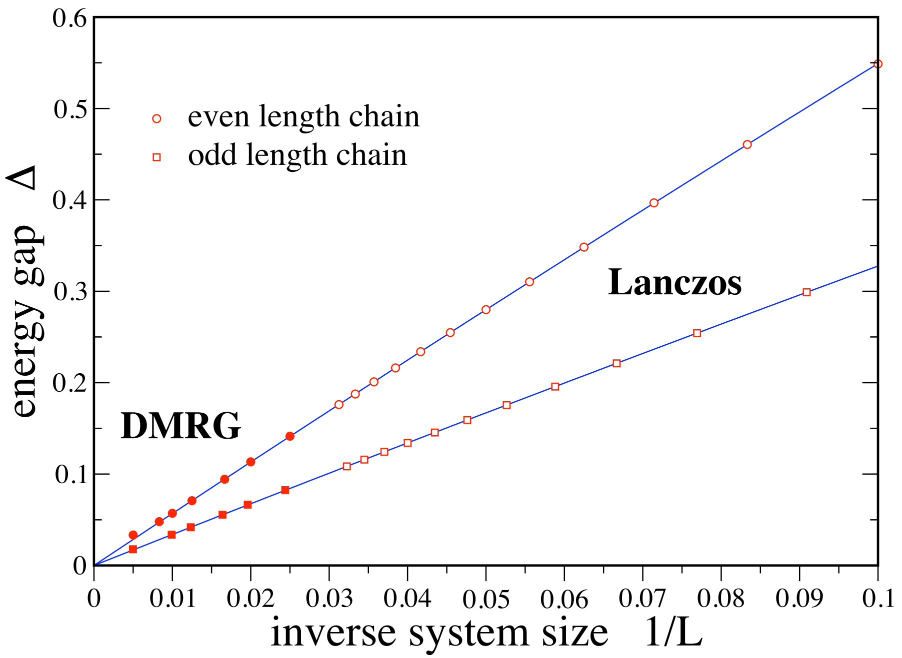
off-diagonal matrix element

Criticality

Finite-size gap

$$\Delta(L) \propto (1/L)^{z=1}$$

conformal field theory description

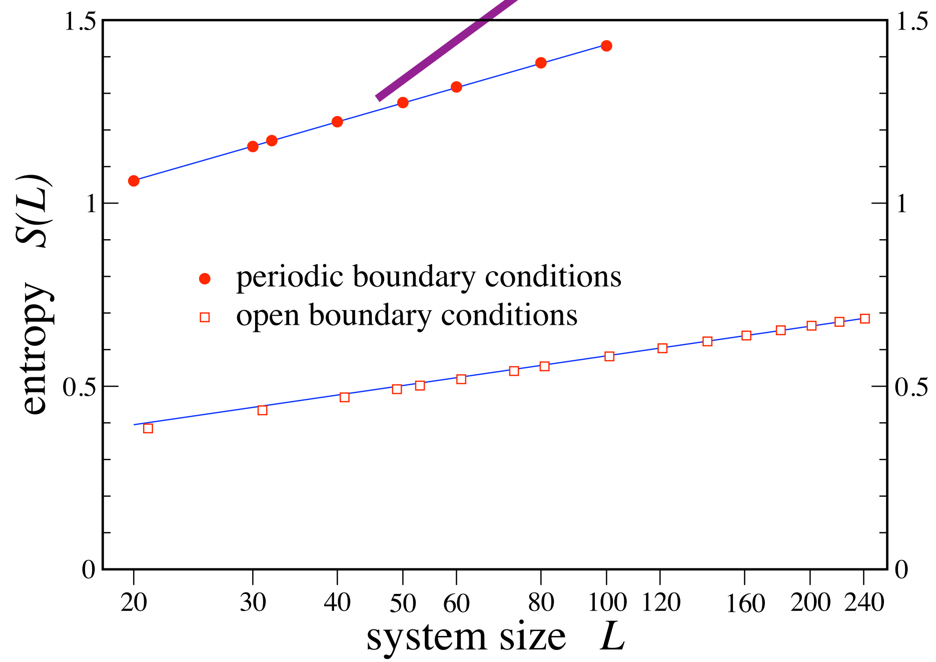


Entanglement entropy

$$S_{\mathrm{PBC}}(L) \propto \frac{c}{3} \log L$$

central charge

$c = 7/10$



Mapping & exact solution

The operators $X_i = -\phi H_i$ form a representation of the **Temperley-Lieb algebra**

$$(X_i)^2 = d \cdot X_i \qquad X_i X_{i\pm 1} X_i = X_i \qquad [X_i, X_j] = 0$$

for $|i - j| \geq 2$

"d-isotopy" parameter
$d = \phi$

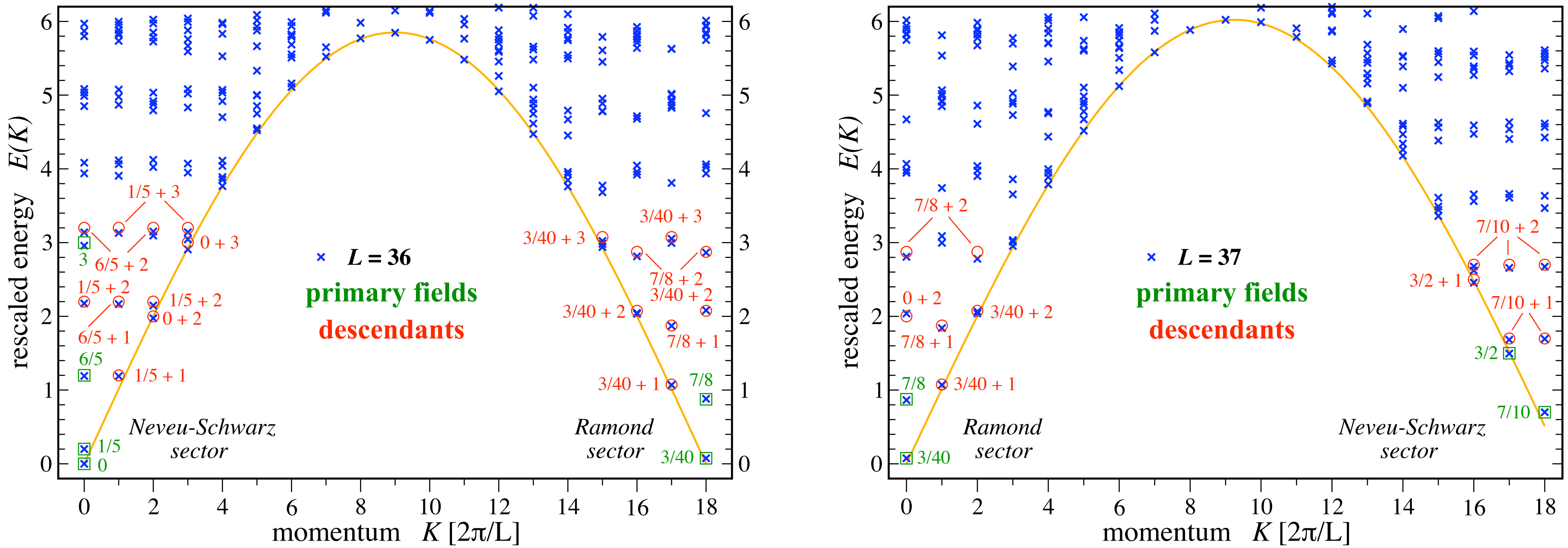
quantum 1D Hamiltonian

integrable lattice model description
restricted-solid-on-solid model (RSOS)

classical 2D
tricritical Ising model

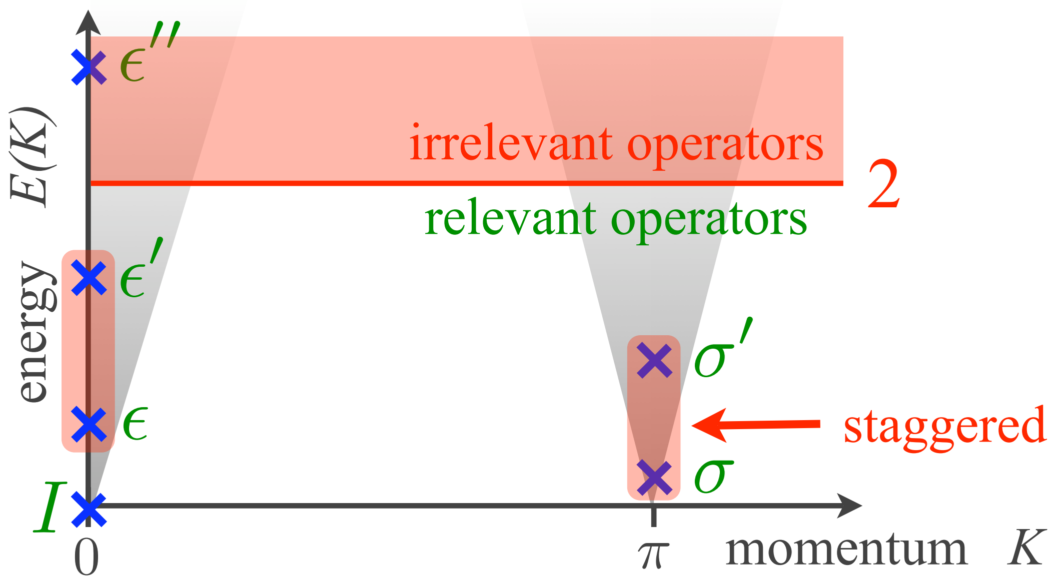
central charge
$c = 7/10$

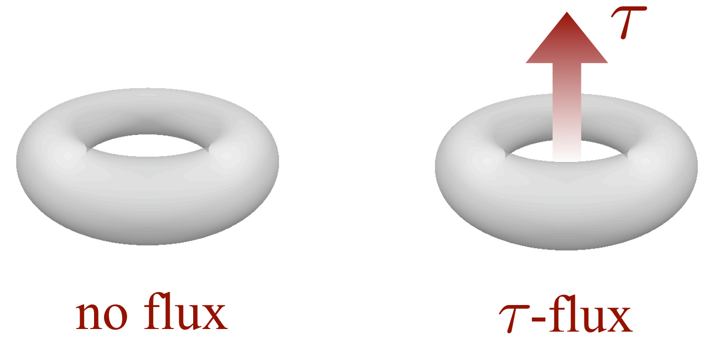
Energy spectra



primary fields	I	ϵ	ϵ'	ϵ''	σ	σ'
scaling dimensions	0	$1/10$	$3/5$	$3/2$	$3/80$	$7/16$
	$K = 0$				$K = \pi$	

Topological symmetry





Relevant perturbations

~~$\sigma_L \sigma_R$~~ ~~$\sigma'_L \sigma'_R$~~ prohibited by **translational symmetry**

~~$\epsilon_L \epsilon_R$~~ ~~$\epsilon'_L \epsilon'_R$~~ prohibited by **topological symmetry**

Symmetry operator

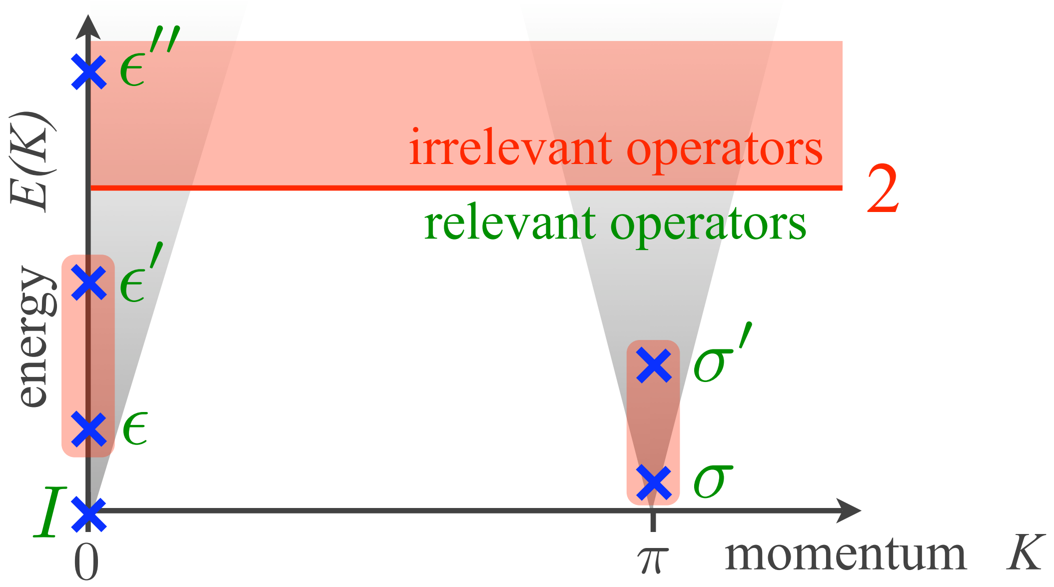
$$\langle x'_1, \ldots, x'_L | Y | x_1, \ldots, x_L \rangle = \prod_{i=1}^{L} \left(F^{x'_{i+1}}_{\tau x_i \tau} \right)^{x'_i}_{x_{i+1}}$$

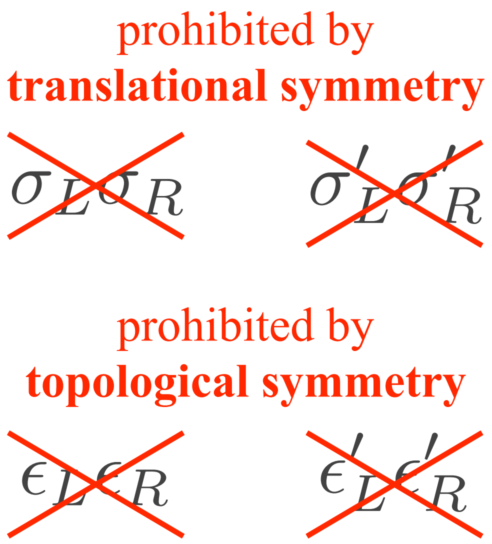
with eigenvalues

$$S_{\tau-\text{flux}} = \phi \quad S_{\text{no flux}} = -\phi^{-1}$$

$$[H, Y] = 0$$

Topological stability



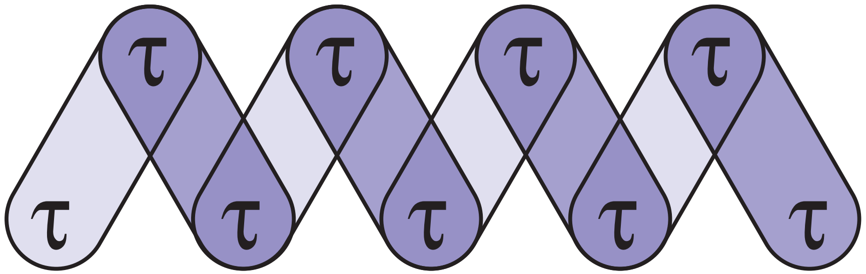


The **criticality** of the chain is **protected** by additional topological symmetry.

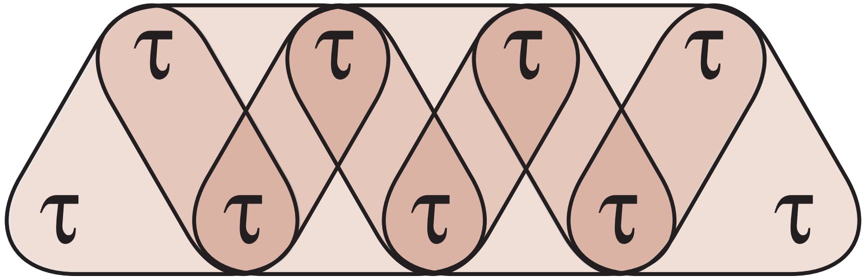
Local perturbations do not gap the system.

Is this special to $SU(2)_3$?

A larger space of models



“dimerized” chain

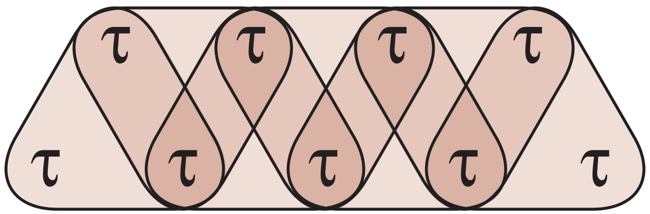


Majumdar-Ghosh chain

Majumdar-Ghosh chain

Consider a (competing) three anyon-fusion term
➠ neither translational nor topological symmetry are broken

SU(2)$_3$ anyons

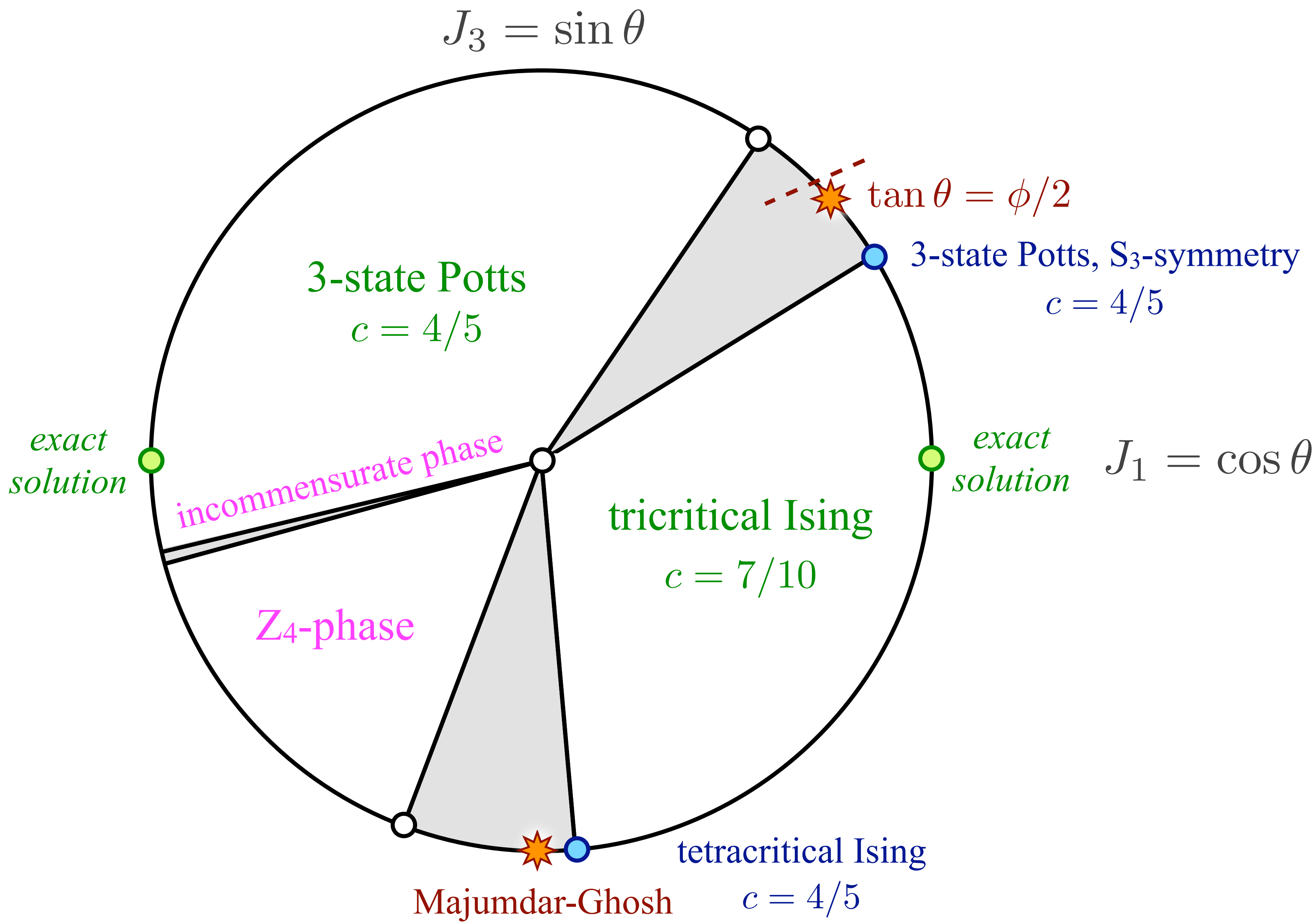


SU(2) spins

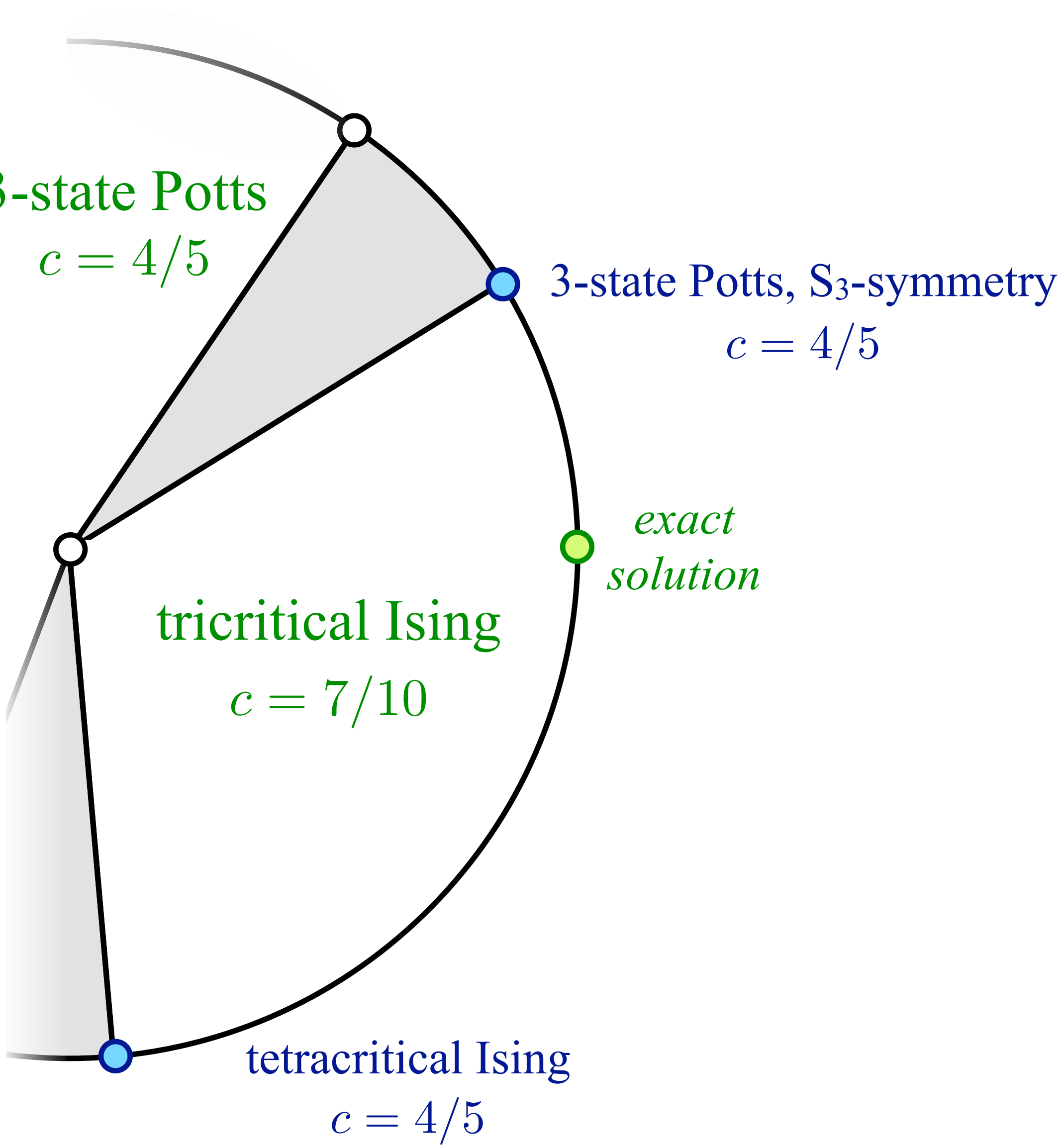
$$
\begin{aligned}
H_{\mathrm{MG}} &= J \sum_i \vec{T}^2_{i-1,i,i+1} \\
&= J \sum_i \vec{S}_i \vec{S}_{i+1} + \frac{J}{2} \sum_i \vec{S}_i \vec{S}_{i+2}
\end{aligned}
$$

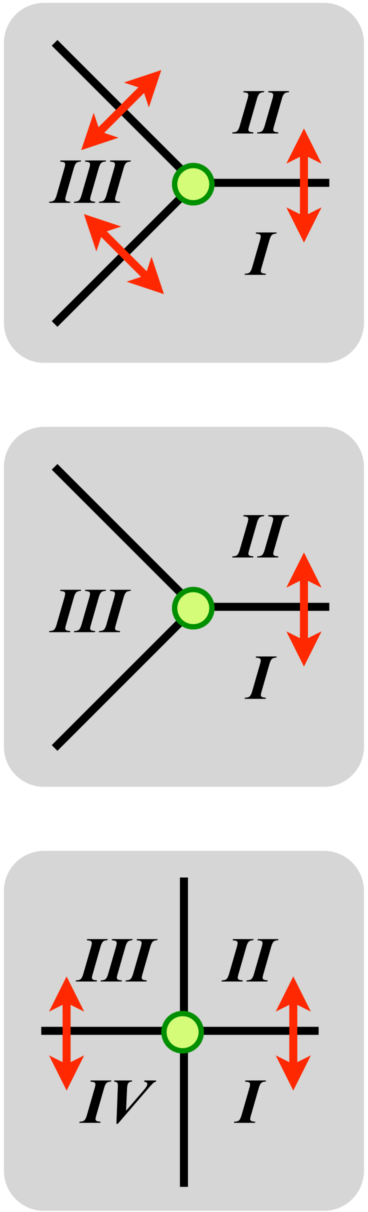
$$
\begin{aligned}
H_i = \; & \mathcal{P}_{\tau 1 \tau 1} + \mathcal{P}_{1 \tau 1 \tau} + \mathcal{P}_{\tau\tau\tau 1} + \mathcal{P}_{1 \tau\tau\tau} + 2\phi^{-2} \mathcal{P}_{\tau\tau\tau\tau} + \\
& \phi^{-1} \left(\mathcal{P}_{\tau 1 \tau\tau} + \mathcal{P}_{\tau\tau 1 \tau} \right) - \phi^{-2} \big(|\tau\tau 1 \tau\rangle \langle \tau 1 \tau\tau| + \text{h.c.} \big) + \\
& \phi^{-5/2} \left(|\tau 1 \tau\tau\rangle \langle \tau\tau\tau\tau| + |\tau\tau 1 \tau\rangle \langle \tau\tau\tau\tau| + \text{h.c.} \right)
\end{aligned}
$$

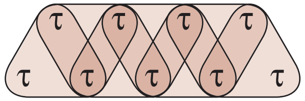
Phase diagram



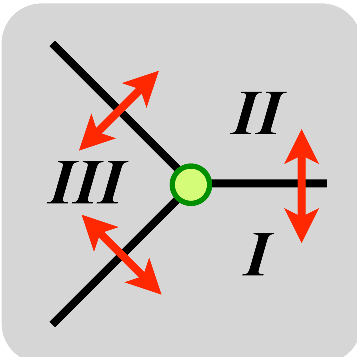
Critical endpoints



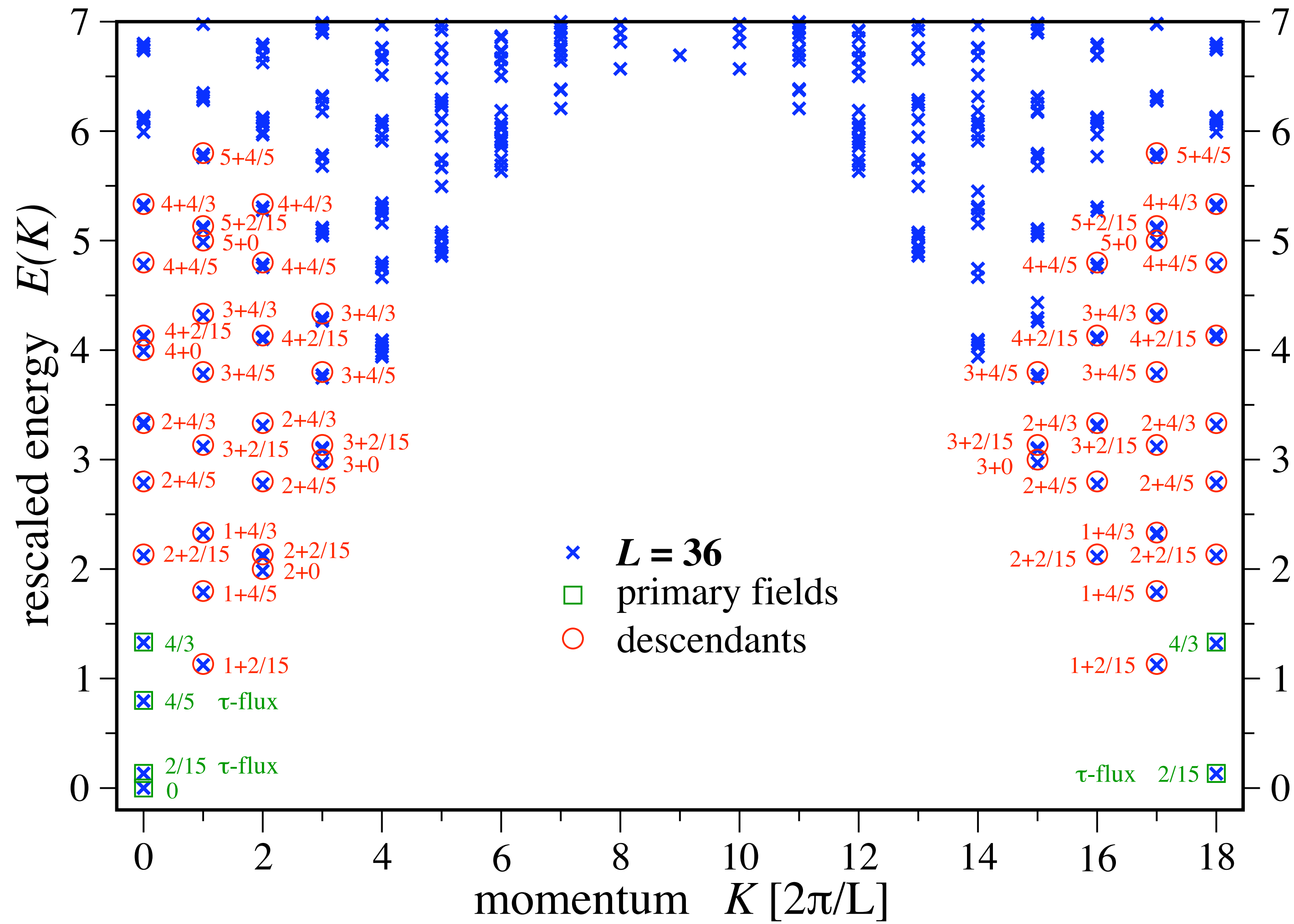




S_3-symmetric point

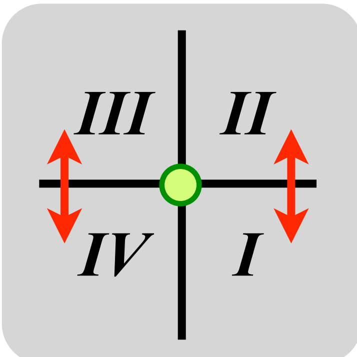


$\theta = 0.176\pi$

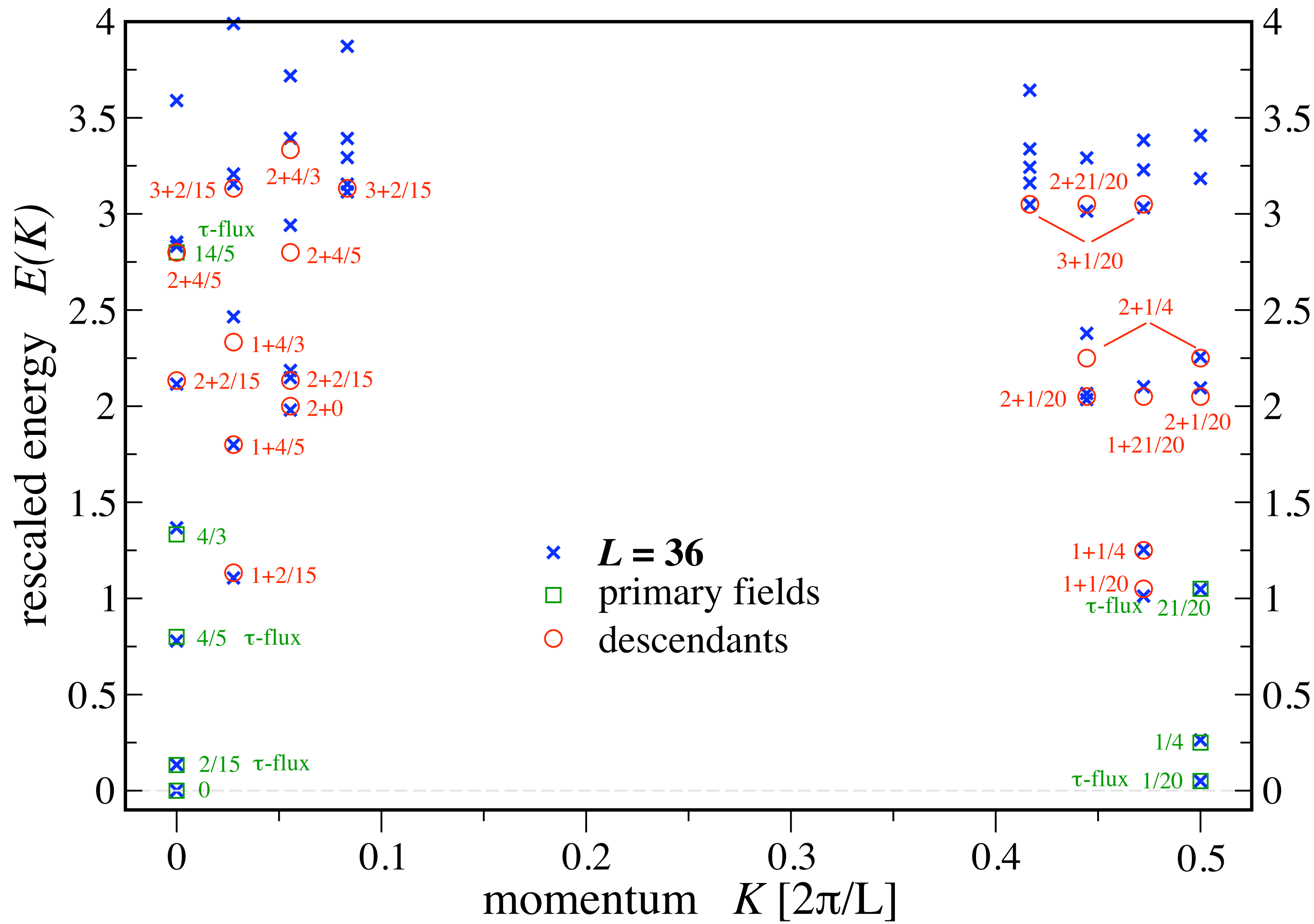


Conformal field theory: parafermions with central charge $c = 4/5$.

Tetracritical Ising

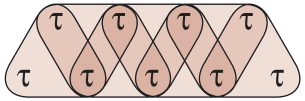


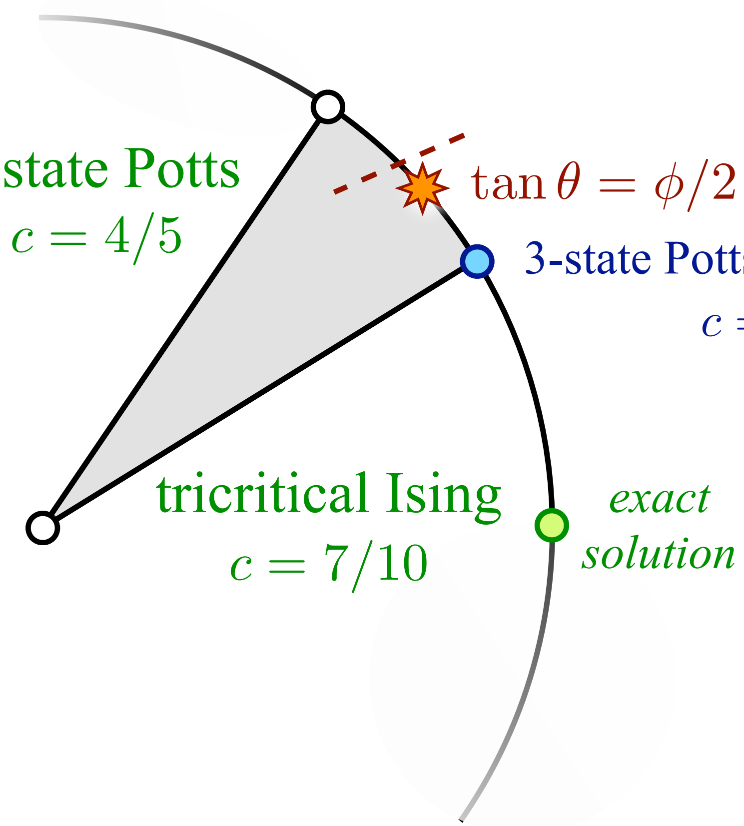
$\theta = 1.528\pi$

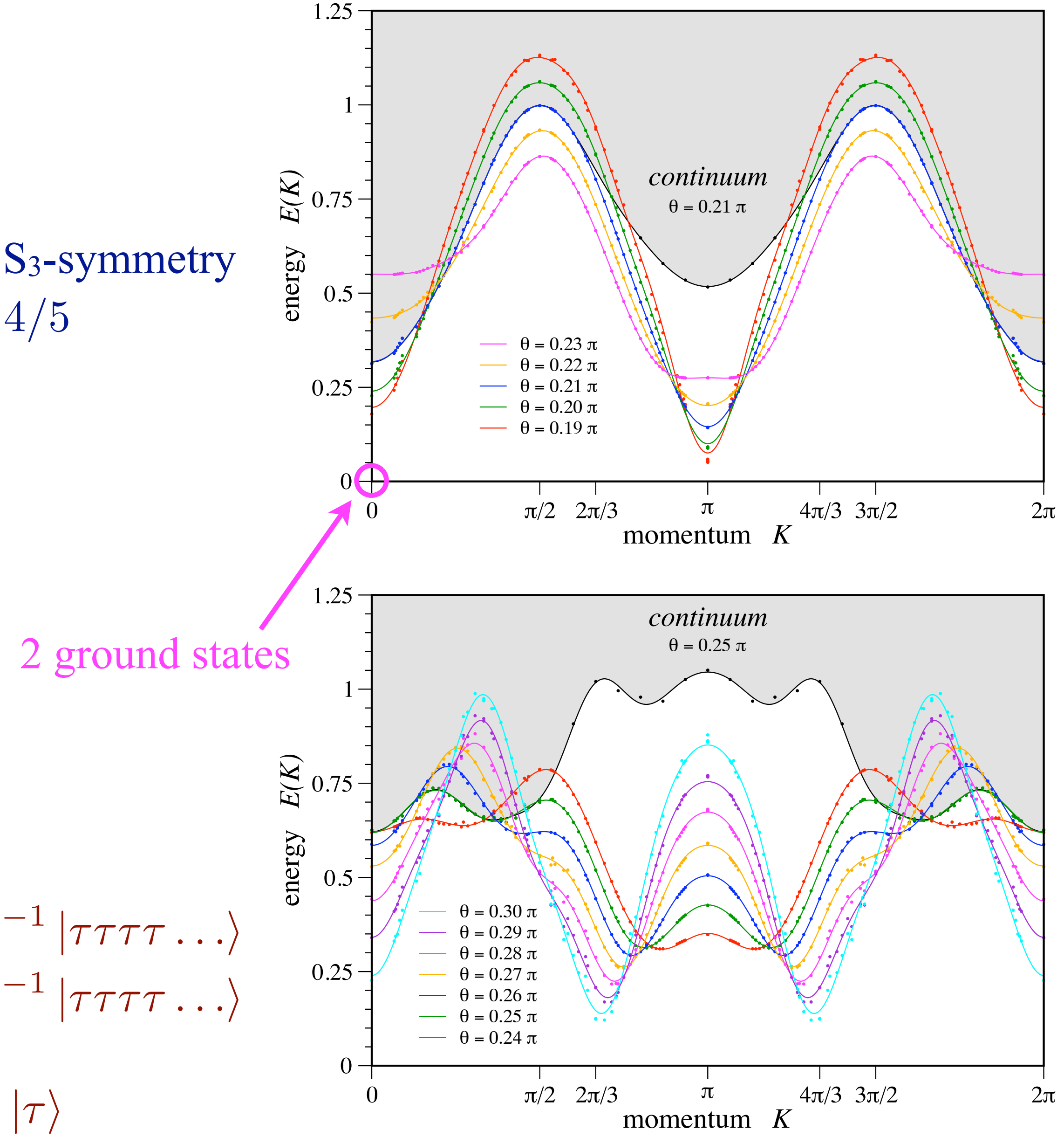


Conformal field theory: minimal model with central charge $c = 4/5$.

Gapped phase (AFM)





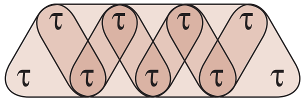


Exact ground states

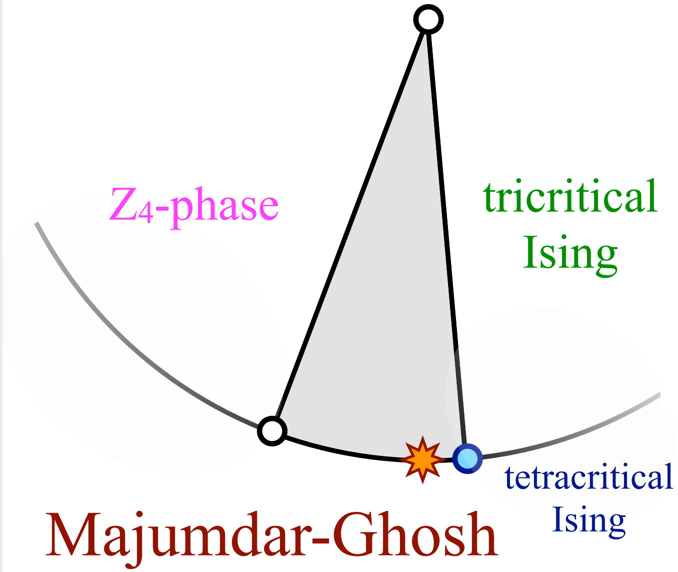
$$|\psi_{\text{no-flux}}\rangle = |\tilde{\tau}_x\tilde{\tau}_x\tilde{\tau}_x\tilde{\tau}_x\ldots\rangle - (-\phi)^{L-1}|\tau\tau\tau\tau\ldots\rangle$$

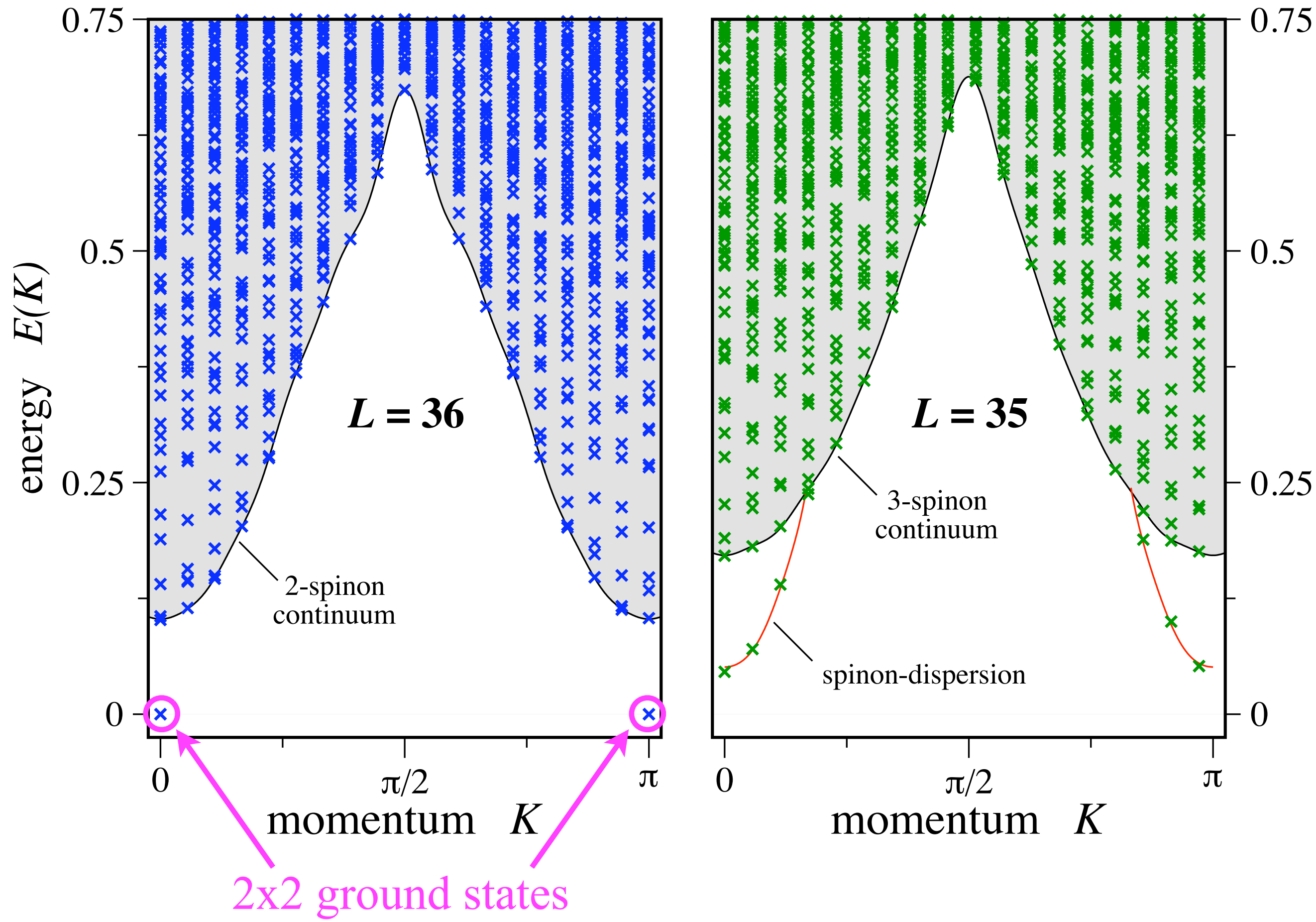
$$|\psi_{\text{flux}}\rangle = |\tilde{\tau}_x\tilde{\tau}_x\tilde{\tau}_x\tilde{\tau}_x\ldots\rangle + (-\phi)^{L-1}|\tau\tau\tau\tau\ldots\rangle$$

$$\tilde{\tau}_x = \phi^{3/2}|1\rangle + |\tau\rangle$$



Majumdar-Ghosh point





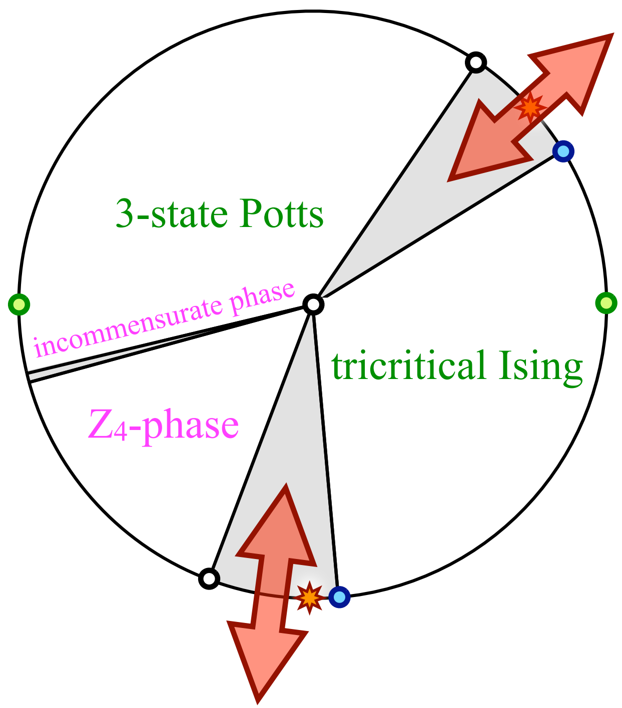
Exact ground states

$$|\psi_{\text{no-flux}}\rangle = |\tau_x\tau\tau_x\tau\tau_x\tau\ldots\rangle + \phi^{L/4-1}|\tau 1\tau 1\tau 1\ldots\rangle \pm |\tau\tau_x\tau\tau_x\tau\tau_x\ldots\rangle + \phi^{L/4-1}|1\tau 1\tau 1\tau\ldots\rangle$$

$$|\psi_{\text{flux}}\rangle = |\tau_x\tau\tau_x\tau\tau_x\tau\ldots\rangle - \phi^{L/4-1}|\tau 1\tau 1\tau 1\ldots\rangle \pm |\tau\tau_x\tau\tau_x\tau\tau_x\ldots\rangle - \phi^{L/4-1}|1\tau 1\tau 1\tau\ldots\rangle$$

$$\tau_x = \phi^{-1/2}|1\rangle + |\tau\rangle$$

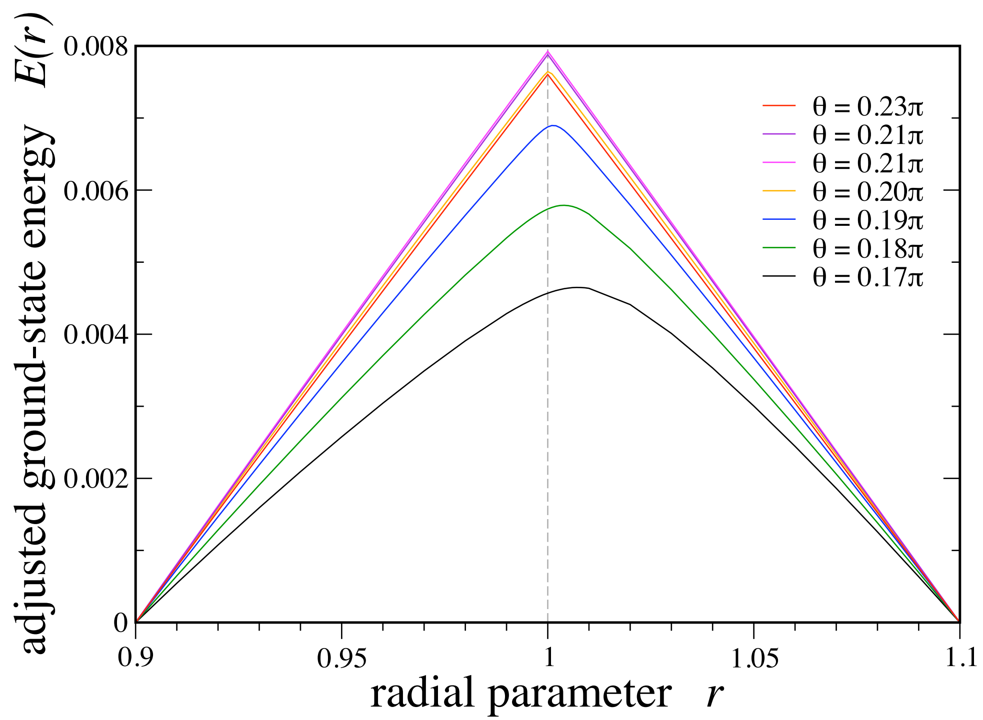
Breaking the topological symmetry



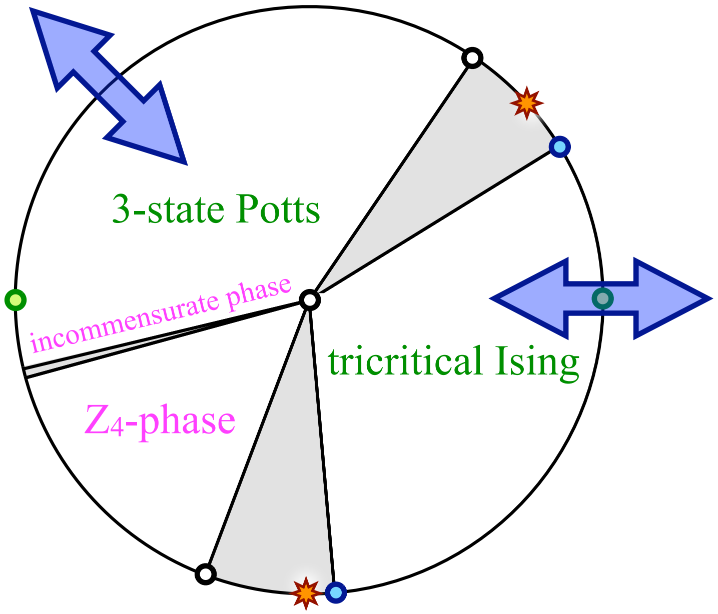
Explicitly break the topological symmetry by going “off” the circle

$$H^i = r \cdot H^i_{\mathrm{diagonal}} + H^i_{\mathrm{off-diagonal}}$$

Gapped phases ➠ **1st order**

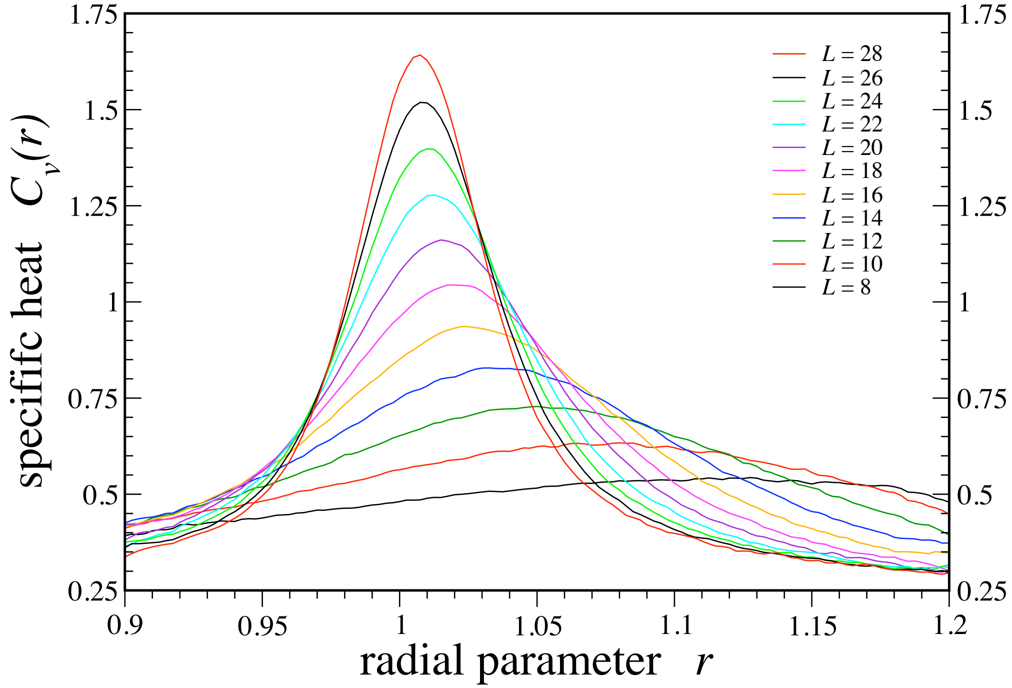


Breaking the topological symmetry



Explicitly break the topological symmetry by going "off" the circle

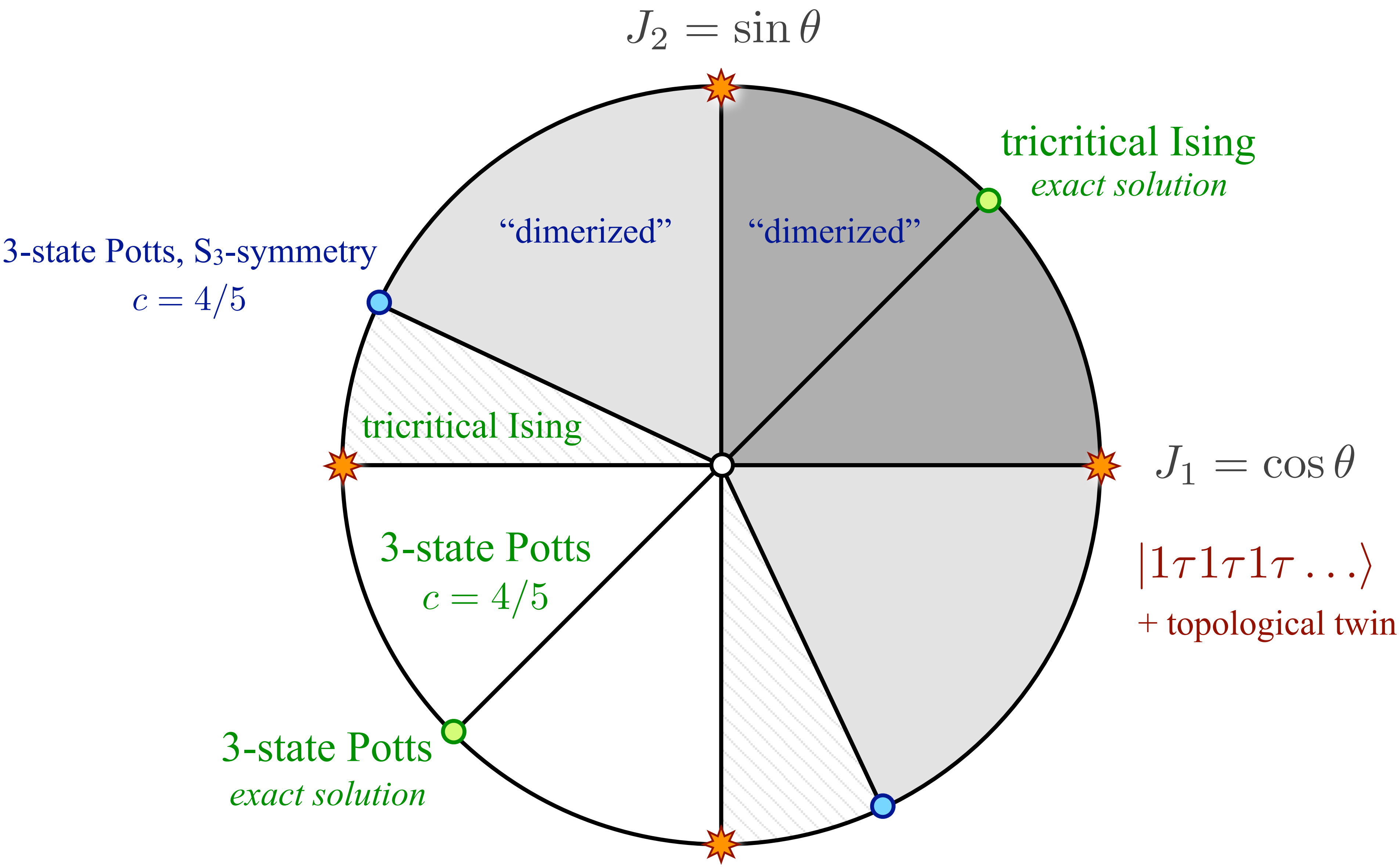
$$H^i = r \cdot H^i_{\mathrm{diagonal}} + H^i_{\mathrm{off-diagonal}}$$



Gapped phases ➡ **1st order**

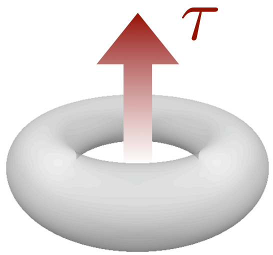
Critical phases ➡ **2nd order**

“Dimerized” chain



Summary and Outlook

- Interacting non-Abelian anyons can support a variety of collective states.
- Interactions modeled by Heisenberg Hamiltonian generalized to anyonic degrees of freedom.
- Exact solutions, CFT descriptions, ...
- Topological symmetry protects critical phases.



Do these observations generalize to $SU(2)_k$?

What happens in two dimensions?

What happens for higher genus surfaces?

arXiv 0801.4602
Phys. Rev. Lett. **98**, 160409 (2007).

Acknowledgments

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- Alexei Kitaev — Caltech
- Andreas Ludwig — UC Santa Barbara
- Matthias Troyer — ETH Zürich
- Zhenghan Wang — Station Q