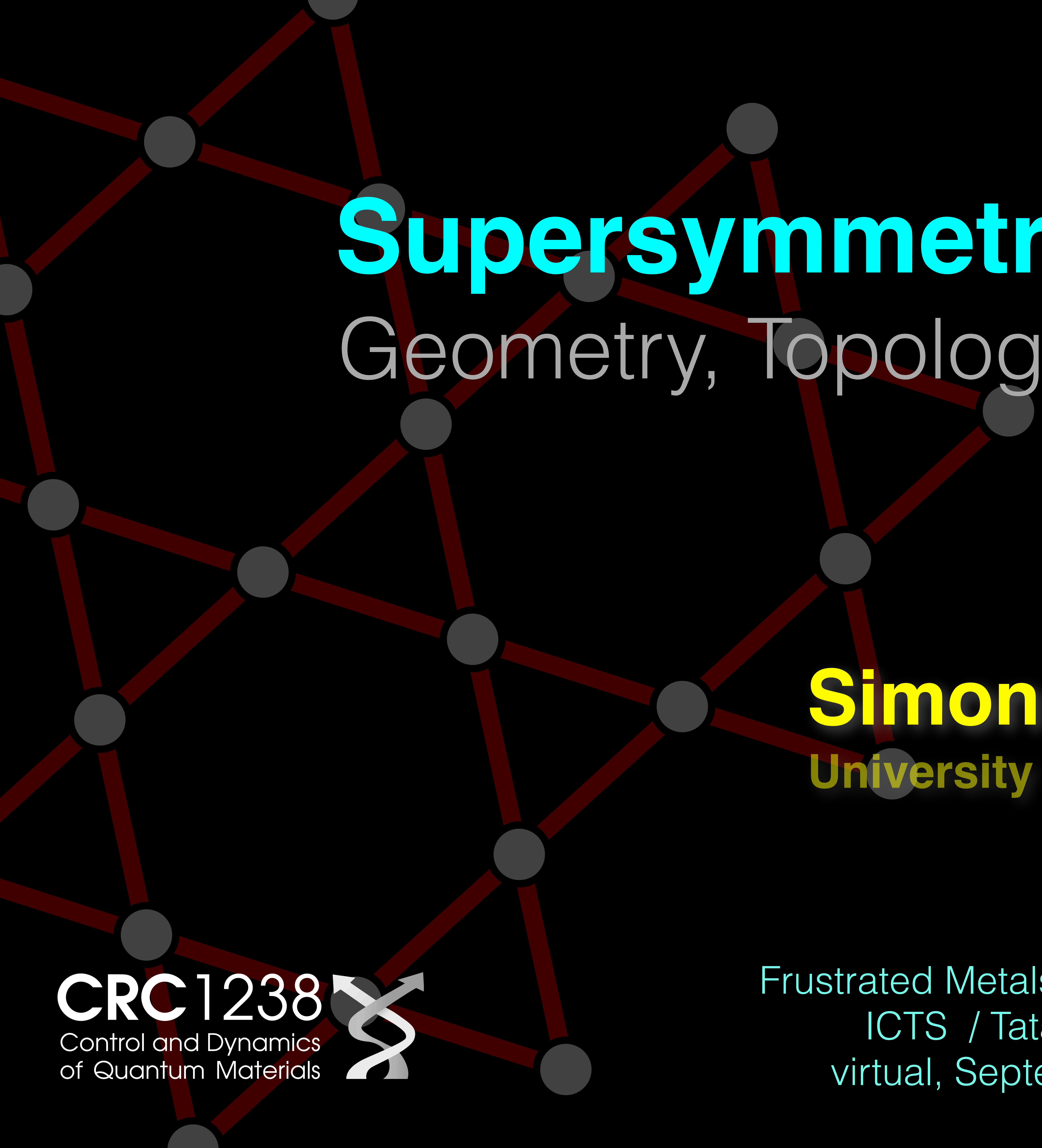


Supersymmetry on the lattice

Geometry, Topology, and Spin Liquids

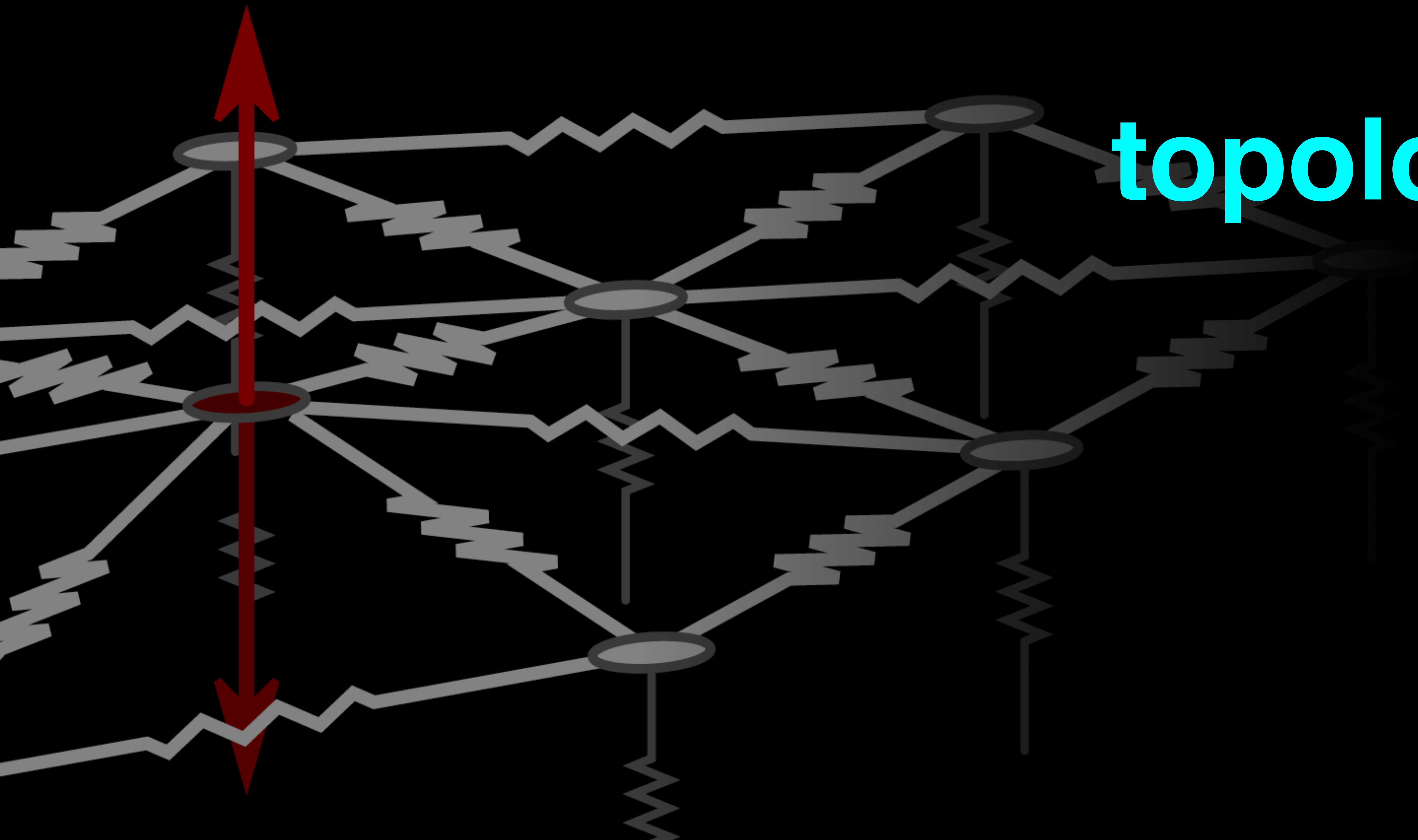
Simon Trebst

University of Cologne

Frustrated Metals and Insulators
ICTS / Tata Institute
virtual, September 2022

CRC 1238
Control and Dynamics
of Quantum Materials

CRC 183
ENTANGLED STATES OF MATTER



topological mechanics

topological mechanics

C.L. Kane & T.C. Lubensky, Nat. Phys. **10**, 39 (2014)

example #1: “floppy modes” in isostatic lattices

Maxwell counting

$$\nu \equiv N_0 - N_{\mathrm{ss}} = d \cdot n_{\mathrm{s}} - n_{\mathrm{b}}$$

degrees of freedom

states of self stress

sites

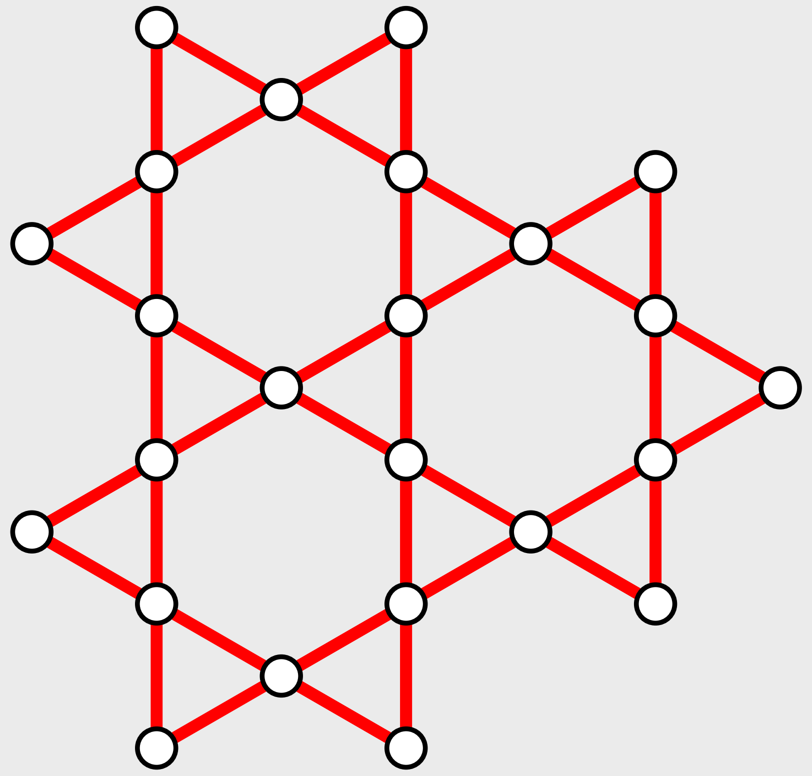
bonds

isostatic lattices

$\nu = 0$

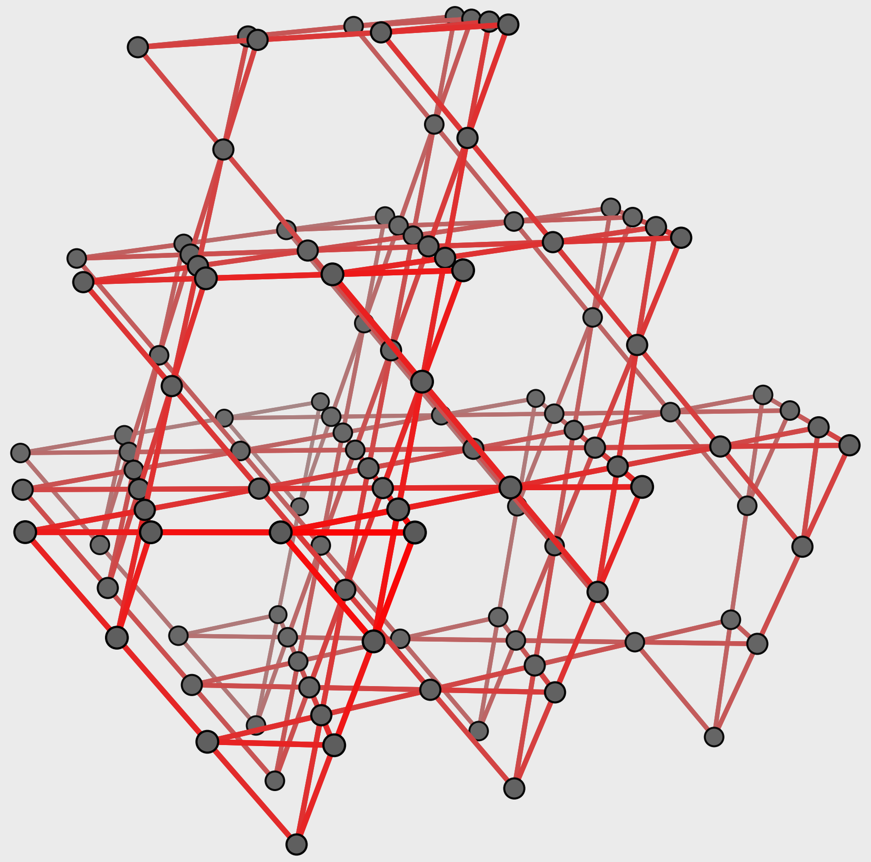
coordination number

$z = 2 \cdot d$



kagome lattice

$d = 2 \quad z = 4$

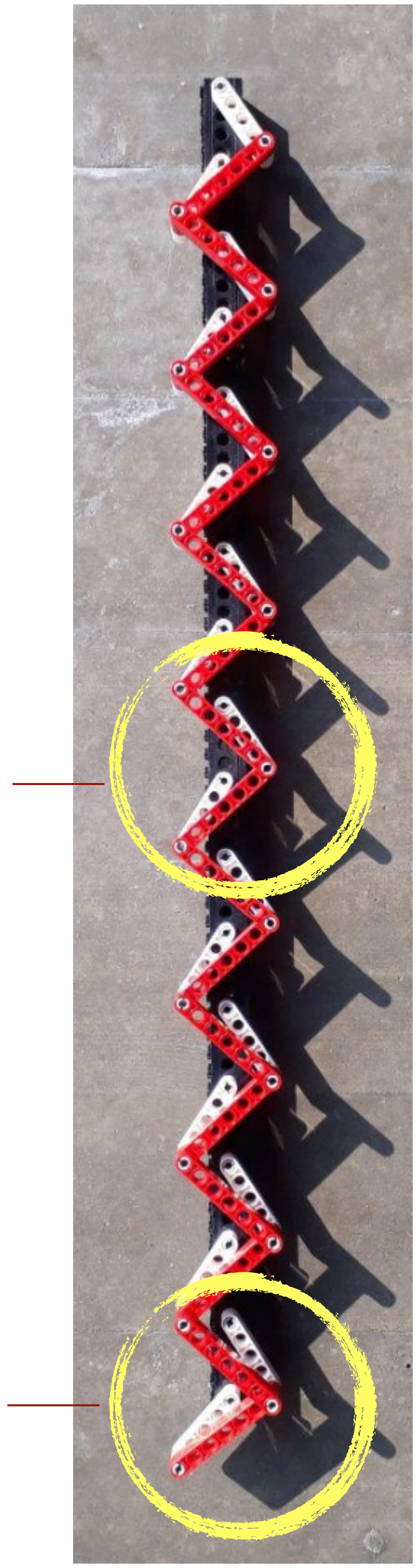


pyrochlore lattice

$d = 3 \quad z = 6$

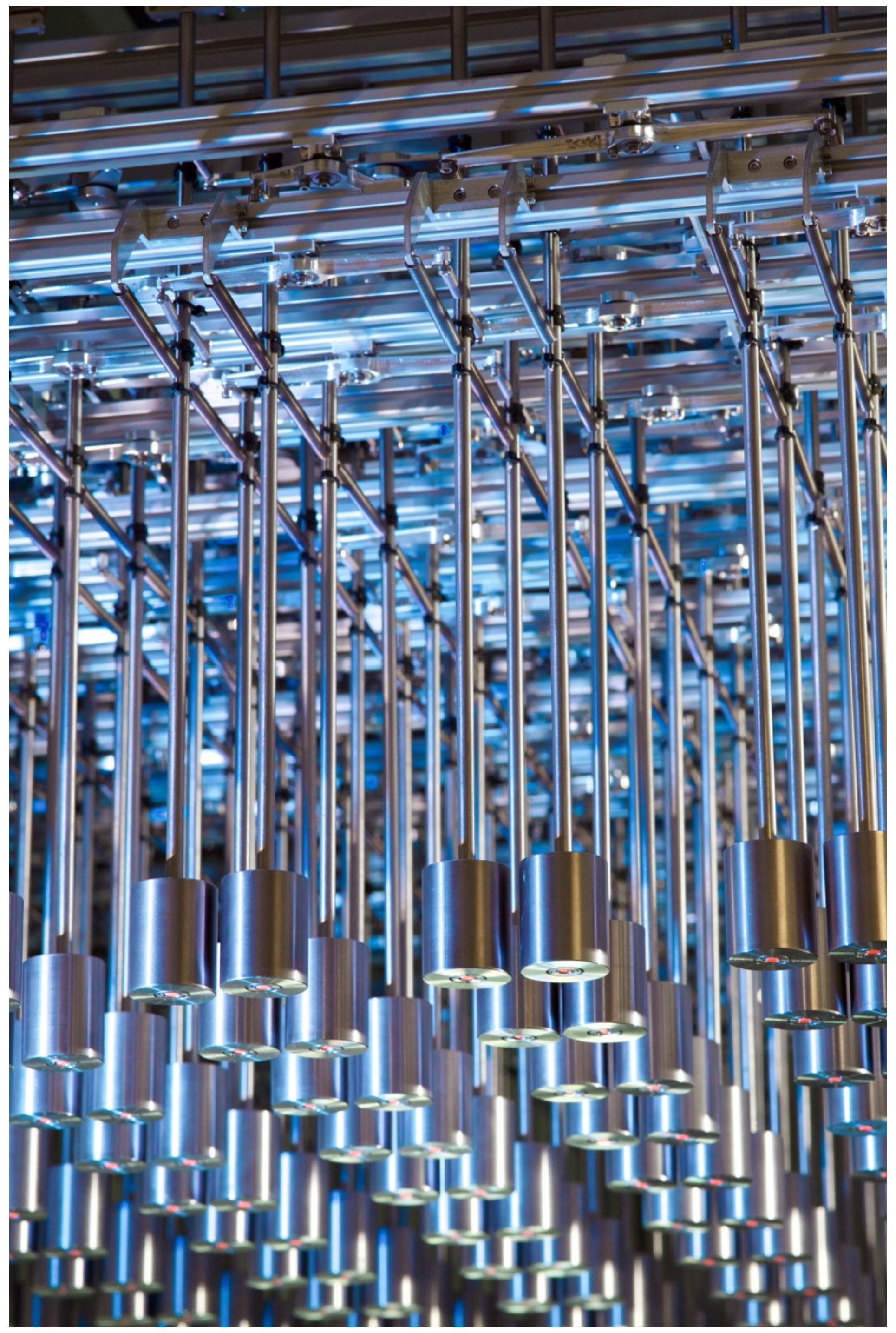
“rigid mode”

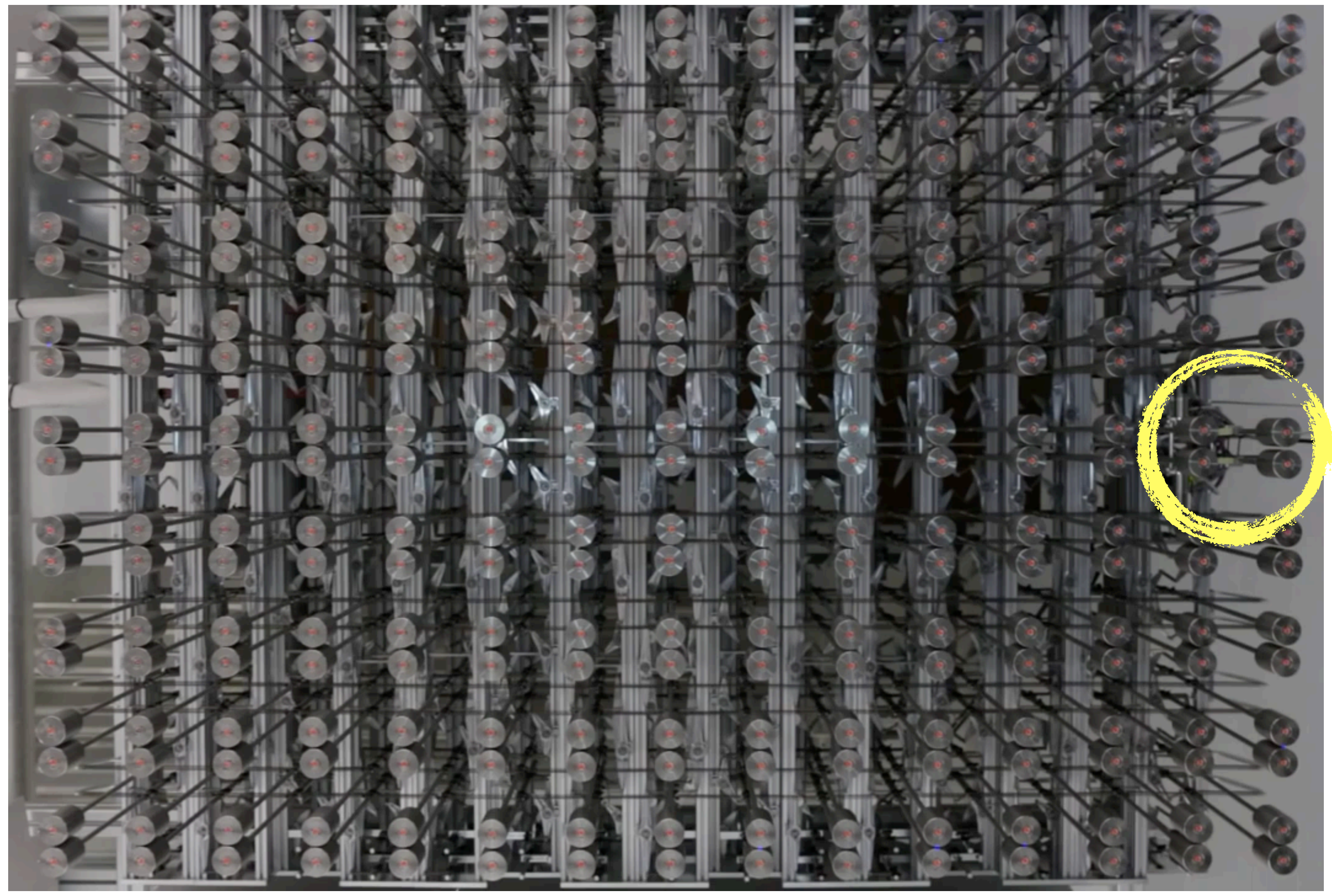
“floppy mode”



topological mechanics

example #2: topological insulator from classical pendula





R. Süsstrunk and S. D. Huber, Science **349**, 47 (2015)
S. D. Huber, Nature Phys. **12**, 621 (2016)

correspondence principles

topological mechanics

electronic system

(matrix) correspondence

mechanical system

Schrödinger's equation

$$i\frac{\partial}{\partial t}\begin{pmatrix}\sqrt{\mathbf{D}}^T x \\ i\dot{x}\end{pmatrix} = \begin{pmatrix}0 & \sqrt{\mathbf{D}}^T \\ \sqrt{\mathbf{D}} & 0\end{pmatrix}\begin{pmatrix}\sqrt{\mathbf{D}}^T x \\ i\dot{x}\end{pmatrix}$$

symmetry class BDI

square root

Newton's equation

$$\ddot{x} = -\mathbf{D}x$$

dynamical matrix

fermions **supersymmetry** **bosons**

meet the team

Jan Attig

Phys. Rev. B 96, 085145 (2017)

Krishanu Roychowdury

Phys. Rev. Res. 1, 032047(R) (2019)

Michael Lawler

arXiv:2207.09475

supersymmetry

arXiv:2207.09475

supersymmetric lattice models

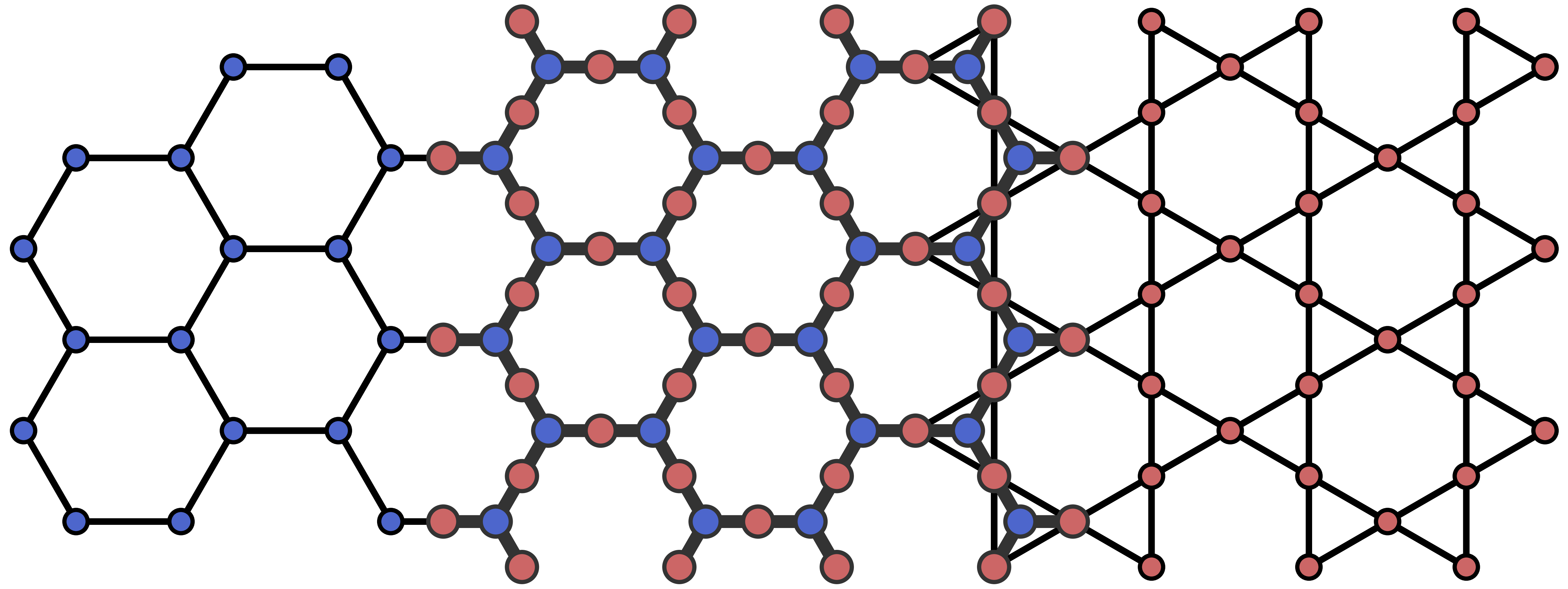
$$\mathbf{H} = \begin{pmatrix} & \mathbf{R} \\ \mathbf{R}^\dagger & \end{pmatrix}$$

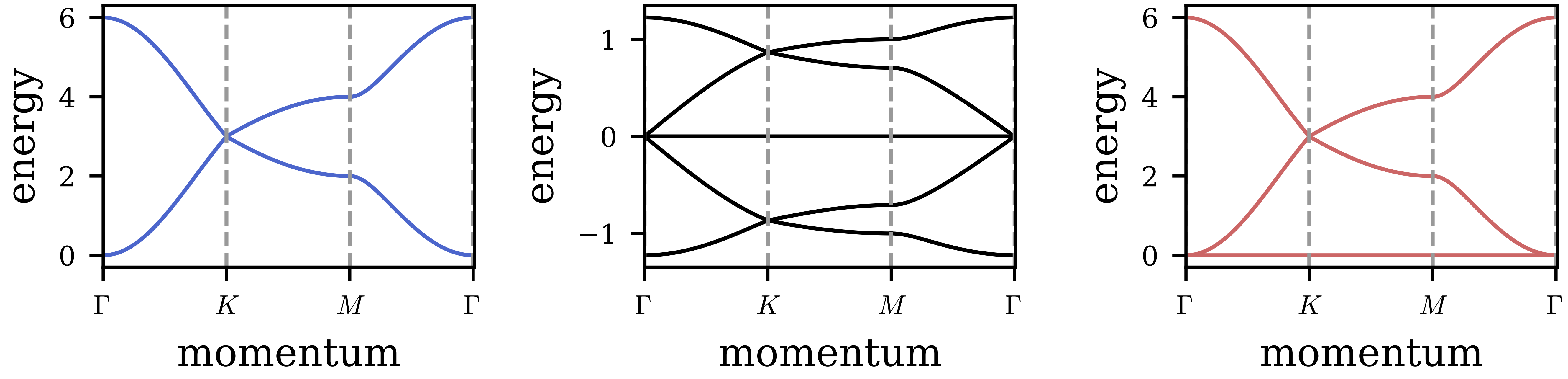
Hermitian matrix

matrix square

$$\mathbf{H}^2 = \begin{pmatrix} \boxed{\mathbf{R}\mathbf{R}^\dagger} & \\ & \boxed{\mathbf{R}^\dagger\mathbf{R}} \end{pmatrix}$$

isospectral blocks





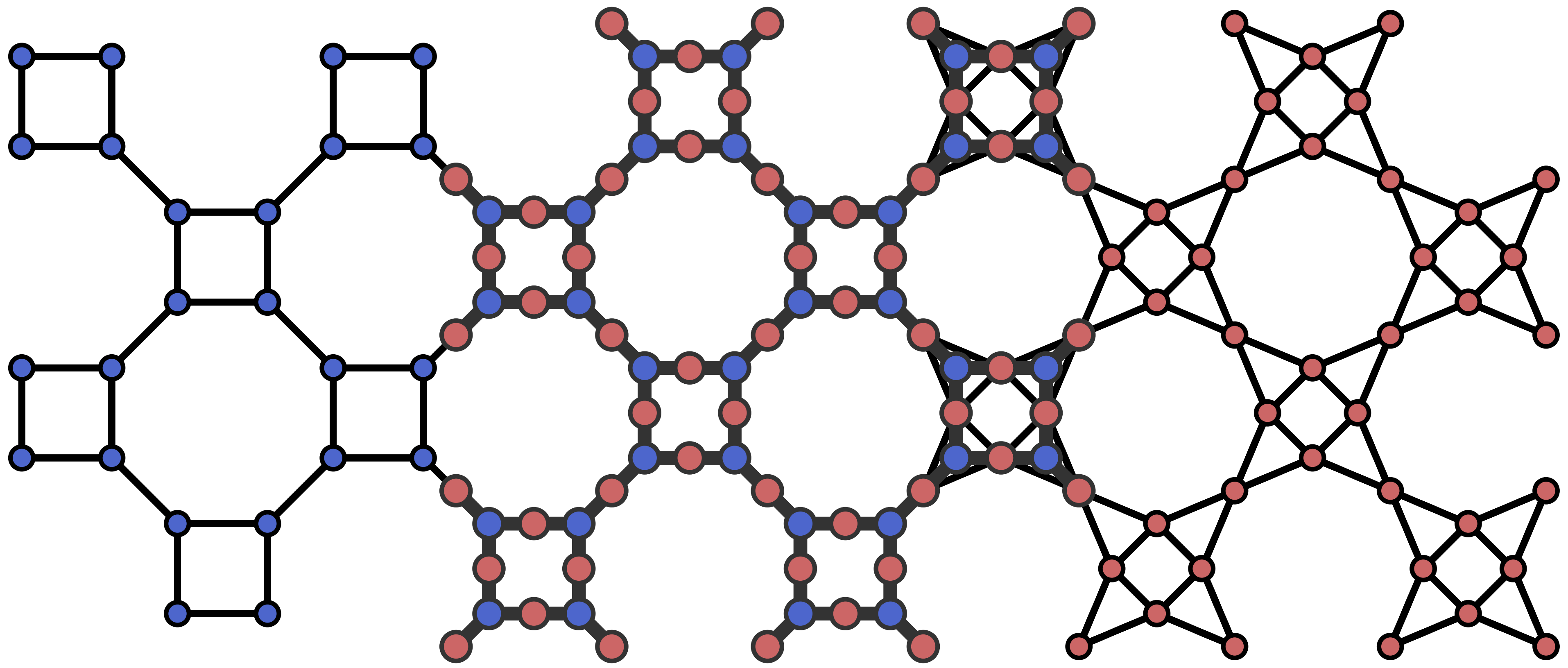
supersymmetry (SUSY)

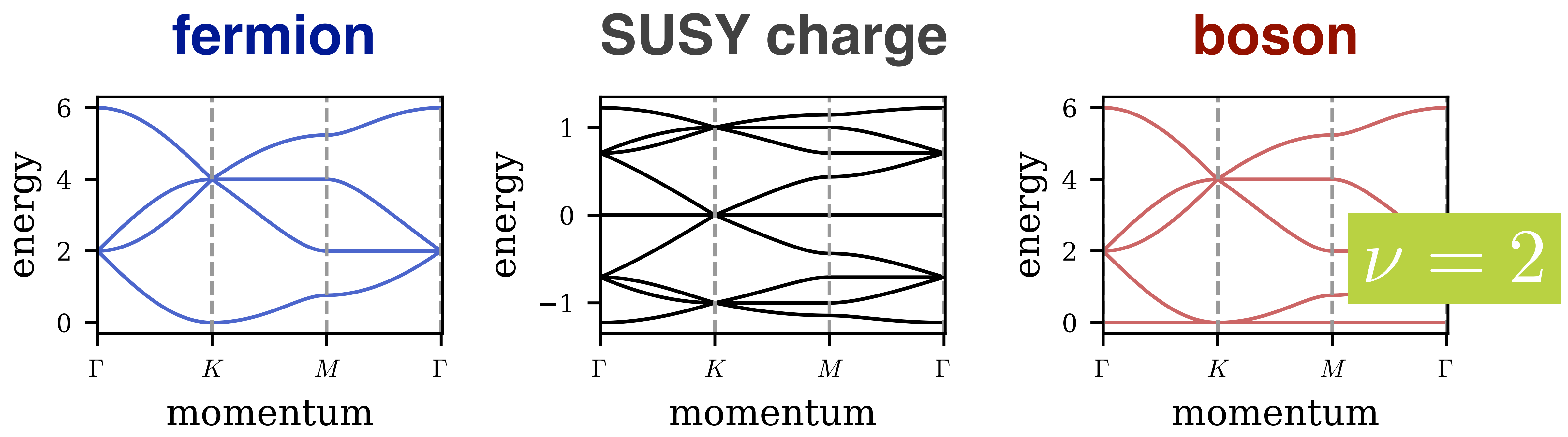
non-hermitian
SUSY charge operator

$$\mathcal{Q} = \underbrace{c_i^\dagger}_{\textbf{fermion}} \underbrace{\mathbf{R}_{ij}}_{\text{arbitrary matrix}} \overbrace{b_j}^{\textbf{boson}}$$

supersymmetric **Hamiltonian**

$$\mathcal{H}_{\text{SUSY}} = \{\mathcal{Q}, \mathcal{Q}^\dagger\} = \underbrace{c^\dagger \mathbf{R}\mathbf{R}^\dagger c}_{\text{fermion}} + \underbrace{b^\dagger \mathbf{R}^\dagger \mathbf{R} b}_{\text{boson}}$$





isospectral
quadratic
Hamiltonians

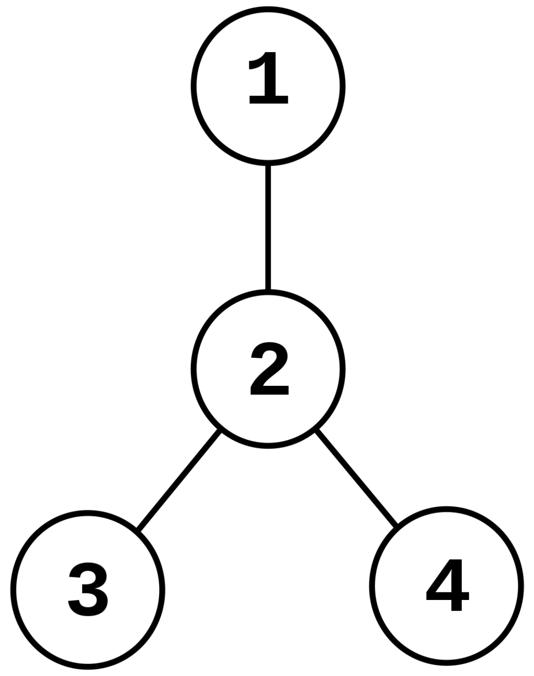
Witten index = # flat bands

$$\nu = \dim(\text{kernel}[\mathbf{R}]) - \dim(\text{kernel}[\mathbf{R}^\dagger])$$
$$= \text{col}[\mathbf{R}] - \text{row}[\mathbf{R}]$$

SUSY graph construction

adjacency matrix

$$\mathbf{A}_{ij} = \begin{cases} 1 & v_i \text{ connected to } v_j \\ 0 & \text{otherwise}\,. \end{cases}$$



$$\begin{array}{c|c} & \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \hline \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \end{array}$$

SUSY graph construction

adjacency matrix

$$\mathbf{A}_{ij} = \begin{cases} 1 & v_i \text{ connected to } v_j \\ 0 & \text{otherwise}\,. \end{cases}$$

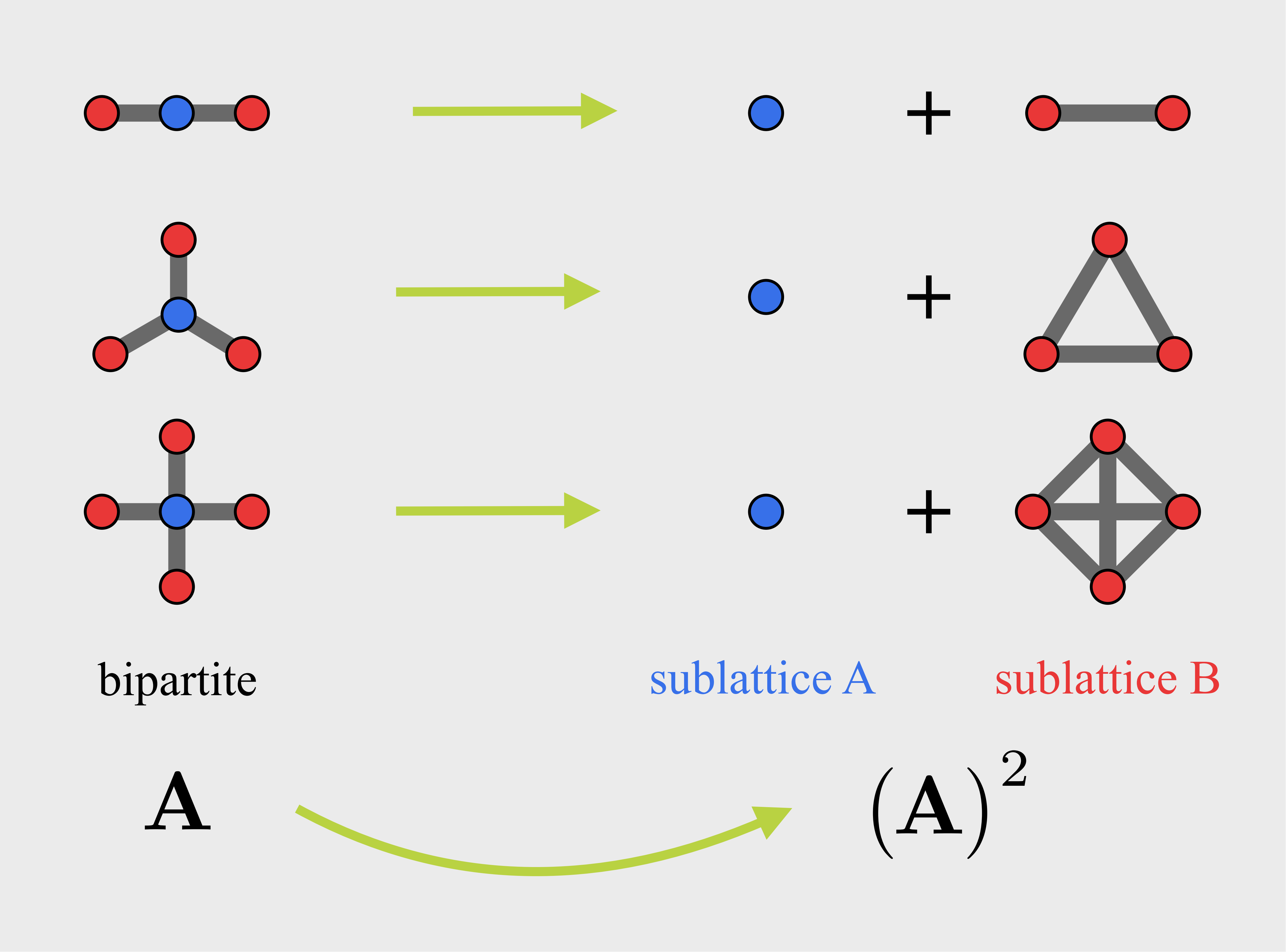
bipartite lattice

$$\mathbf{A} = \begin{pmatrix} & \mathbf{A}_{\mathrm{I-II}} \\ \mathbf{A}_{\mathrm{II-I}} & \end{pmatrix}$$

lattice "squaring"

$$\mathbf{A}^2 = \begin{pmatrix} \mathbf{A}_{\mathrm{I}} & \\ & \mathbf{A}_{\mathrm{II}} \end{pmatrix}$$

lattice "squaring"



SUSY graph construction

SUSY graph correspondence

boson

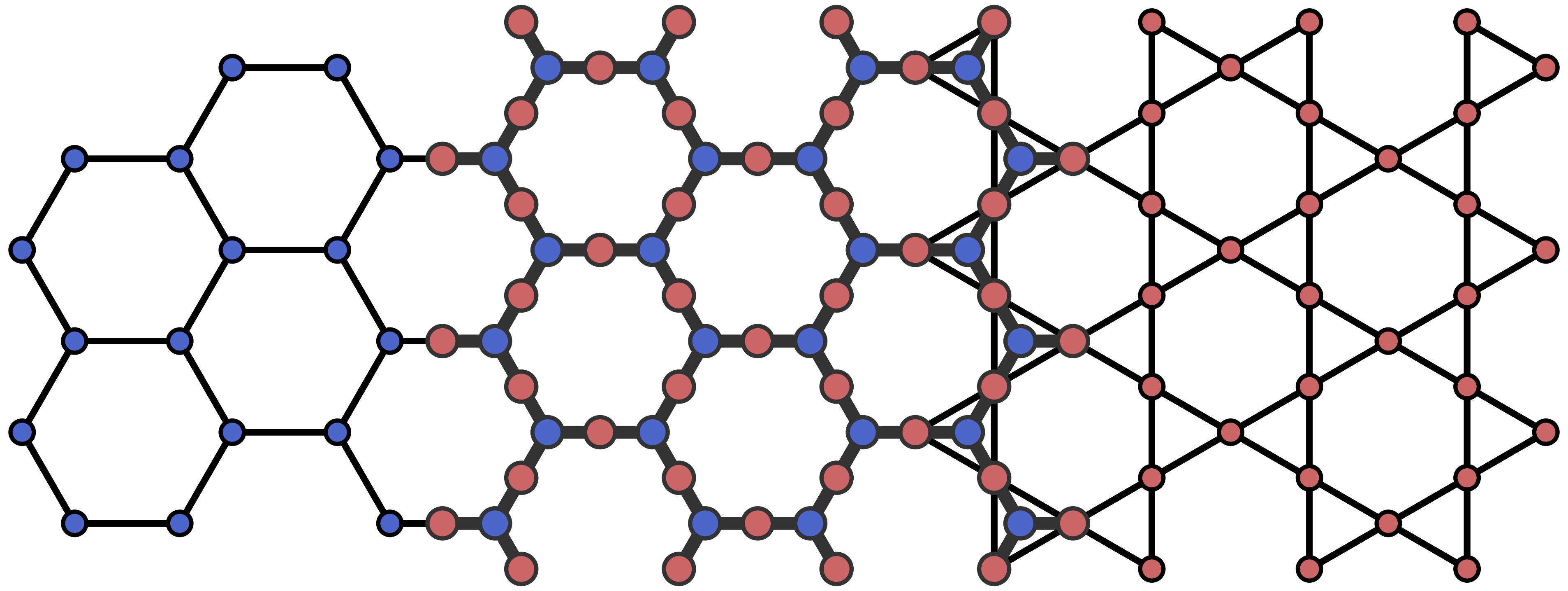
$$\begin{pmatrix} \boxed{\mathbf{R}\mathbf{R}^\dagger} & \\ & \boxed{\mathbf{R}^\dagger\mathbf{R}} \end{pmatrix} = \begin{pmatrix} \boxed{\mathcal{H}_F} & \\ & \boxed{\mathcal{H}_B} \end{pmatrix} = \begin{pmatrix} \boxed{\mathbf{A}_{\mathrm{I}}} & \\ & \boxed{\mathbf{A}_{\mathrm{II}}} \end{pmatrix}$$

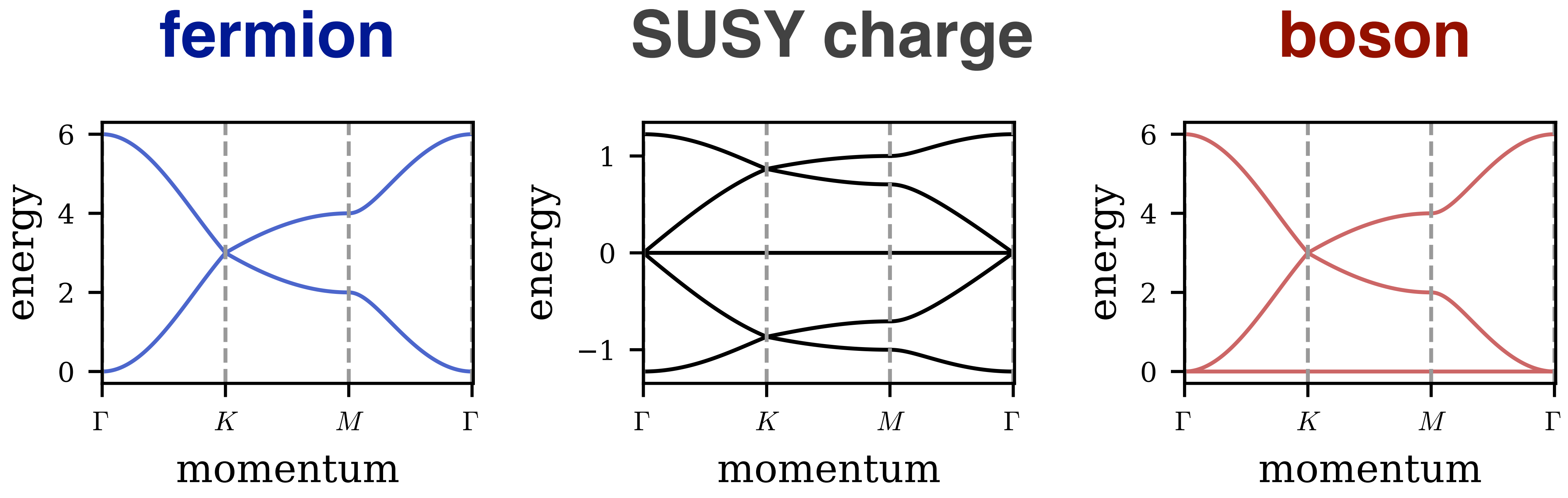
fermion

graph “square root”

$$\begin{pmatrix} & \mathbf{R} \\ \boxed{\mathbf{R}^\dagger} & \end{pmatrix} = \begin{pmatrix} & \mathcal{Q} \\ \boxed{\mathcal{Q}^\dagger} & \end{pmatrix} = \begin{pmatrix} & \mathbf{A}_{\mathrm{I-II}} \\ \boxed{\mathbf{A}_{\mathrm{II-I}}} & \end{pmatrix}$$

SUSY charge





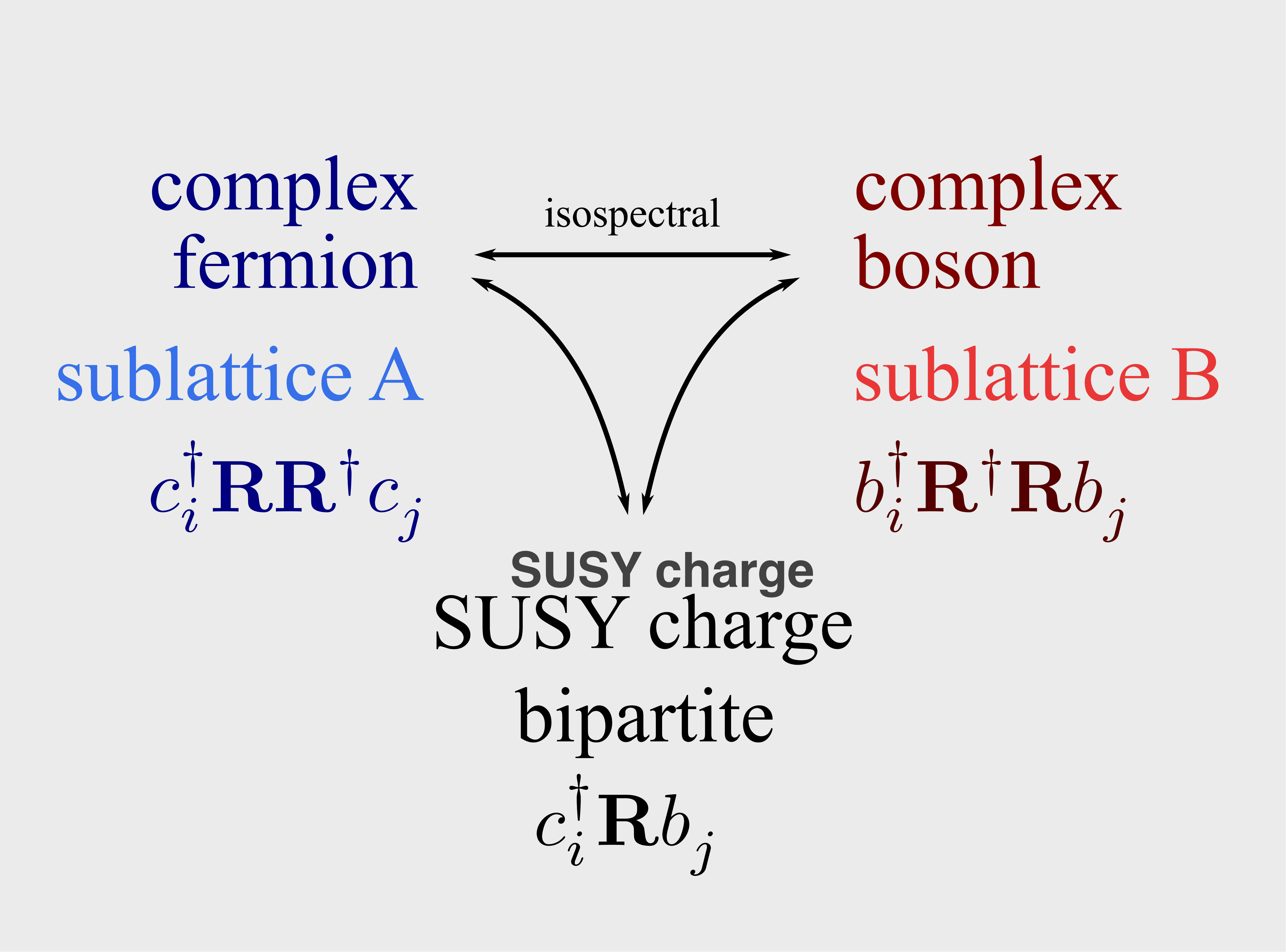
SUSY graph construction

SUSY graph correspondence

$$\begin{pmatrix} \boxed{\mathbf{R}\mathbf{R}^\dagger} & \\ & \boxed{\mathbf{R}^\dagger\mathbf{R}} \end{pmatrix} = \begin{pmatrix} \boxed{\mathcal{H}_F} & \\ & \boxed{\mathcal{H}_B} \end{pmatrix} = \begin{pmatrix} \underset{\text{fermion}}{\boxed{\mathbf{A}_{\mathrm{I}}}} & \\ & \overset{\text{boson}}{\boxed{\mathbf{A}_{\mathrm{II}}}} \end{pmatrix}$$

graph "square root"

$$\begin{pmatrix} & \mathbf{R} \\ \boxed{\mathbf{R}^\dagger} & \end{pmatrix} = \begin{pmatrix} & \mathcal{Q} \\ \boxed{\mathcal{Q}^\dagger} & \end{pmatrix} = \begin{pmatrix} & \mathbf{A}_{\mathrm{I-II}} \\ \underset{\text{SUSY charge}}{\boxed{\mathbf{A}_{\mathrm{II-I}}}} & \end{pmatrix}$$



complex fermion

sublattice A

$c_i^\dagger \mathbf{R}\mathbf{R}^\dagger c_j$

isospectral

complex boson

sublattice B

$b_i^\dagger \mathbf{R}^\dagger \mathbf{R} b_j$

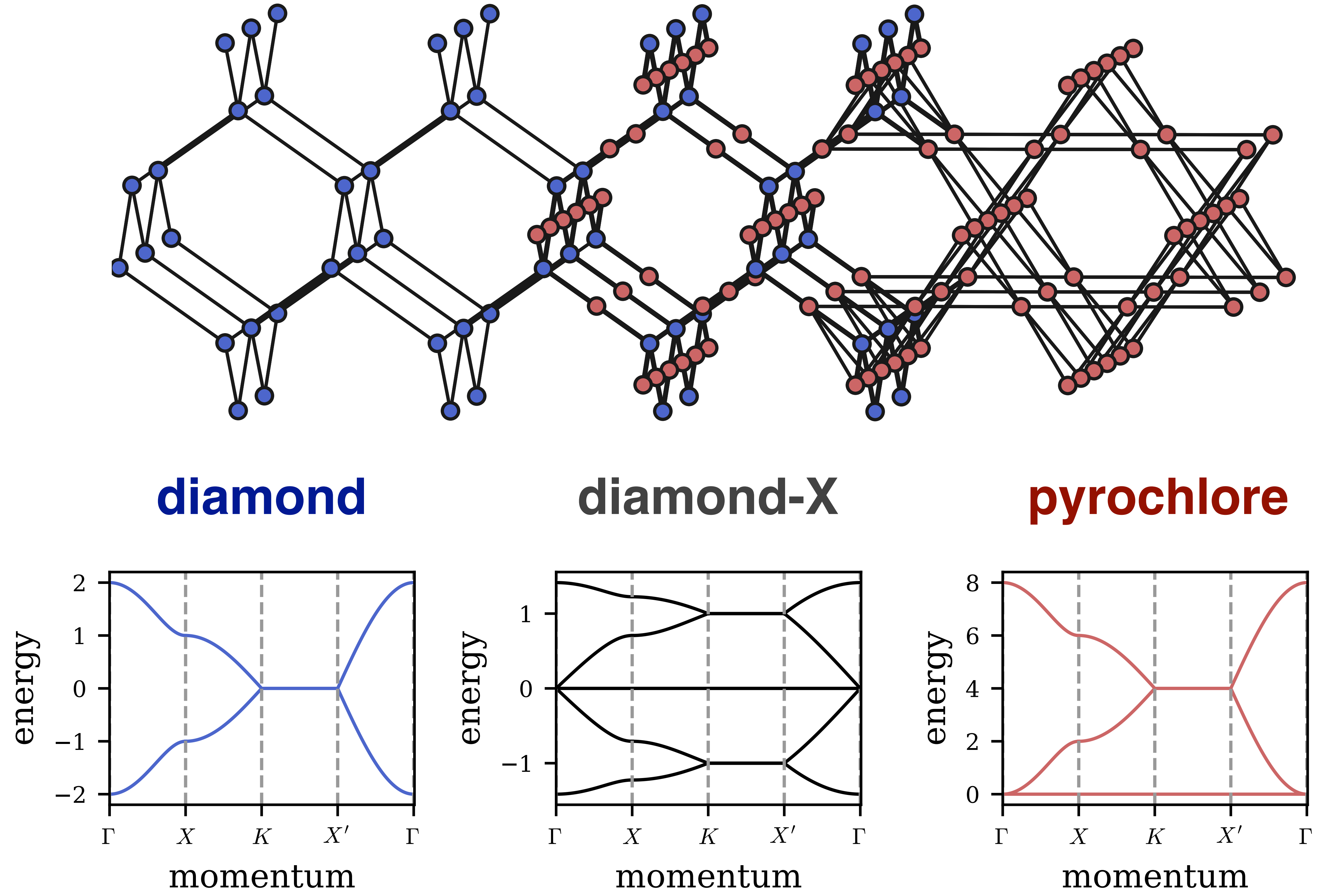
SUSY charge

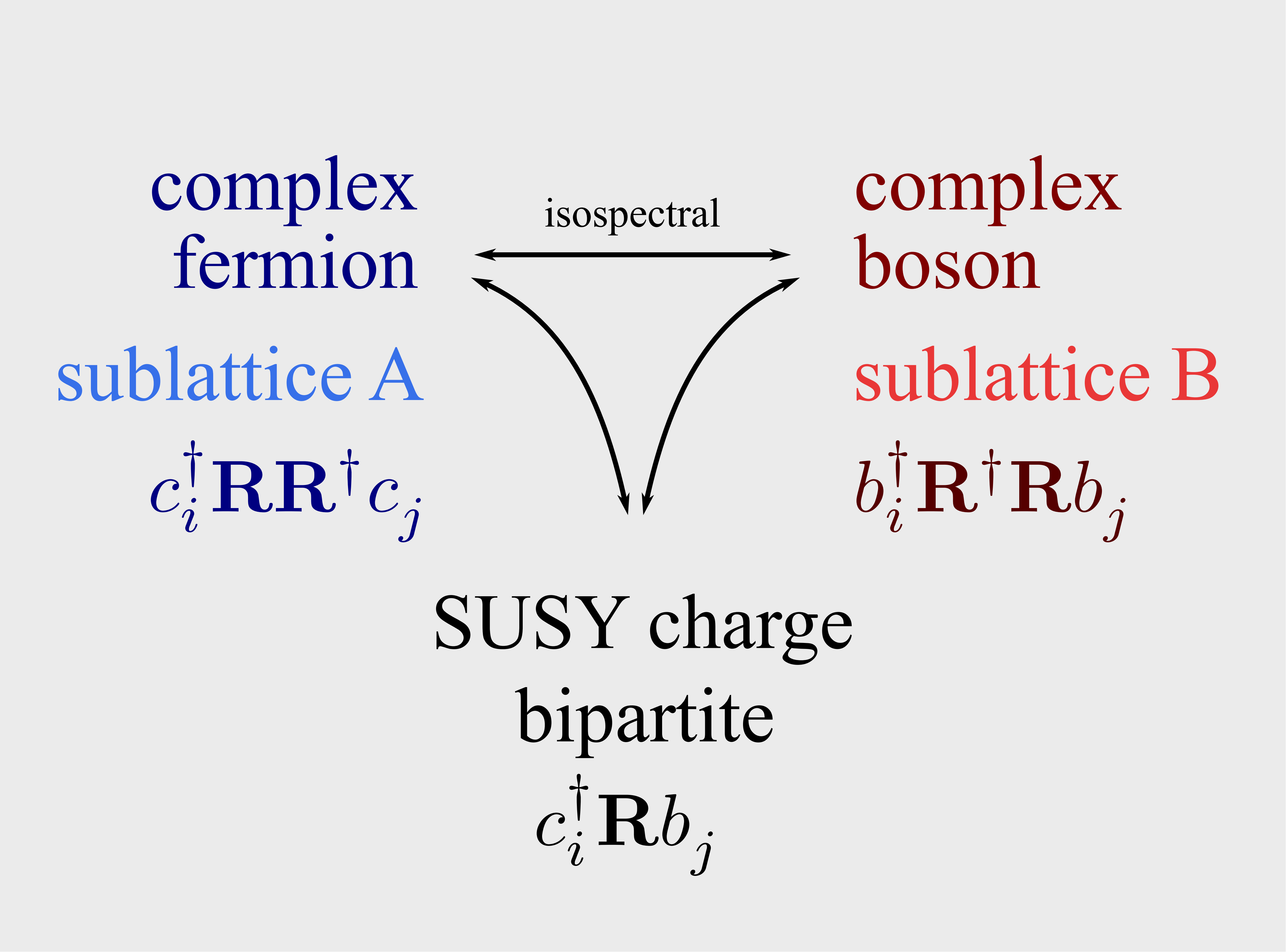
SUSY charge

bipartite

$c_i^\dagger \mathbf{R} b_j$

SUSY graph construction





complex fermion

sublattice A

$c_i^\dagger \mathbf{R}\mathbf{R}^\dagger c_j$

isospectral

complex boson

sublattice B

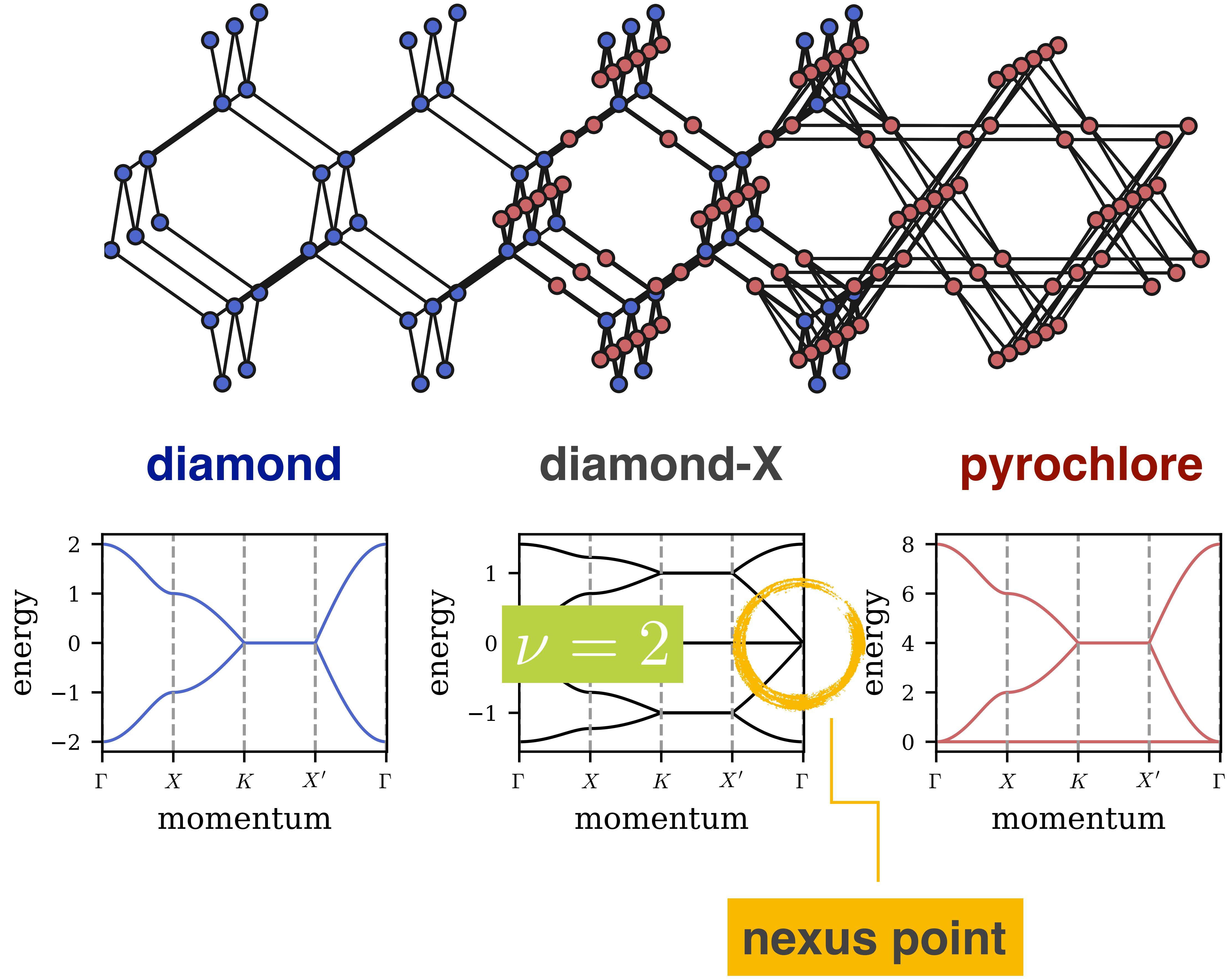
$b_i^\dagger \mathbf{R}^\dagger \mathbf{R} b_j$

SUSY charge

bipartite

$c_i^\dagger \mathbf{R} b_j$

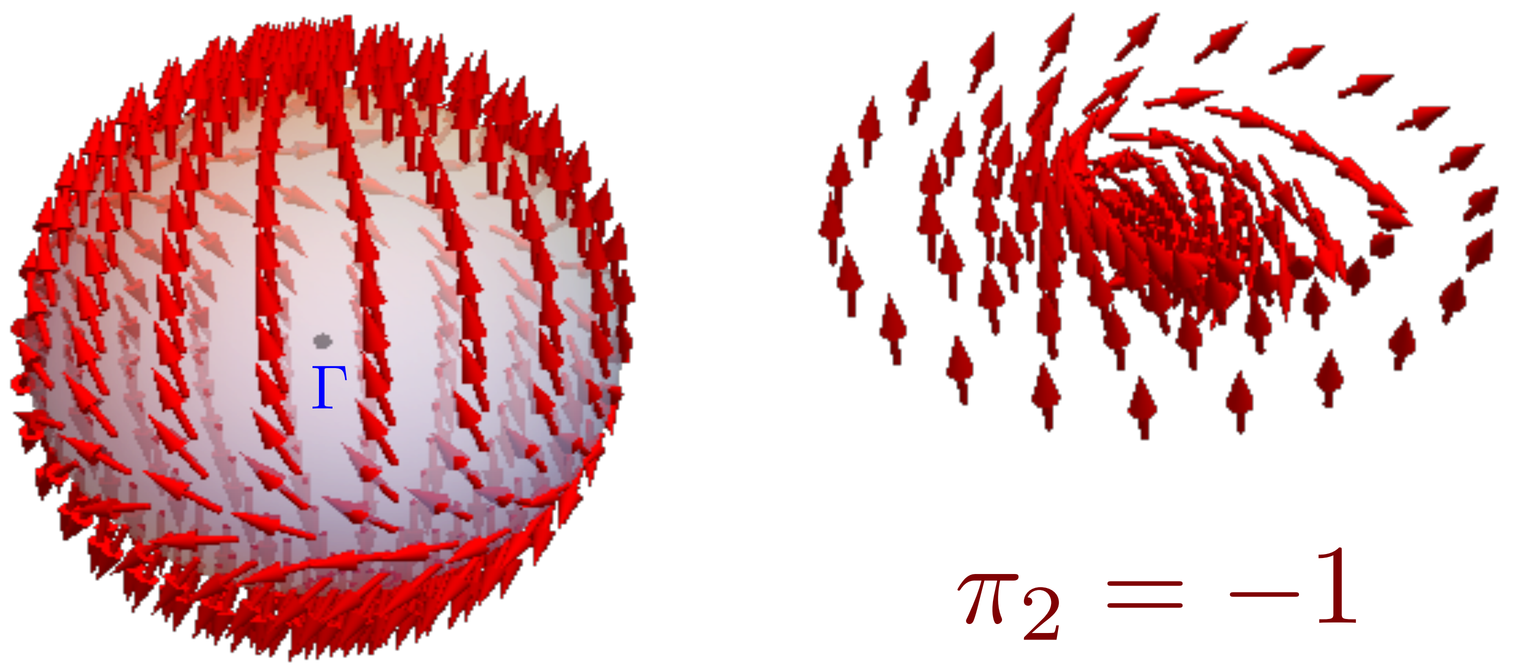
SUSY & topology

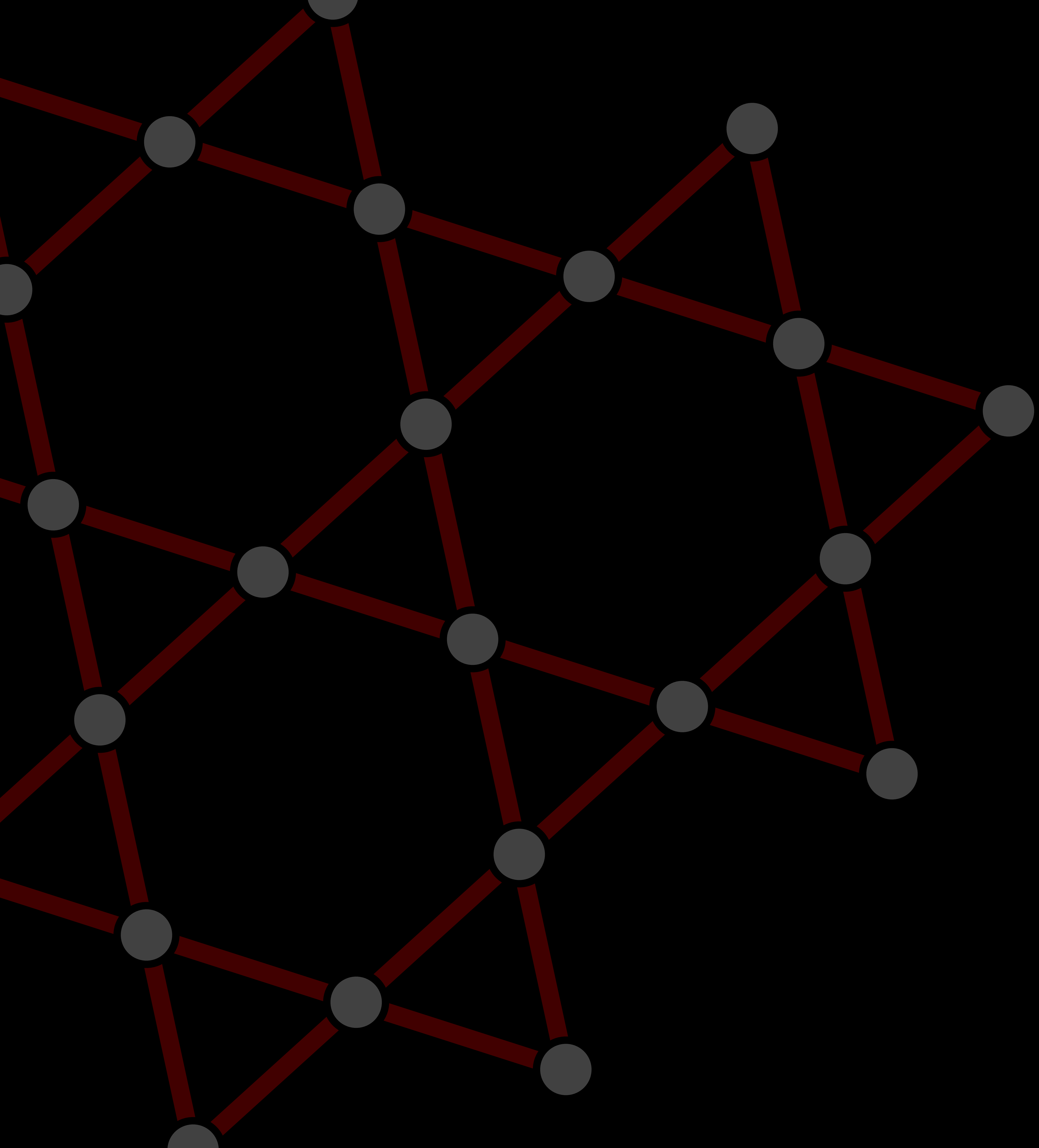


topological classification of nexus points
for varying Witten index

BDI

ν	π_1	π_2	π_3
1	$\mathbb{Z}_2$	0	$\mathbb{Z}$
2	0	$\mathbb{Z}$	$\mathbb{Z}$
3	0	0	$\mathbb{Z}$
4	0	0	0





frustrated magnets

ground-state manifolds

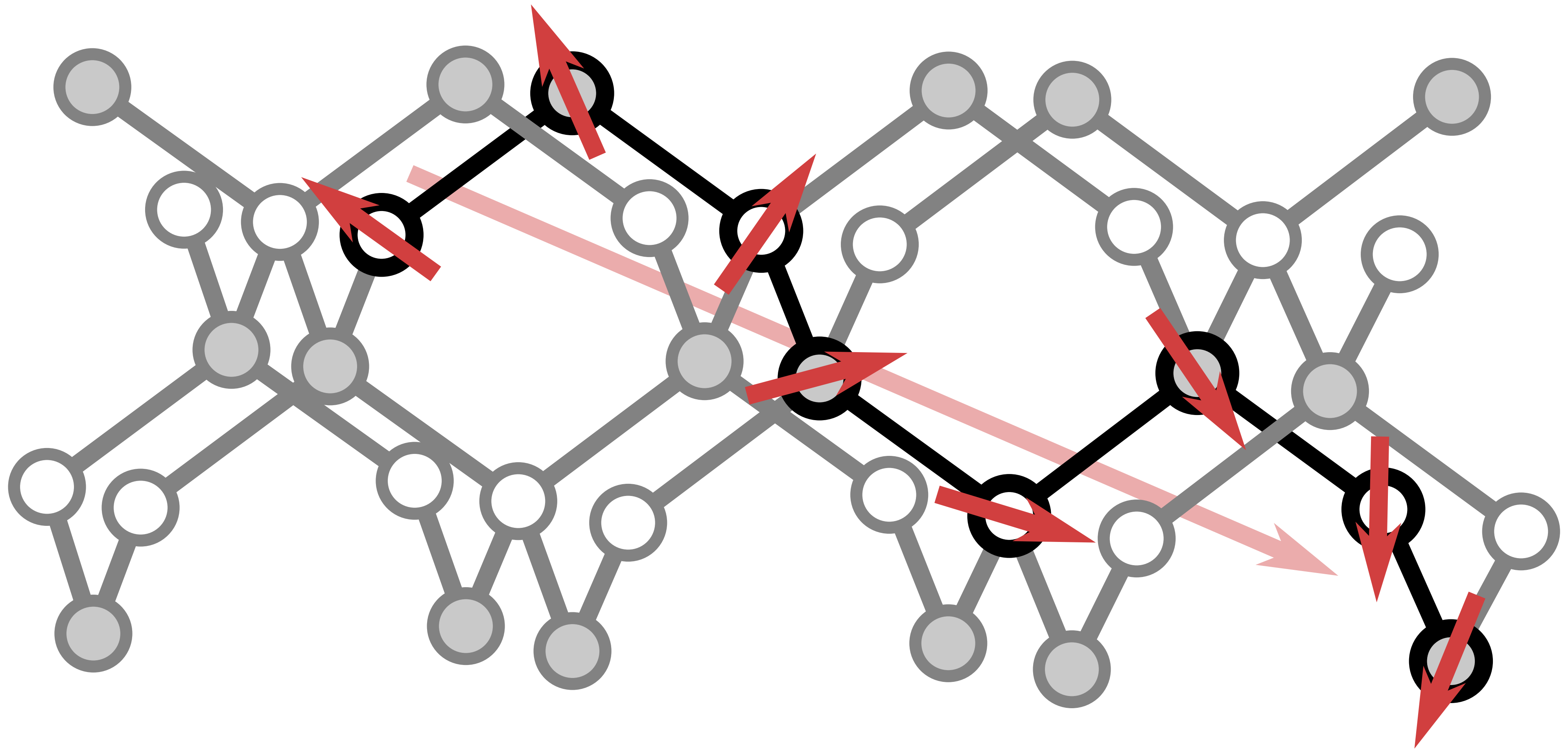
Phys. Rev. B 96, 085145 (2017)

spin spirals

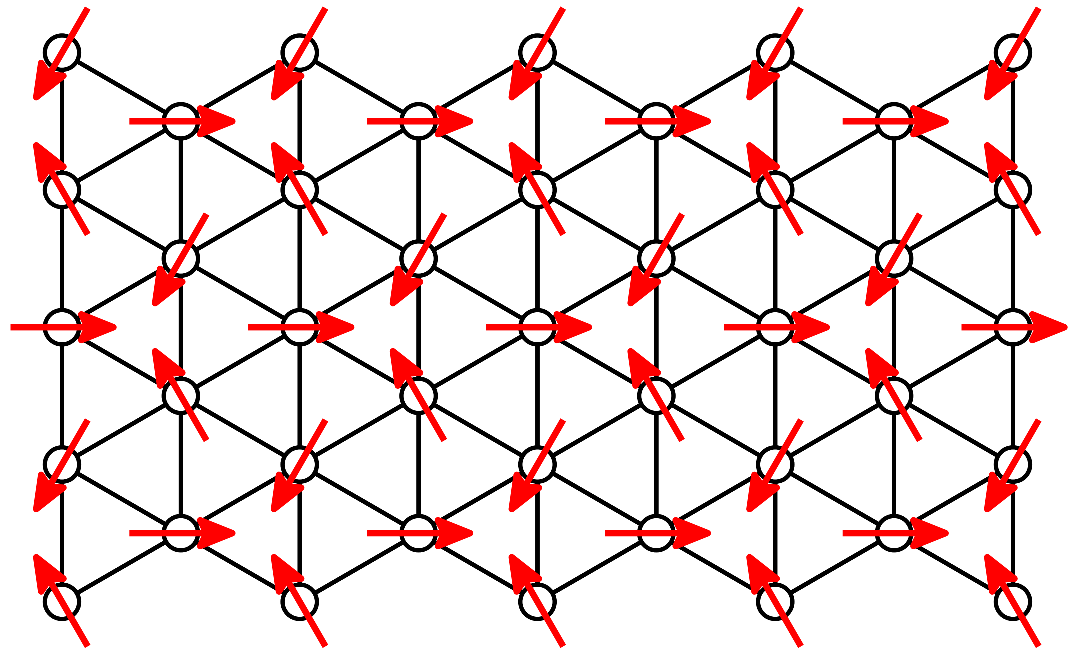
Coplanar spirals typically arise as ground state(s) of Heisenberg antiferromagnets.

elementary ingredient for

- multiferroics
- spin textures & multi-q states
- spiral spin liquids

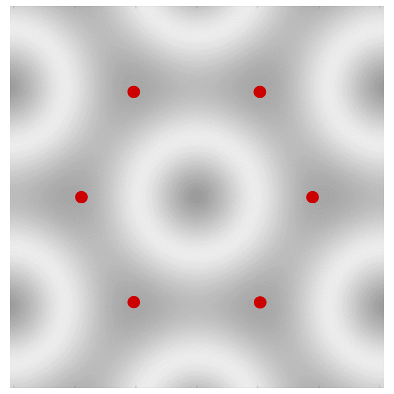


120° order of Heisenberg AFM on triangular lattice



$$\vec{S}(\vec{r}) = \mathrm{Re}\left(\left(\vec{S}_1 + i\vec{S}_2\right) e^{i\vec{q}\vec{r}}\right)$$

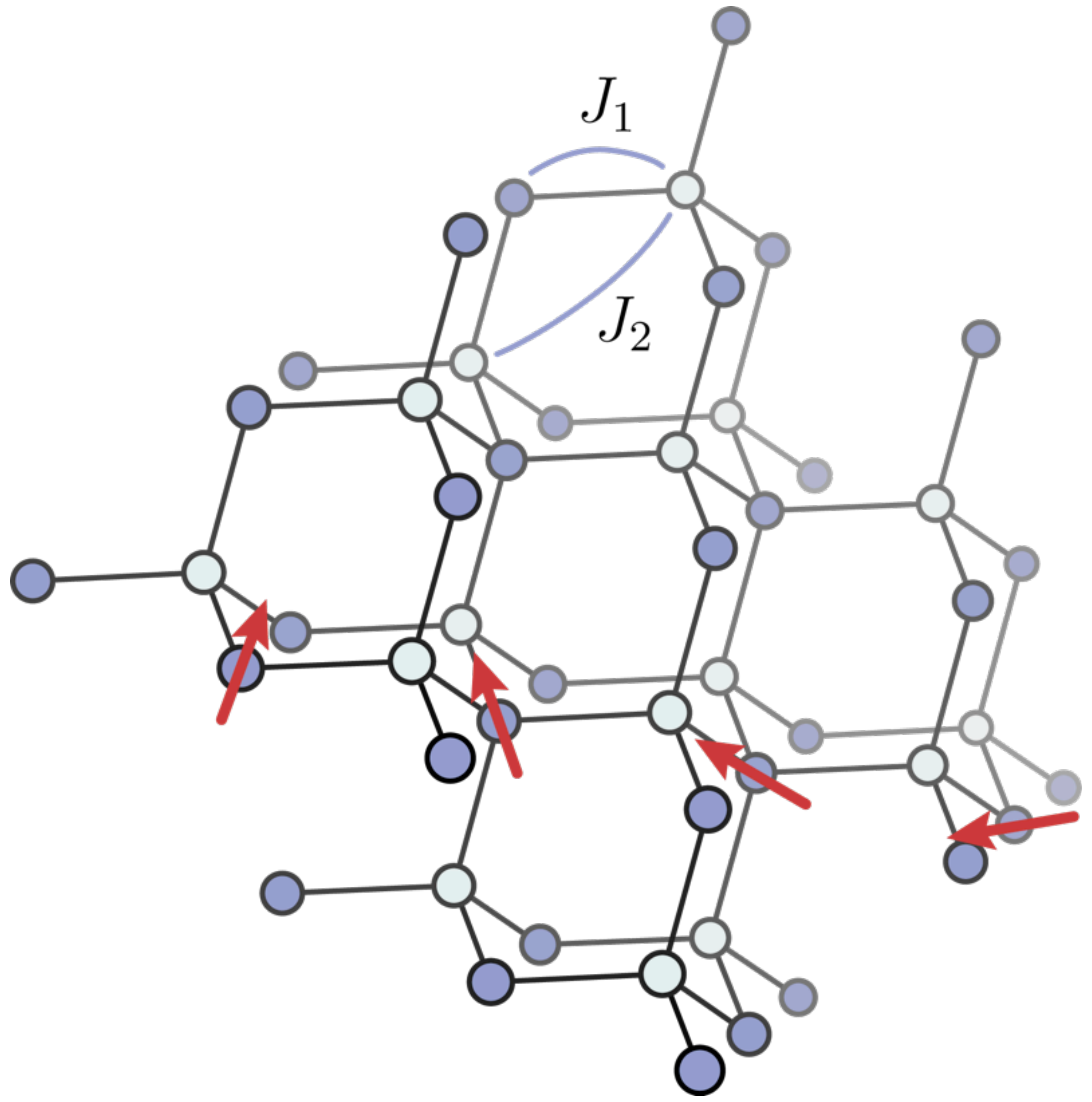
$$\vec{q} = \left(\pm\frac{2\pi}{3}, \frac{2\pi}{\sqrt{3}}\right)$$



description in terms of a single **wavevector**

spin spiral liquids / materials

Frustrated diamond lattice antiferromagnets

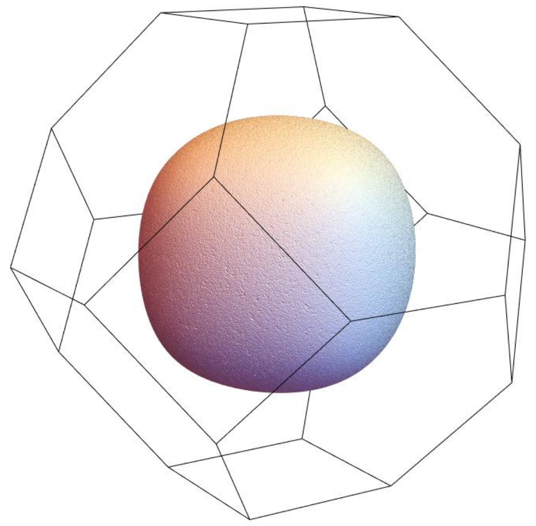


A-site spinels

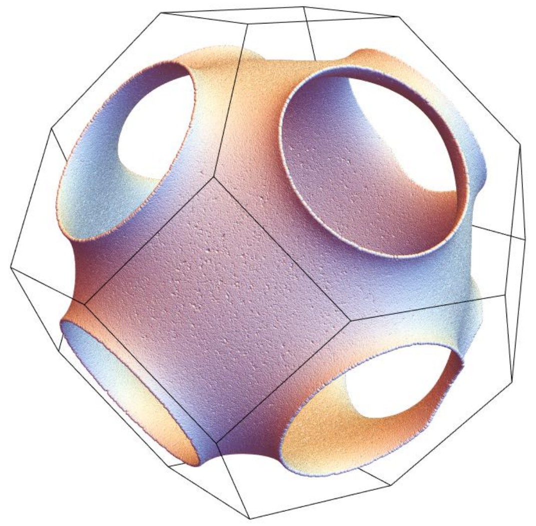
$MnSc_2S_4$	S=5/2
$FeSc_2S_4$	S=2
$CoAl_2O_4$	S=3/2
$NiRh_2O_4$	S=1

$$\mathcal{H} = J_1 \sum_{\langle i,j \rangle} \vec{S}_i \vec{S}_j + J_2 \sum_{\langle\langle i,j \rangle\rangle} \vec{S}_i \vec{S}_j$$

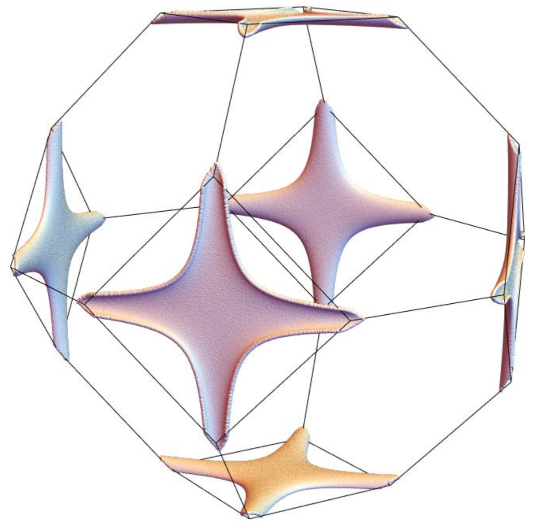
degenerate coplanar spirals form **spin spiral surfaces** in *k*-space



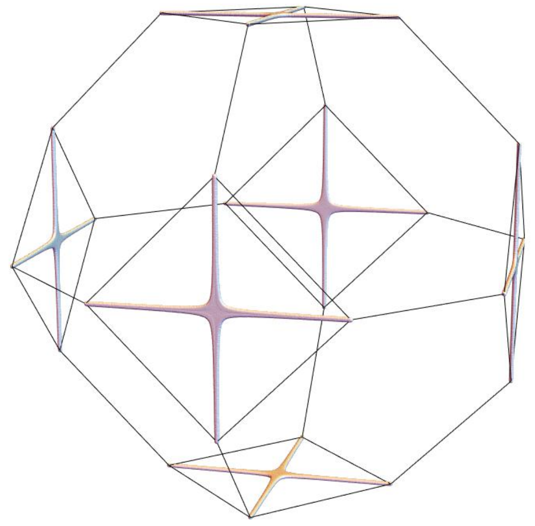
$J_2/J_1 = 0.2$



$J_2/J_1 = 0.4$

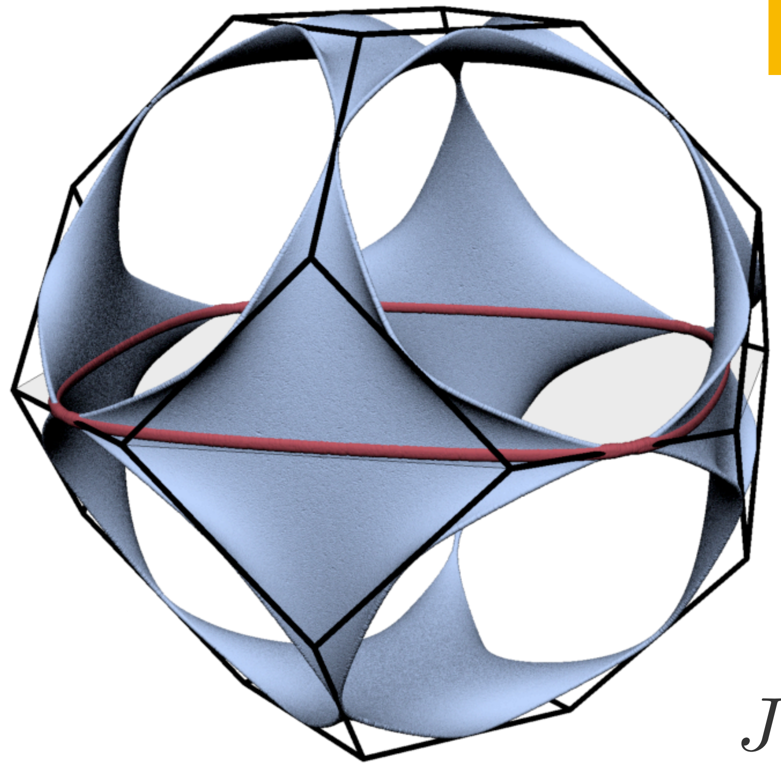


$J_2/J_1 = 3$



$J_2/J_1 = 100$

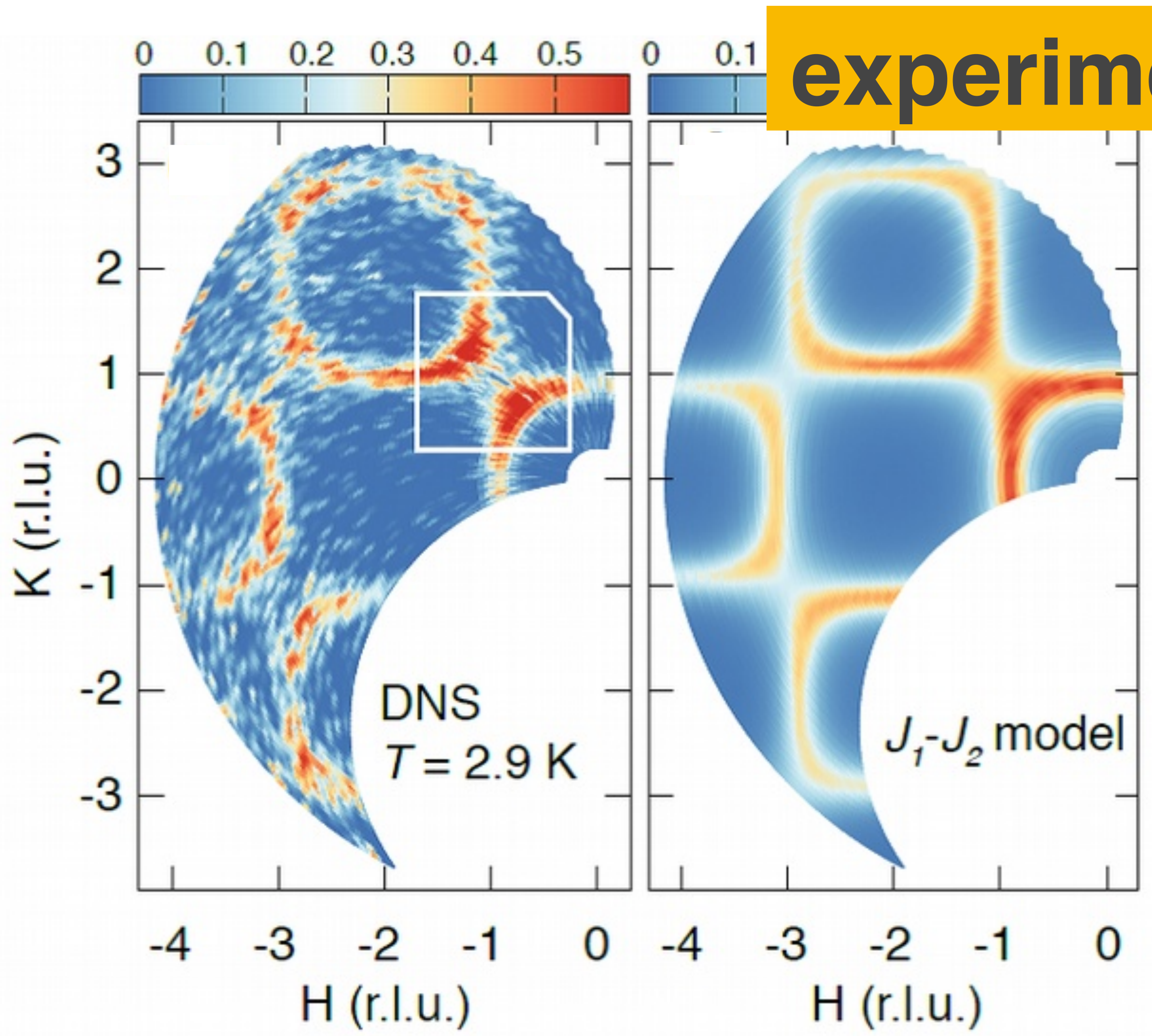
theory



$J_2/J_1 = 0.85$

Nature Phys. **3**, 487 (2007)

experiment

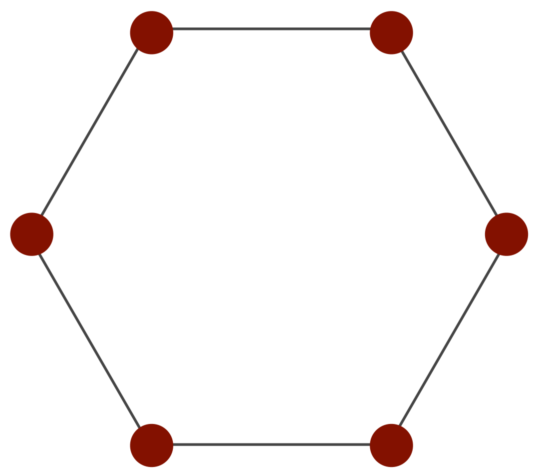


Nature Phys. **13**, 157 (2017)

spin spiral manifolds

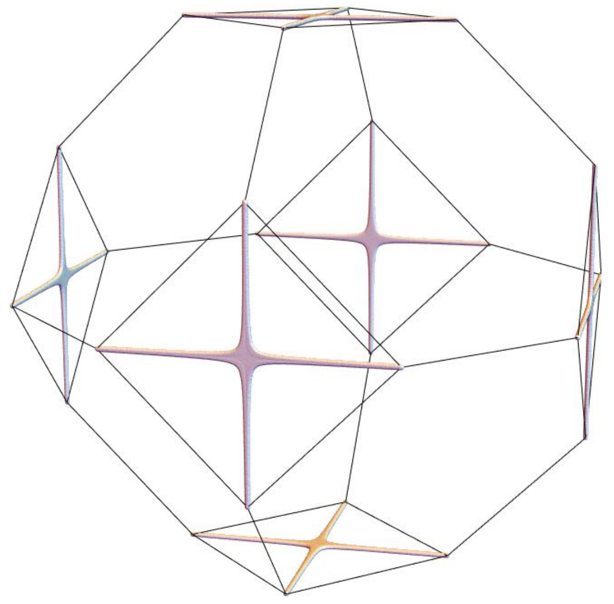
Spiral manifolds are extremely reminiscent of **Fermi surfaces**

triangular lattice



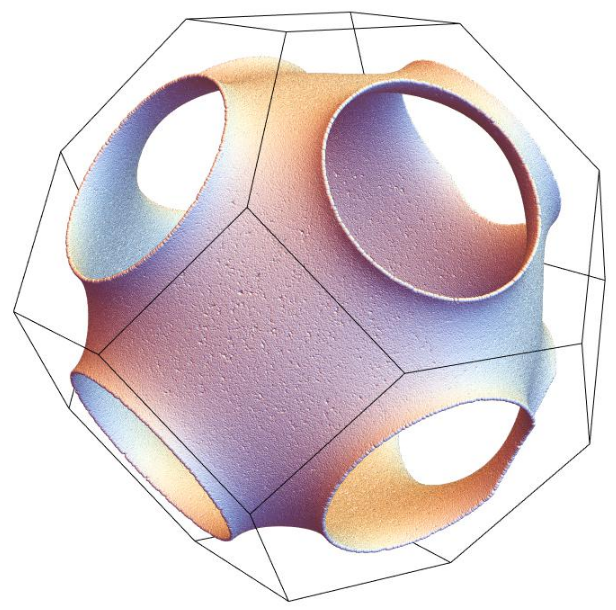
Dirac points

FCC lattice



nodal lines

diamond lattice



Fermi surface

Note, however:

Spiral manifolds – **ground-state** property of classical spin system
Fermi surfaces – **mid-spectrum** feature of an electronic system.

spin spiral manifolds

Spiral manifolds – **ground-state** property of classical spin system

Fermi surfaces – **mid-spectrum** feature of an electronic system.

Luttinger-Tisza approach

$$\mathcal{H}_{\text{Heisenberg}} = \sum_{ij} \boxed{M_{ij}}\, \mathbf{S}_i \cdot \mathbf{S}_j$$

interaction matrix

$$\boxed{\mathbf{M} = \mathbf{R}^\dagger \mathbf{R}}$$

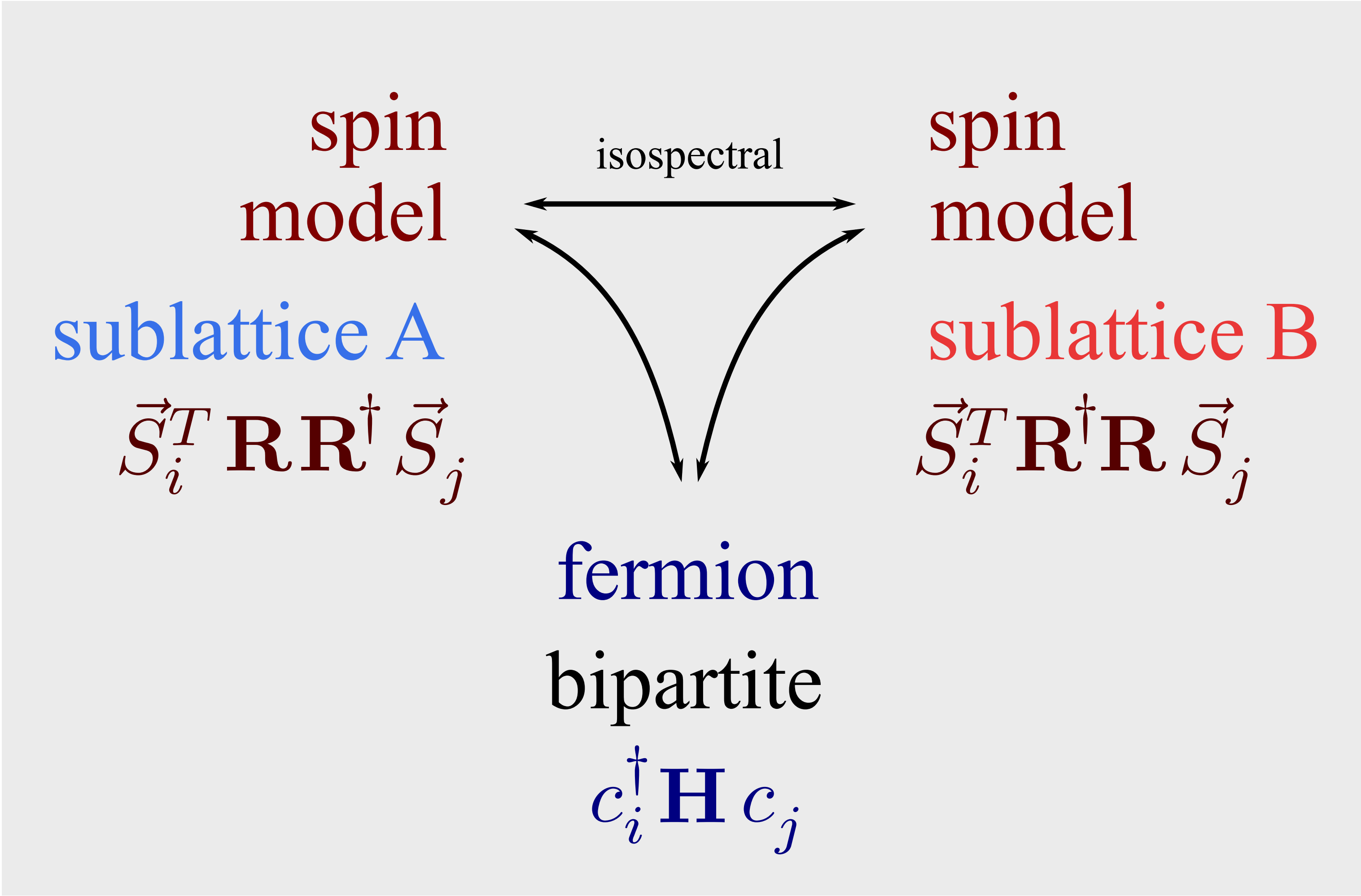
bosonic side
of SUSY construction

chiral fermion lattice model

$$\mathcal{H} = \begin{pmatrix} c_A^\dagger & c_B^\dagger \end{pmatrix} \boxed{\begin{pmatrix} & \mathbf{R} \\ \mathbf{R}^\dagger & \end{pmatrix}} \begin{pmatrix} c_A \\ c_B \end{pmatrix}$$

$$\boxed{\mathcal{Q} = c_A^\dagger\, \mathbf{R}\, c_B}$$

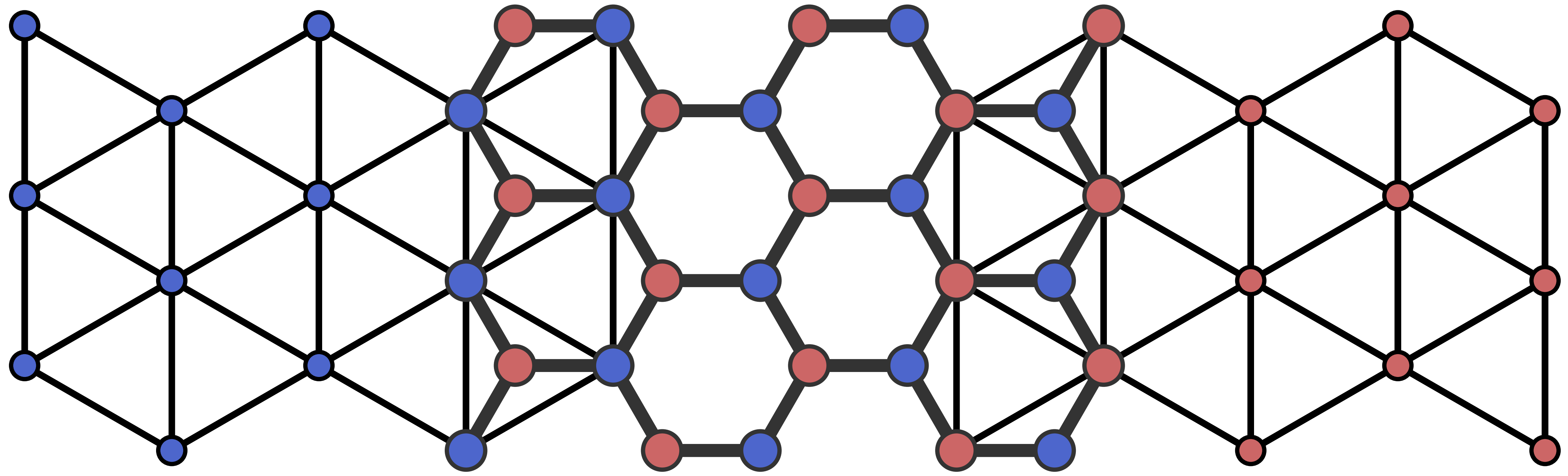
SUSY **charge**
in our construction

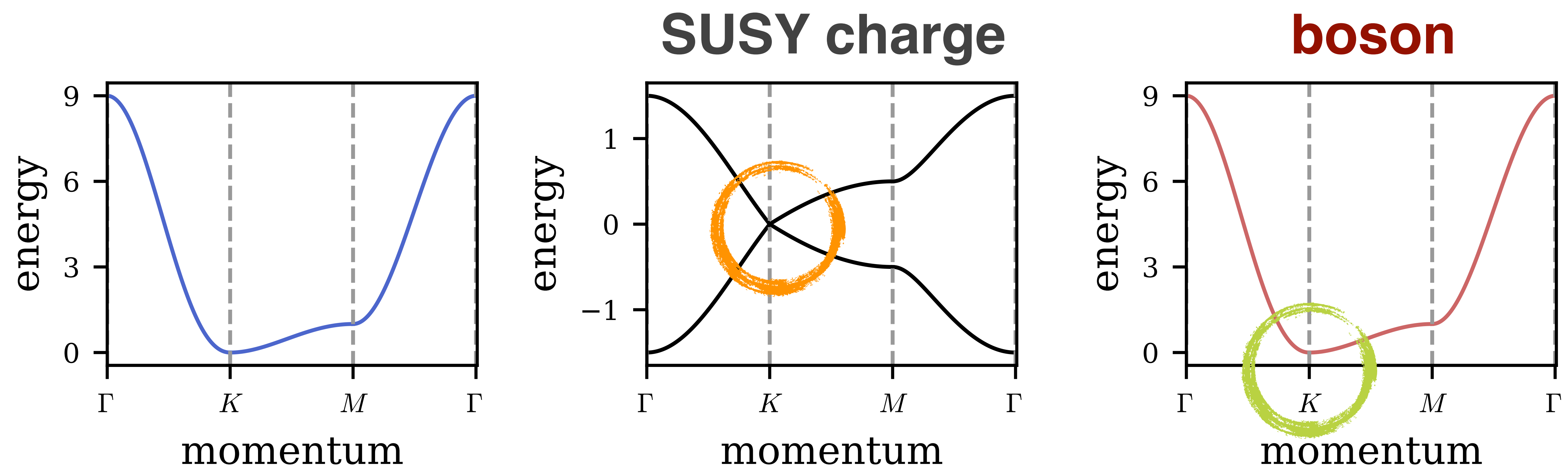


spin spiral manifolds

Spiral manifolds – **ground-state** property of classical spin system

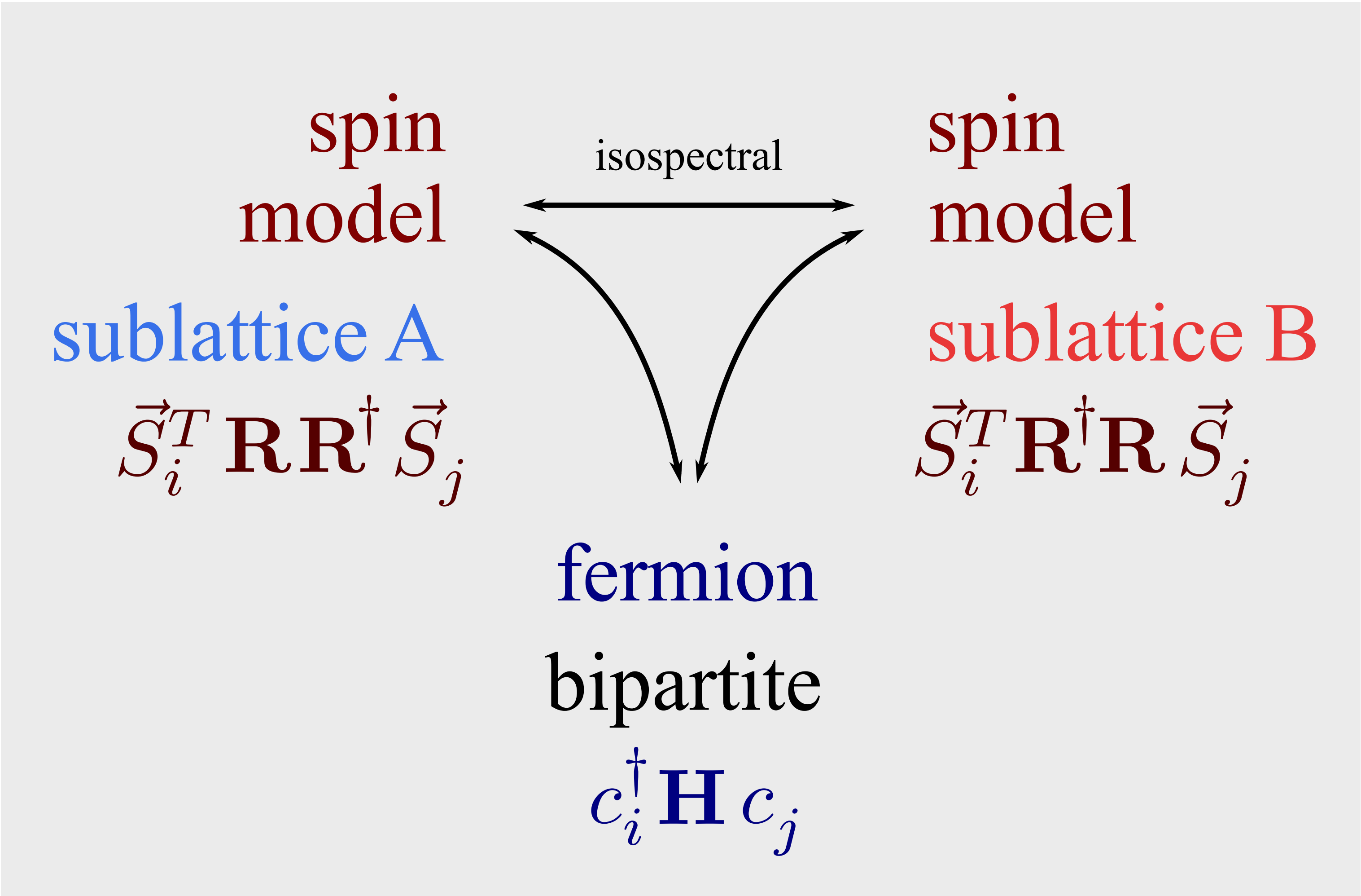
Fermi surfaces – **mid-spectrum** feature of an electronic system.





SUSY charge

boson



spin model

sublattice A

$\vec{S}_i^T \mathbf{R}\mathbf{R}^\dagger \vec{S}_j$

spin model

sublattice B

$\vec{S}_i^T \mathbf{R}^\dagger\mathbf{R} \vec{S}_j$

fermion

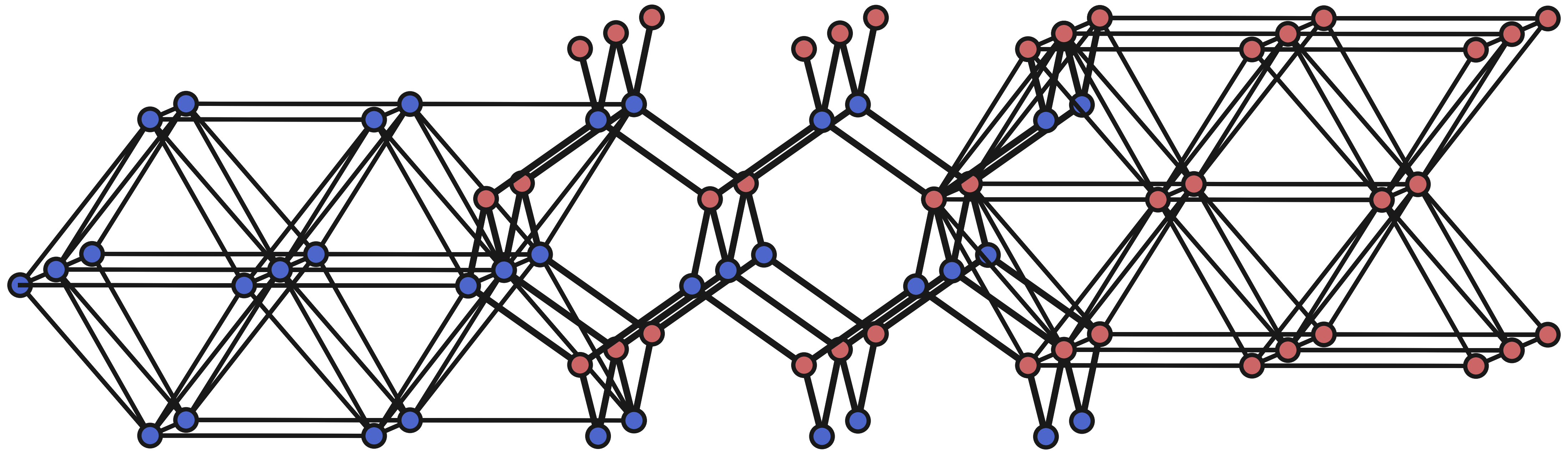
bipartite

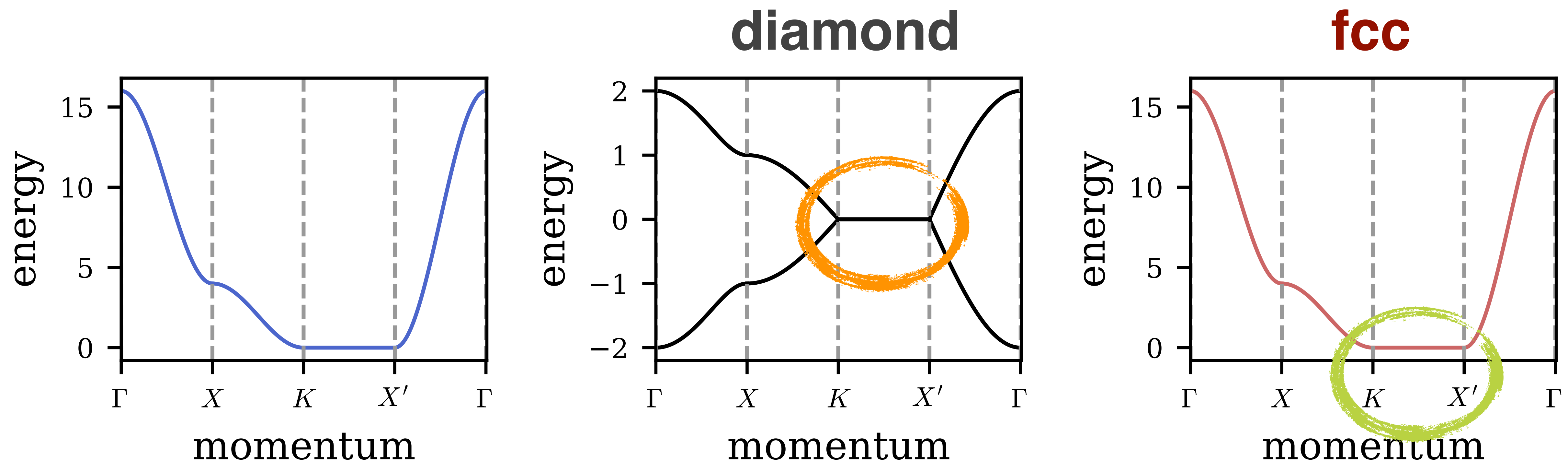
$c_i^\dagger \mathbf{H} c_j$

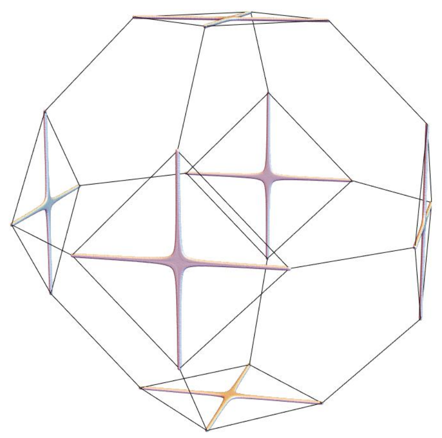
spin spiral manifolds

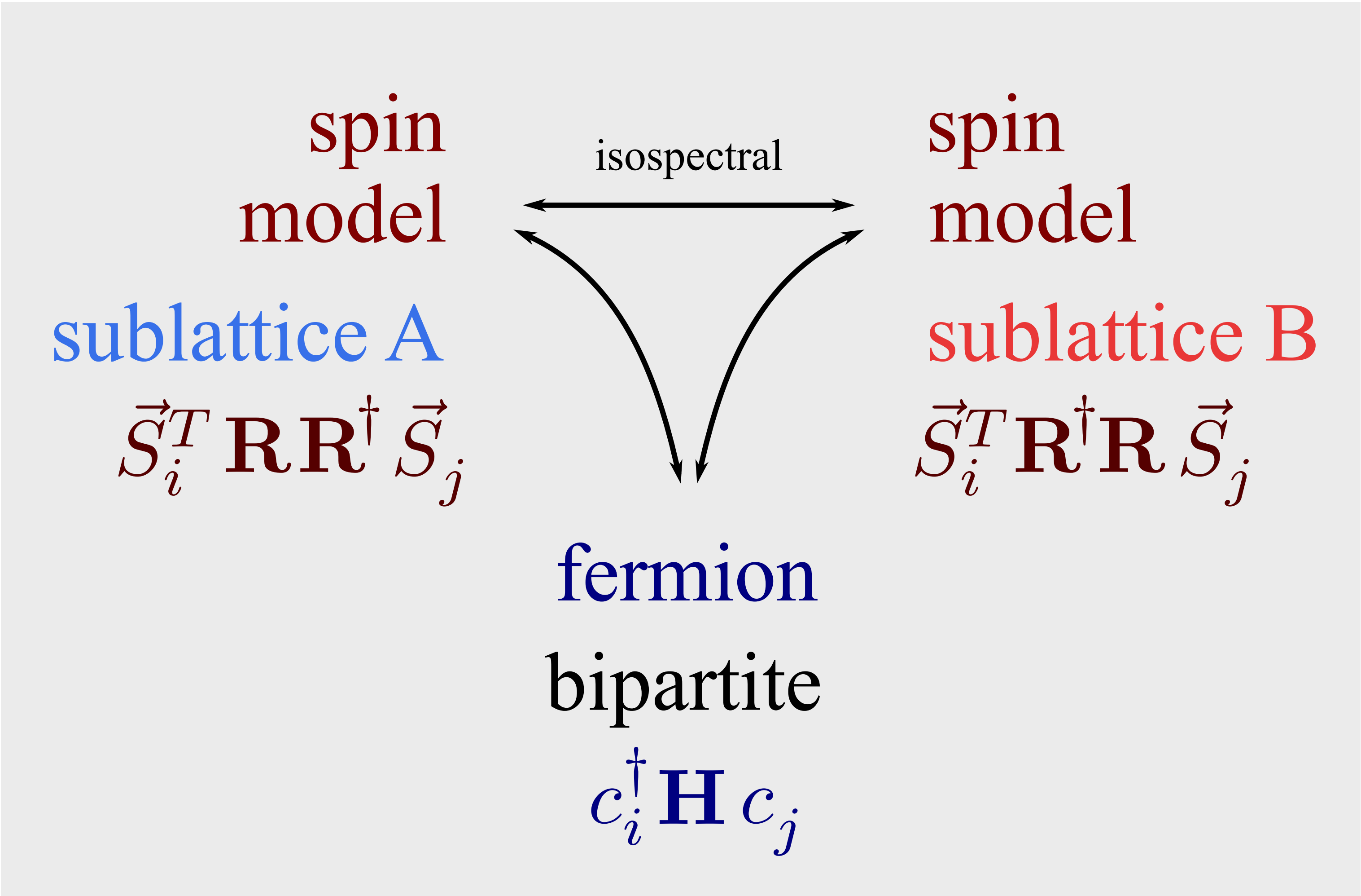
Spiral manifolds – **ground-state** property of classical spin system

Fermi surfaces – **mid-spectrum** feature of an electronic system.









spin model

sublattice A

$\vec{S}_i^T \, \mathbf{R}\mathbf{R}^\dagger \, \vec{S}_j$

isospectral

spin model

sublattice B

$\vec{S}_i^T \, \mathbf{R}^\dagger\mathbf{R} \, \vec{S}_j$

fermion

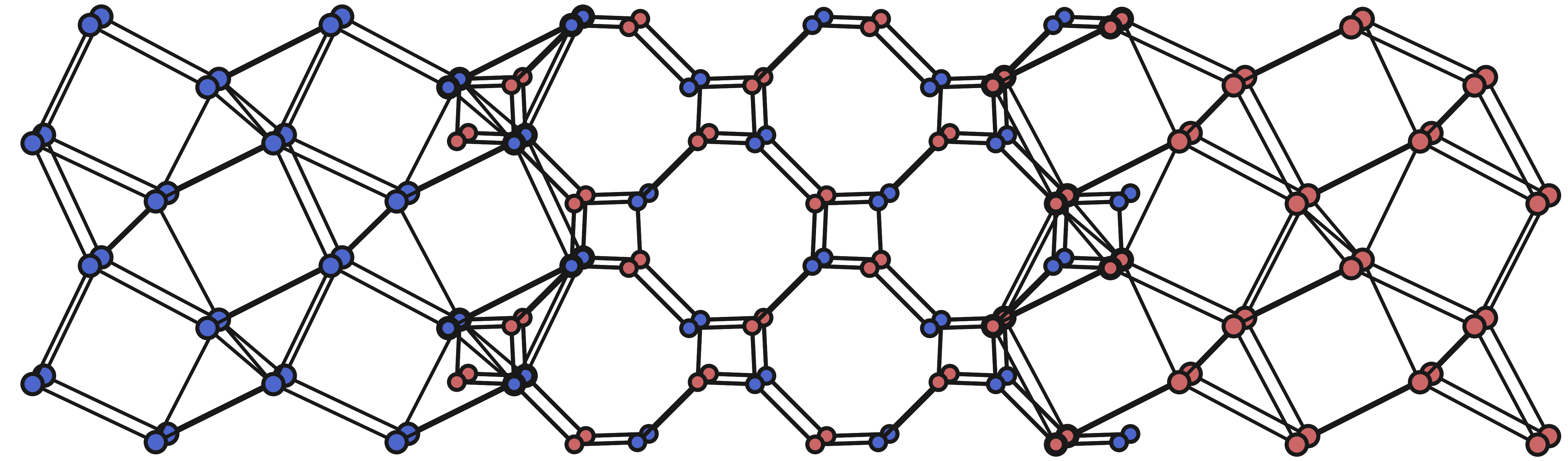
bipartite

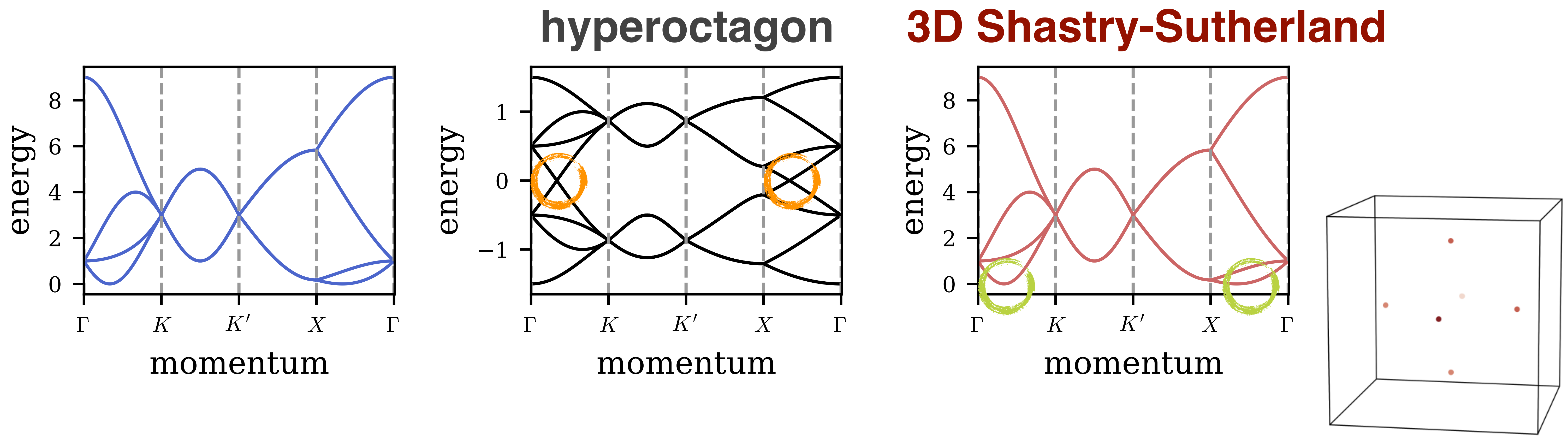
$c_i^\dagger \, \mathbf{H} \, c_j$

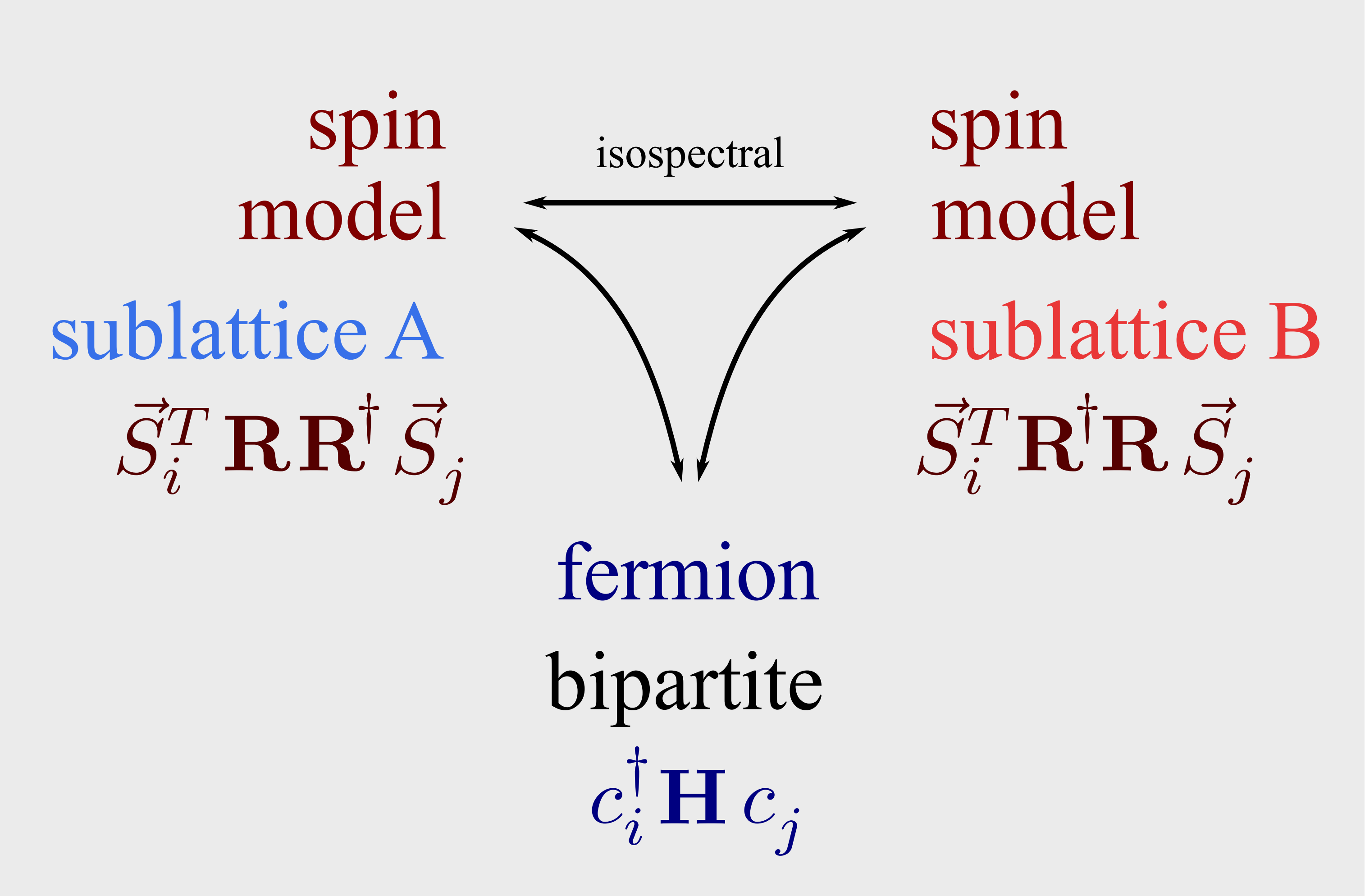
spin spiral manifolds

Spiral manifolds – **ground-state** property of classical spin system

Fermi surfaces – **mid-spectrum** feature of an electronic system.







spin model

sublattice A

$\vec{S}_i^T \, \mathbf{R}\mathbf{R}^\dagger \, \vec{S}_j$

spin model

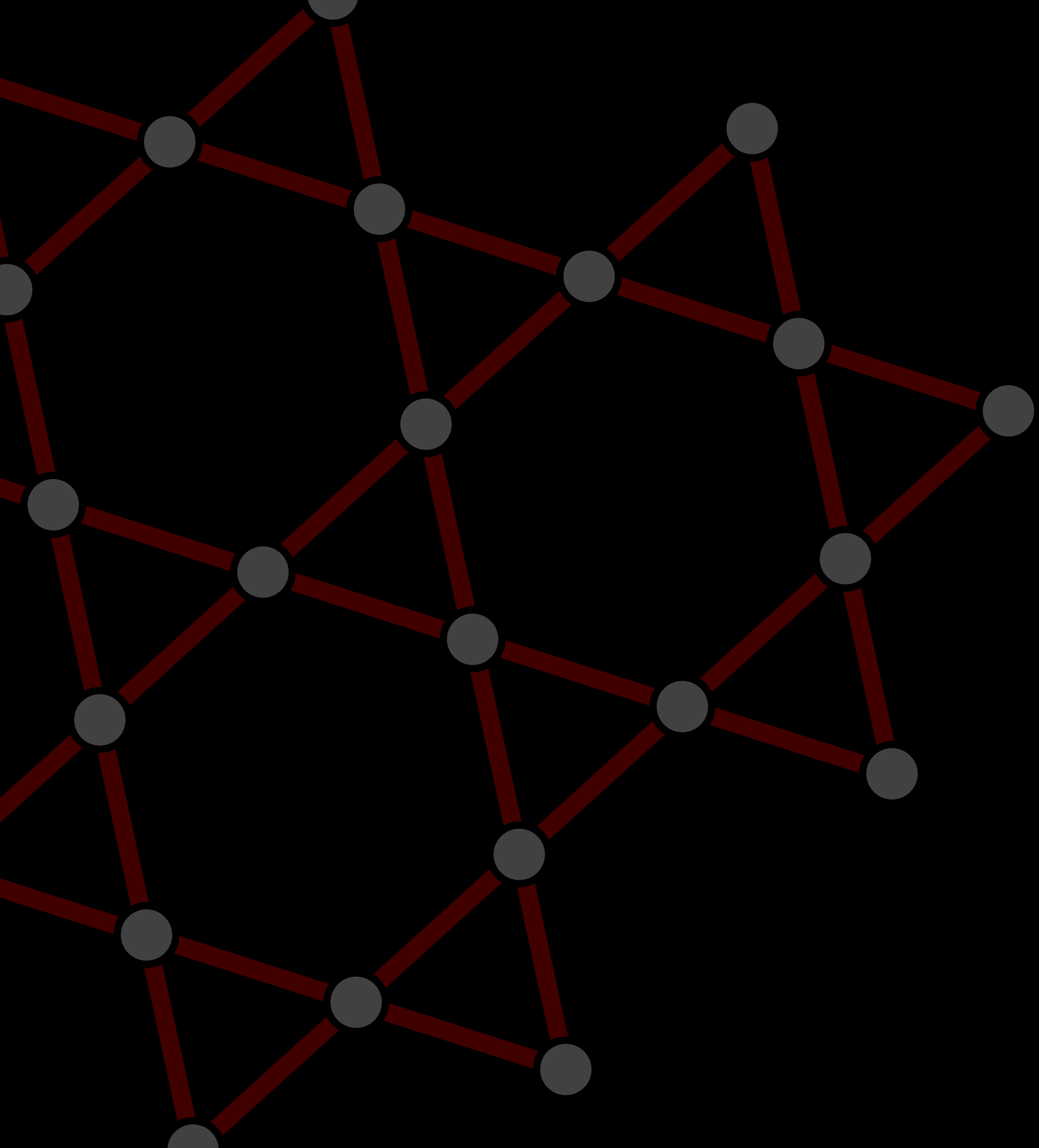
sublattice B

$\vec{S}_i^T \, \mathbf{R}^\dagger\mathbf{R} \, \vec{S}_j$

fermion

bipartite

$c_i^\dagger \, \mathbf{H} \, c_j$

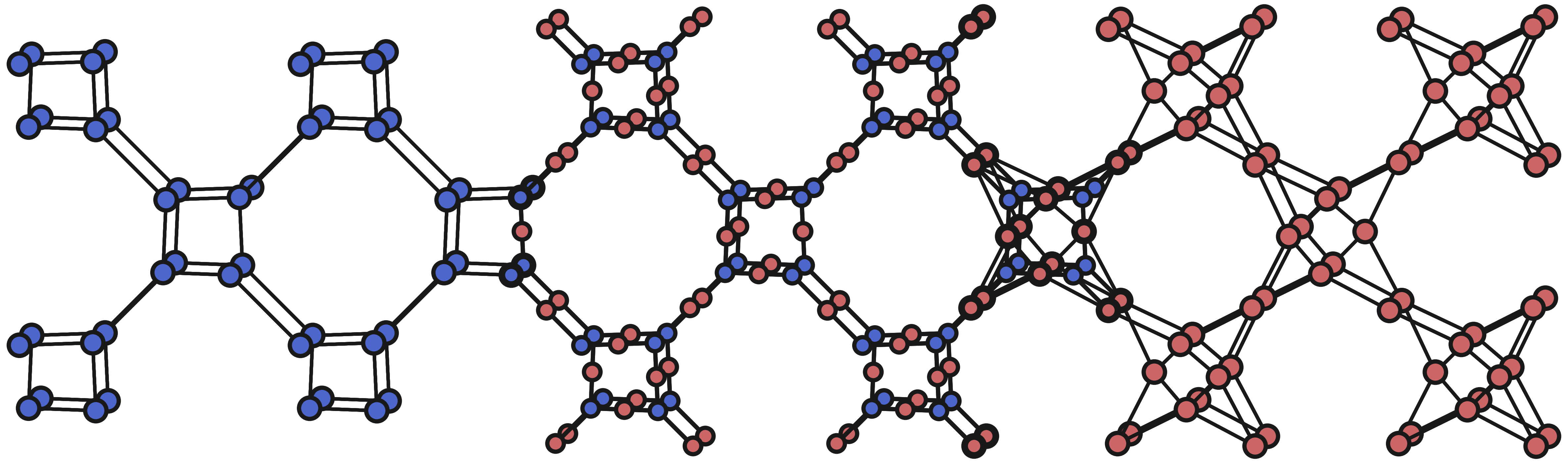


frustrated magnets

spin liquids & parton dispersions

quantum spin liquids

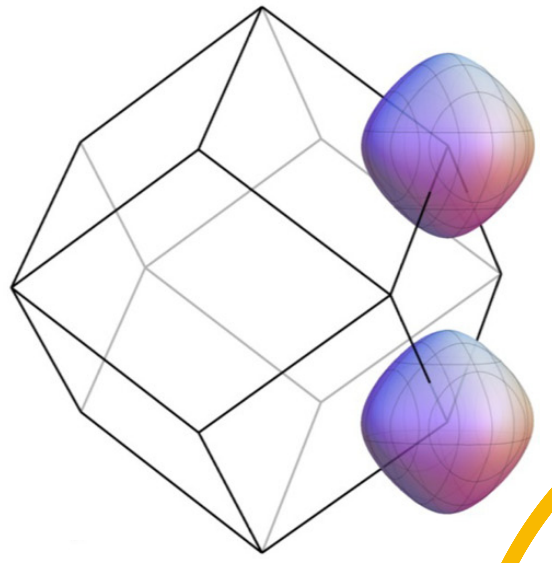
3D Kitaev materials
(akin to β,γ-$LiIr_2O_3$)

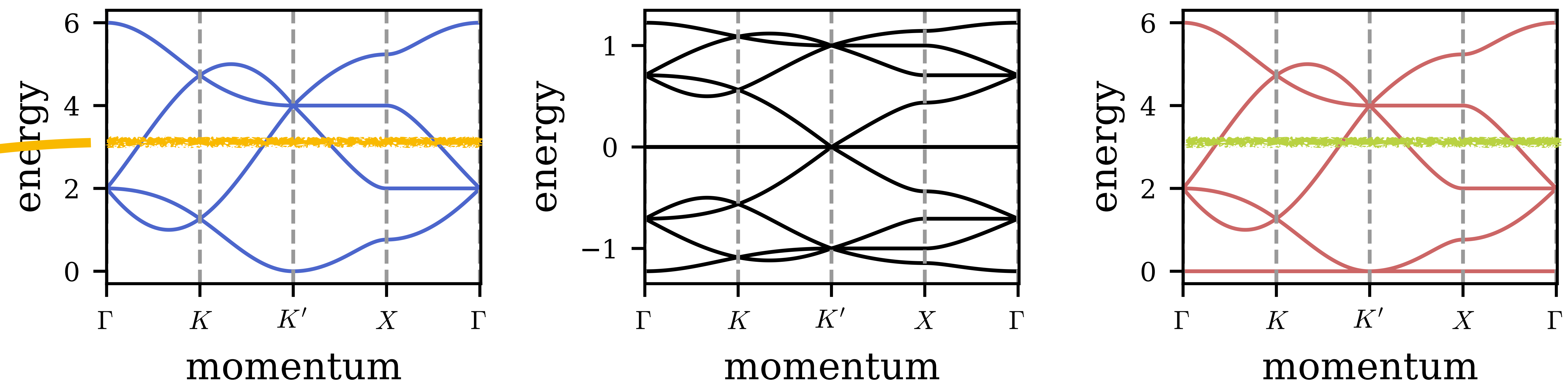


j=1/2 Mott insulator
$Na_4Ir_3O_8$

hyperoctagon

hyperkagome





Majorana Fermi surface

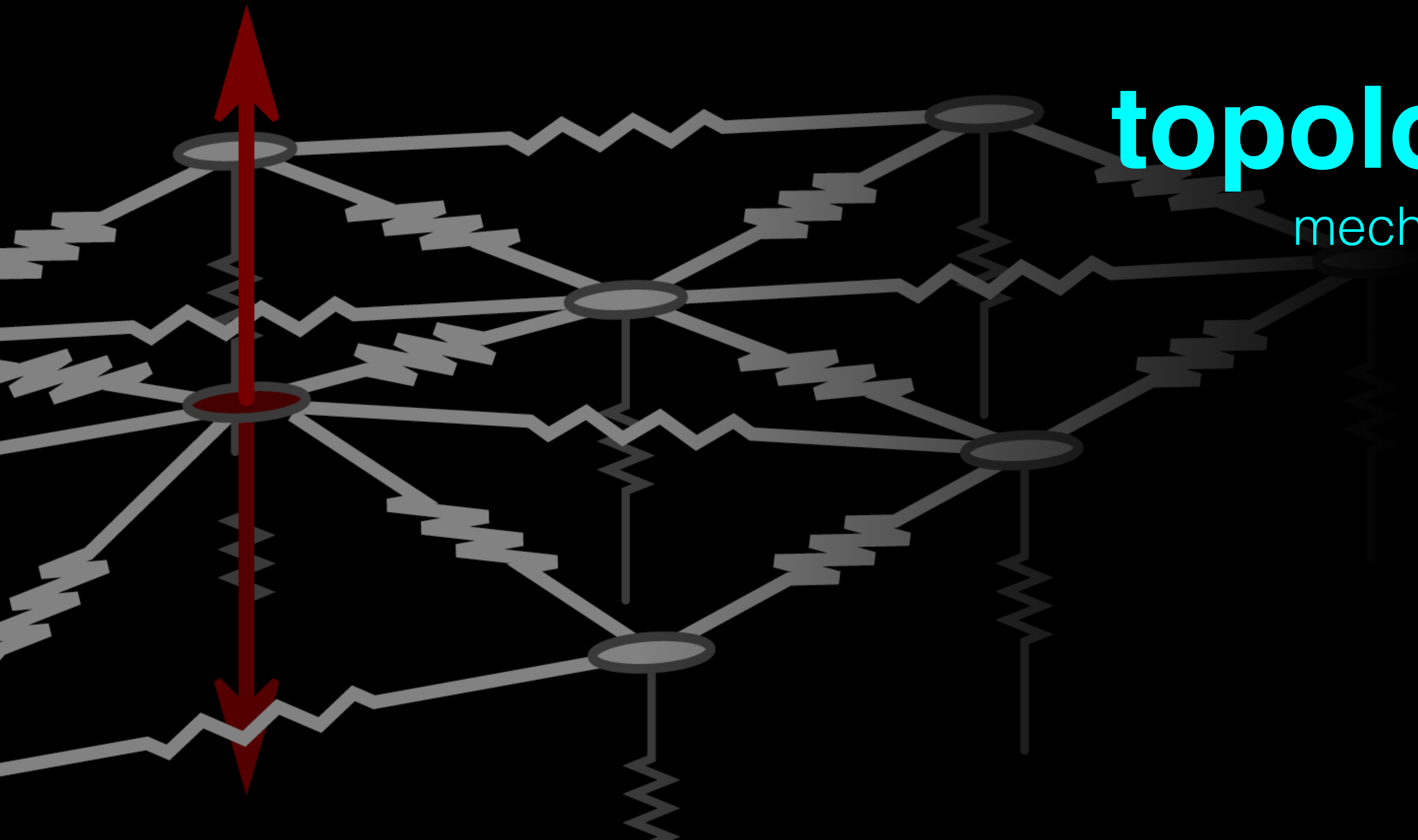
Z_2 gauge theory

M. Hermanns, ST, PRB **89**, 235102 (2014)

spinon Fermi surface

U(1) gauge theory

M. Lawler et al, PRL **101**, 197202 (2008)



topological mechanics

mechanical analogues of Kitaev spin liquids

Phys. Rev. Res. 1, 032047(R) (2019)

SUSY & topological mechanics

topological mechanics – phase space coordinates (p, q)
as bosonic degrees of freedom

real bosons $[\hat{q}_i, \hat{p}_j] = i\delta_{i,j}$

time-reversal symmetric dynamics

natural SUSY partners

real fermions
= Majorana fermions

symmetry class
BDI

$$\mathcal{H}_F = -\frac{i}{2}\underbrace{\gamma_i^A \mathbf{A}_{ij} \gamma_j^B}_{\text{bipartite lattice}} + \text{h.c.} = \frac{i}{2}\begin{pmatrix}\gamma^A & \gamma^B\end{pmatrix} \underbrace{\begin{pmatrix} & -\mathbf{A} \\ \mathbf{A}^T & \end{pmatrix}}_{\text{block off-diagonal}} \begin{pmatrix}\gamma^A \\ \gamma^B\end{pmatrix}$$

SUSY & topological mechanics

Majorana fermions

$$\mathcal{H}_F = -\frac{i}{2}\underbrace{\gamma_i^A \mathbf{A}_{ij}\gamma_j^B}_{\text{bipartite lattice}} + \text{h.c.} = \frac{i}{2}\left(\gamma^A \quad \gamma^B\right)\underbrace{\begin{pmatrix} & -\mathbf{A} \\ \mathbf{A}^T & \end{pmatrix}}_{\text{block off-diagonal}}\begin{pmatrix}\gamma^A \\ \gamma^B\end{pmatrix}$$

H_{SUSY}

$= \{\mathcal{Q}, \mathcal{Q}^\dagger\}$

local SUSY charge

$$\mathcal{Q} = \underbrace{\gamma_i^A \mathbf{A}_{ij}\hat{q}_j}_{\text{real fermions}} + \overbrace{\gamma_i^B \mathbf{1}_{ij}\hat{p}_j}^{\text{real bosons}} = \overbrace{\left(\gamma^A \quad \gamma^B\right)}^{\text{two sublattices}}\underbrace{\begin{pmatrix}\mathbf{A} & \\ & \mathbf{1}\end{pmatrix}}_{\text{block-diagonal } \mathbf{R}}\begin{pmatrix}\hat{q} \\ \hat{p}\end{pmatrix}$$

H_{SUSY}

$= \{\mathcal{Q}, \mathcal{Q}^\dagger\}$

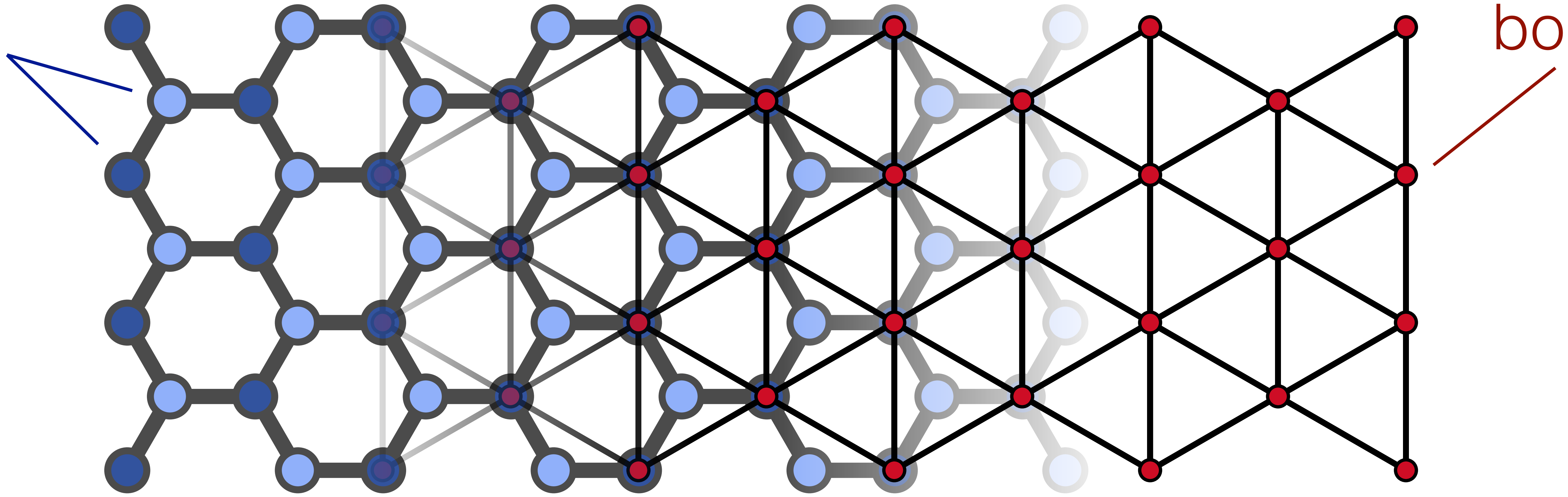
$$\mathcal{H}_B = \frac{1}{2}\sum_{ij}\hat{q}_i(\mathbf{A}^T\mathbf{A})_{ij}\hat{q}_j + \frac{1}{2}\sum_i \hat{p}_i\hat{p}_i = \frac{1}{2}\left(\hat{q} \quad \hat{p}\right)\underbrace{\overbrace{\begin{pmatrix}\mathbf{A}^T\mathbf{A} & \\ & \mathbf{1}\end{pmatrix}}^{\text{dynamical matrix}}}_{\text{block-diagonal = decoupling}}\begin{pmatrix}\hat{q} \\ \hat{p}\end{pmatrix}$$

real bosons

classical limit

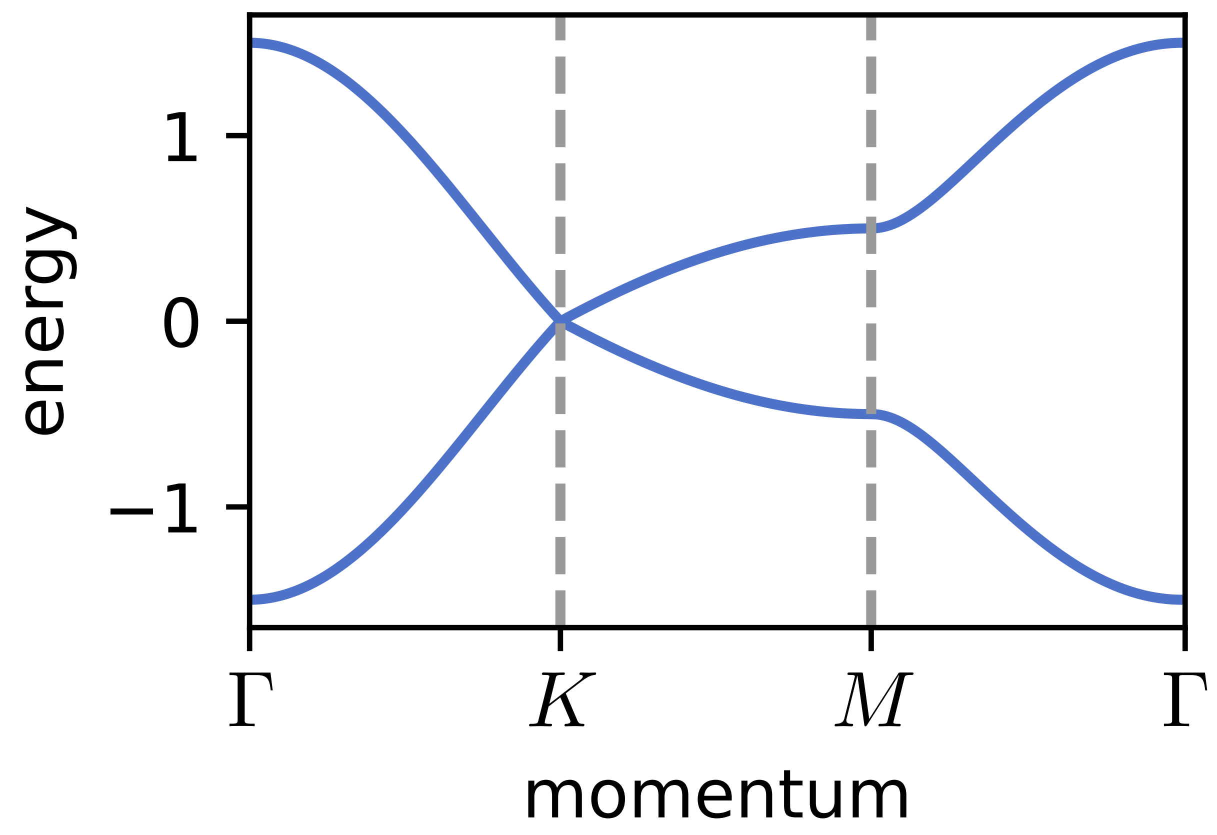
© Simon Trebst

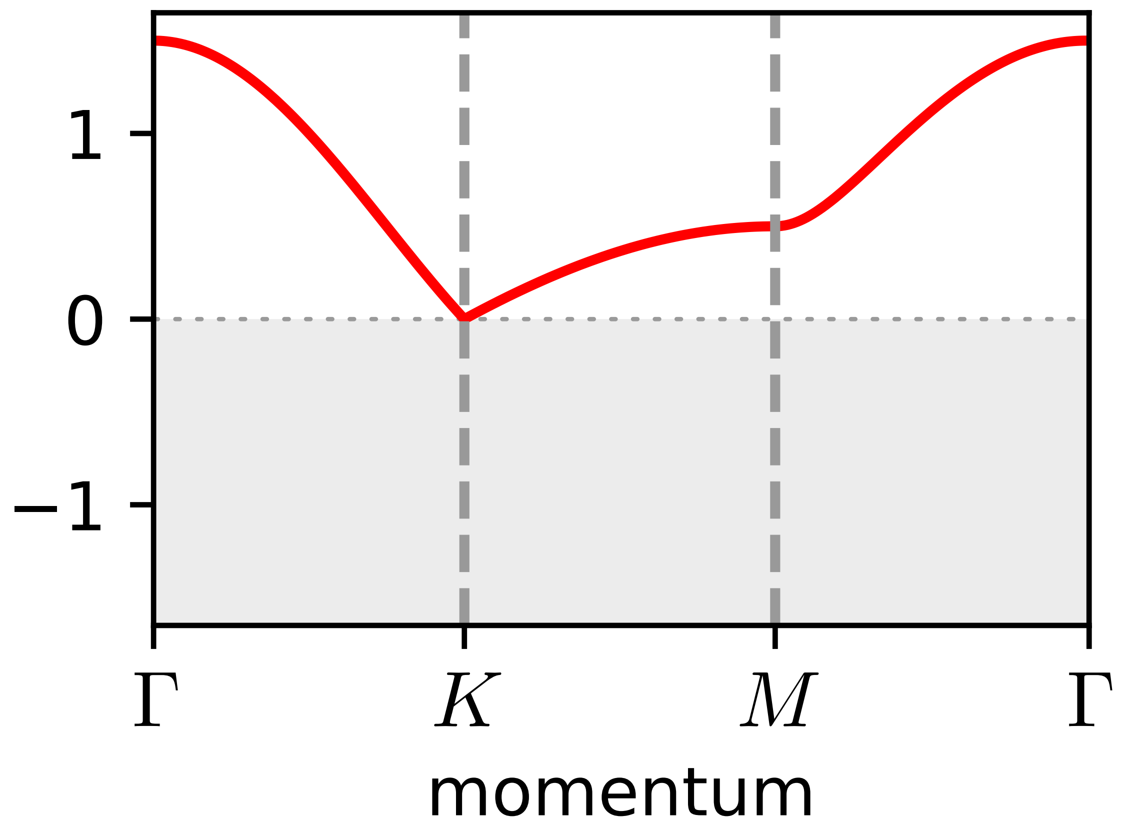
SUSY & topological mechanics



Majorana
fermions

real
bosons





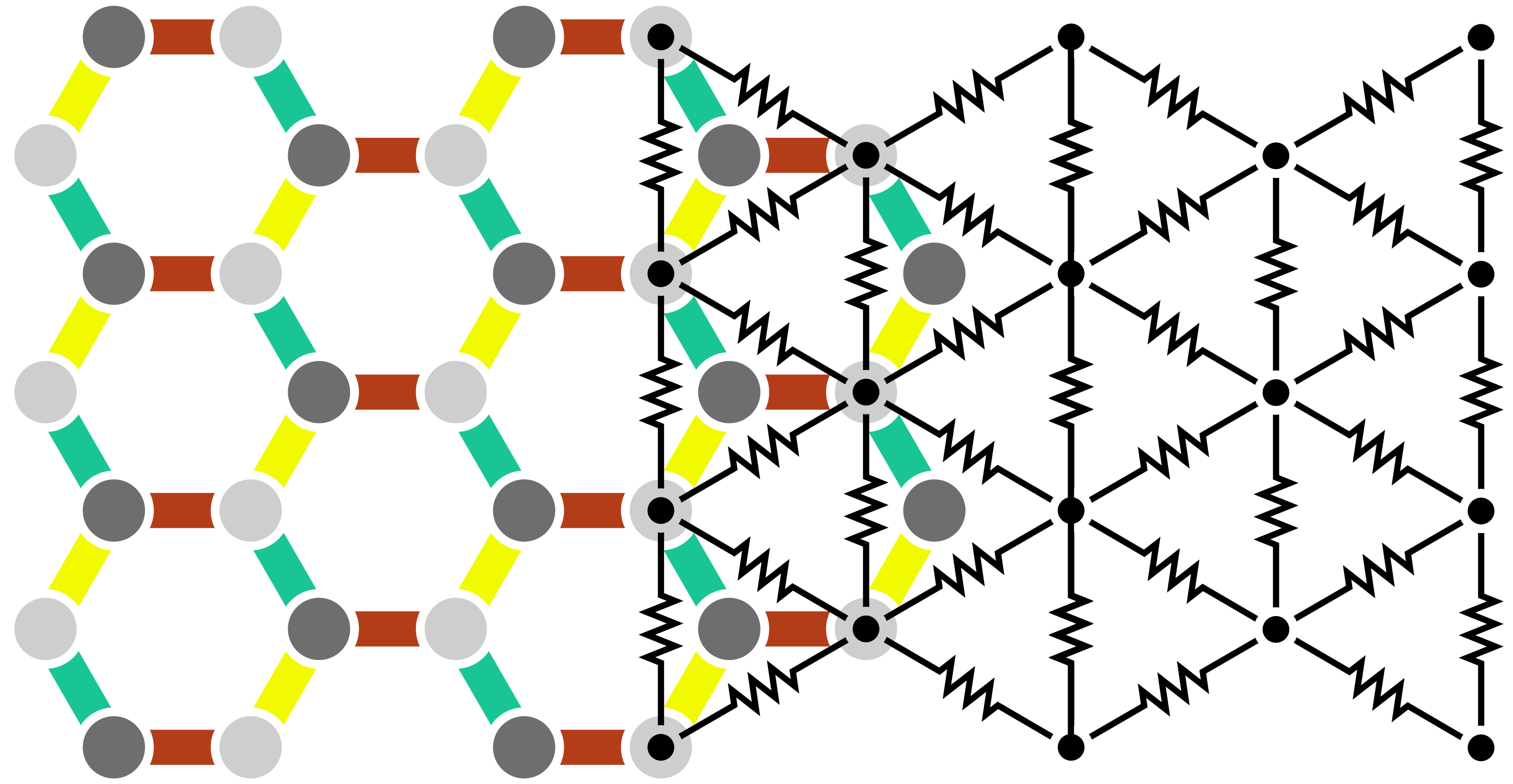
equations of motion

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix}\gamma_A\\ \gamma_B\end{pmatrix} = i\begin{pmatrix} & -\mathbf{A}(\mathbf{k})\\ \mathbf{A}^\dagger(\mathbf{k}) & \end{pmatrix}\begin{pmatrix}\gamma_A\\ \gamma_B\end{pmatrix}$$

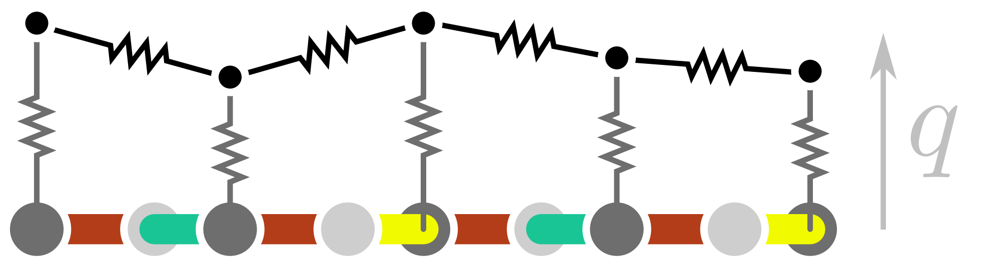
$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix}\hat{p}\\ \hat{q}\end{pmatrix} = i\begin{pmatrix} & -1\\ \mathbf{A}^\dagger(\mathbf{k})\mathbf{A}(\mathbf{k}) & \end{pmatrix}\begin{pmatrix}\hat{p}\\ \hat{q}\end{pmatrix}$$

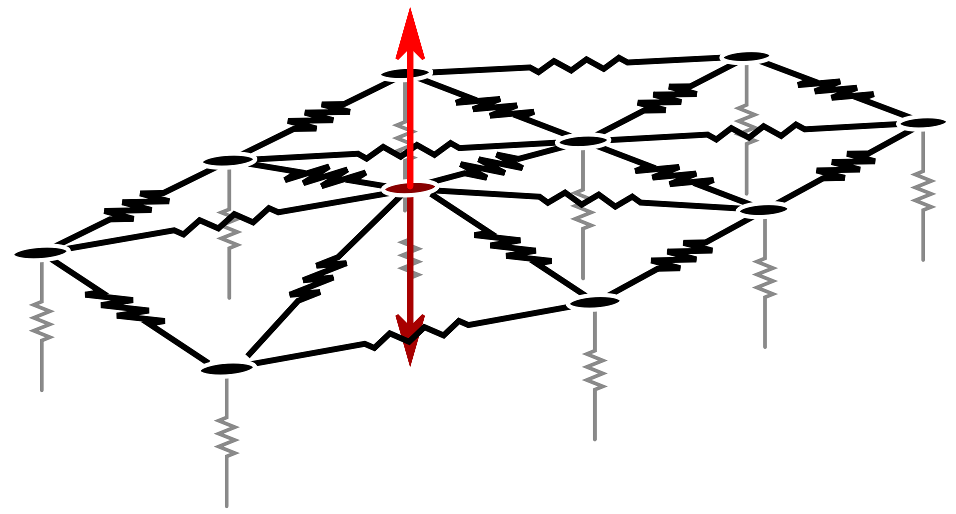
balls & springs Kitaev model

Majorana fermions
on **honeycomb** lattice



balls & springs
on **triangular** lattice



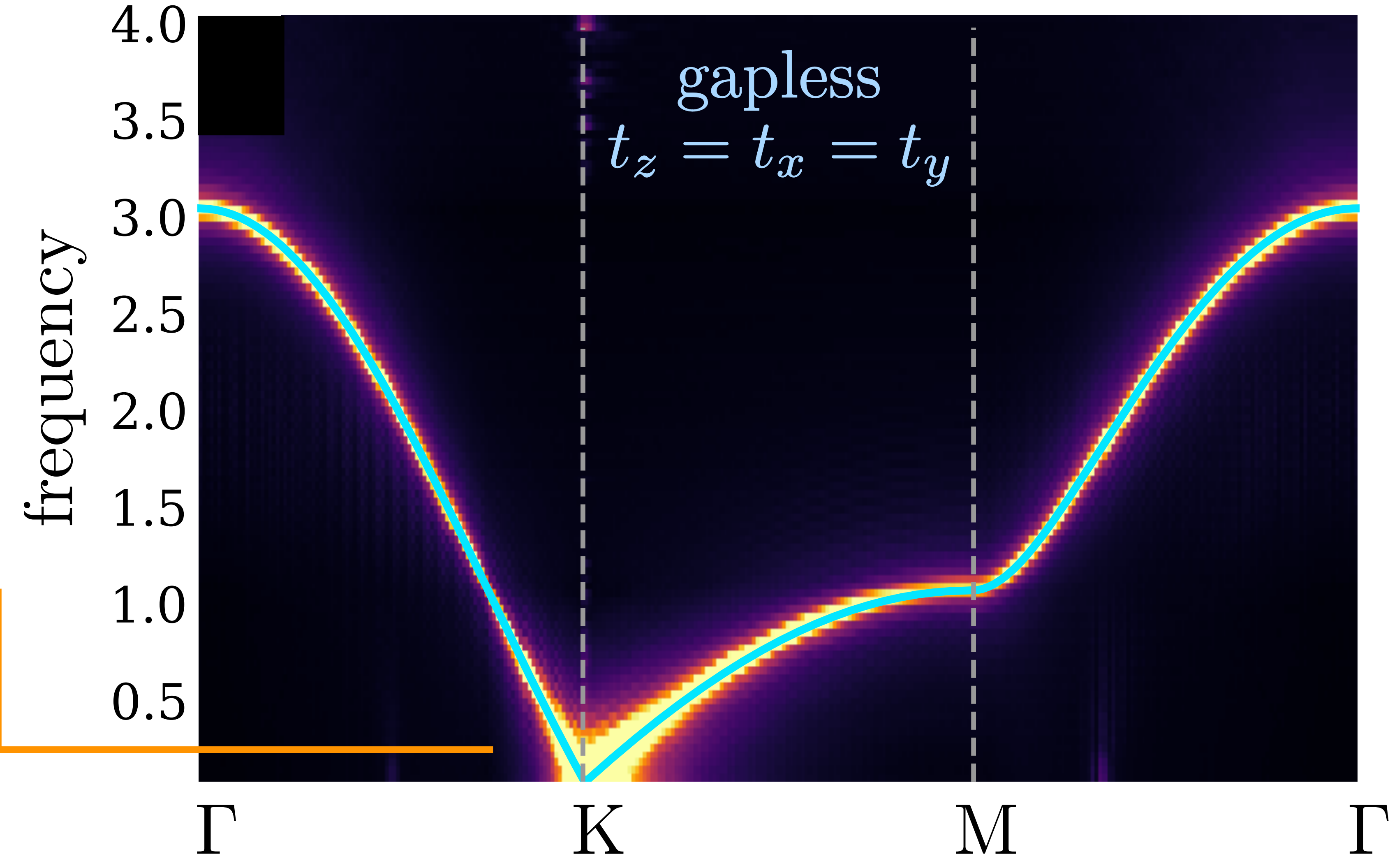


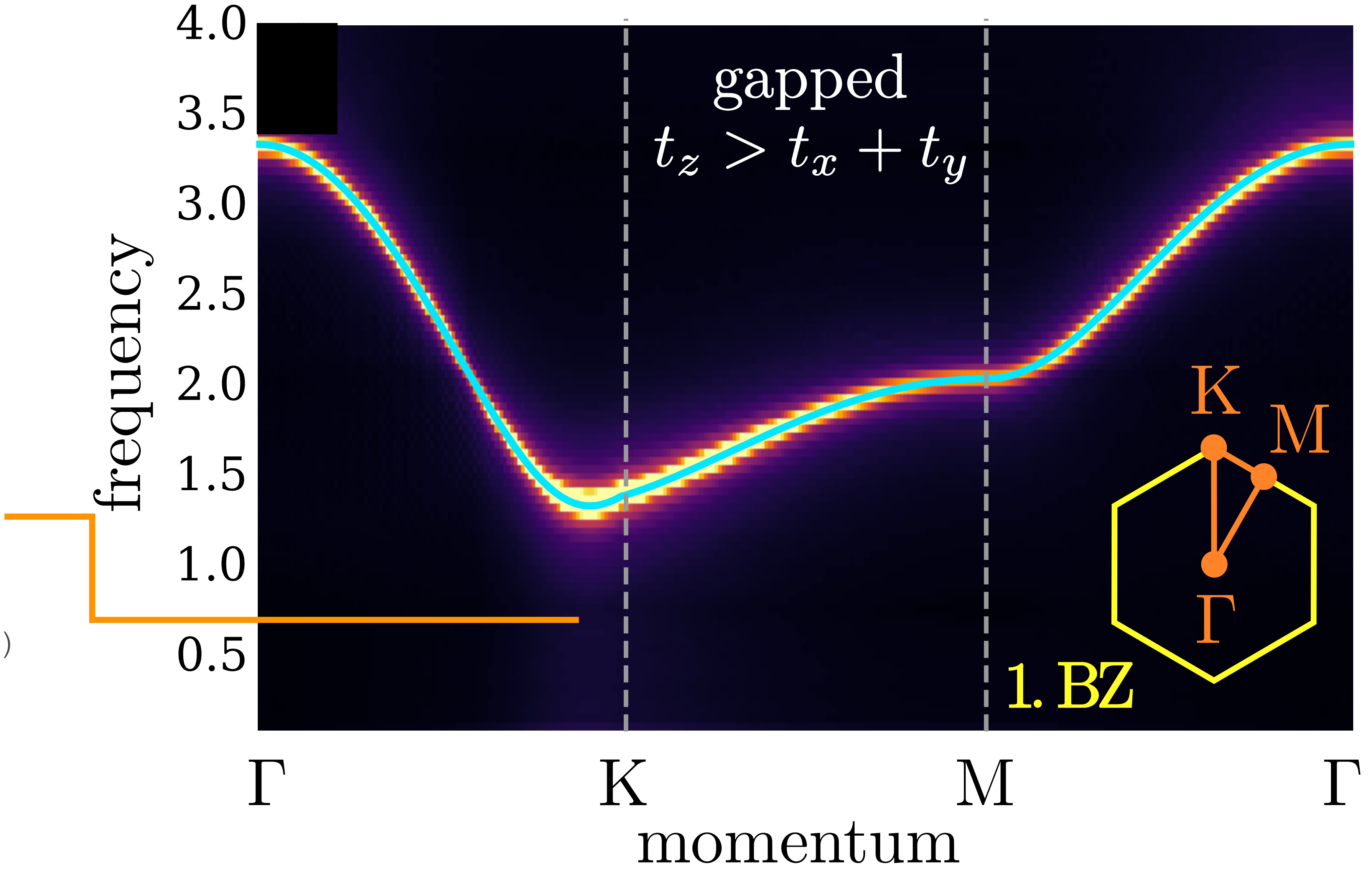
Dirac cone with **linear** dispersion
= evidence of many-body physics

(every single spring has a quadratic dispersion)

energy **gap** = low-frequency rigidity
of a mechanical system (unusual)

(no Goldstone modes here, explicit symmetry breaking)





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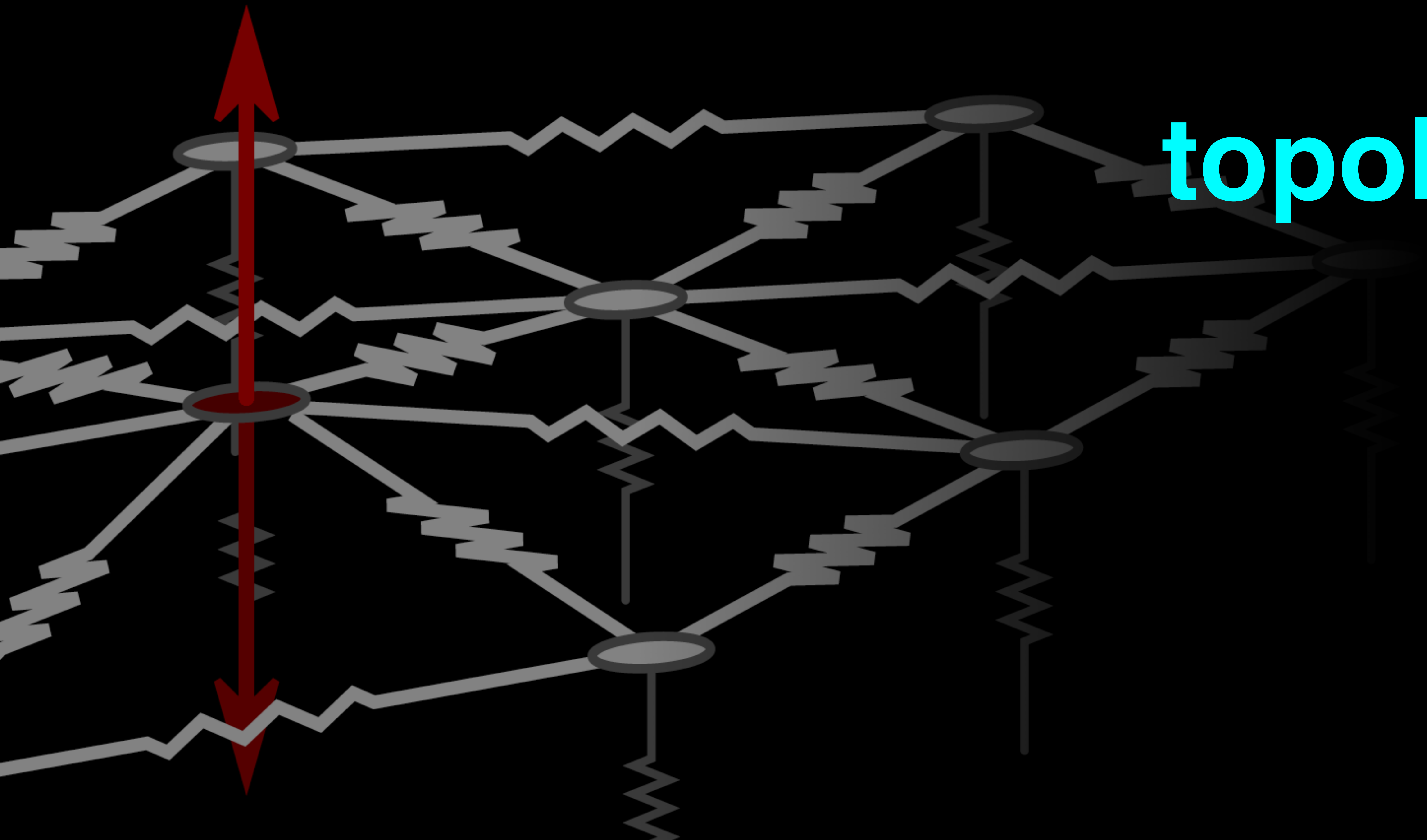


topological invariants

Phys. Rev. Res. 1, 032047(R) (2019)

SUSY topological invariants for bosons

Phys. Rev. Res. 1, 032047(R) (2019)

Our SUSY construction allows to explore

topological properties of bosonic systems

by connecting the symplectic bosonic eigenfunctions

with a **fermionic Berry phase** of its SUSY partner.

SUSY topological invariants for bosons

Phys. Rev. Res. 1, 032047(R) (2019)

Our SUSY construction allows to explore
topological properties of bosonic systems
by connecting the symplectic bosonic eigenfunctions
with a **fermionic Berry phase** of its SUSY partner.

fermionic Berry curvature

$$\mathcal{A} = \langle u_m(\mathbf{k}) | i \nabla_k | u_n(\mathbf{k}) \rangle$$

fermionic eigenstates

SUSY

via **rigidity matrix**

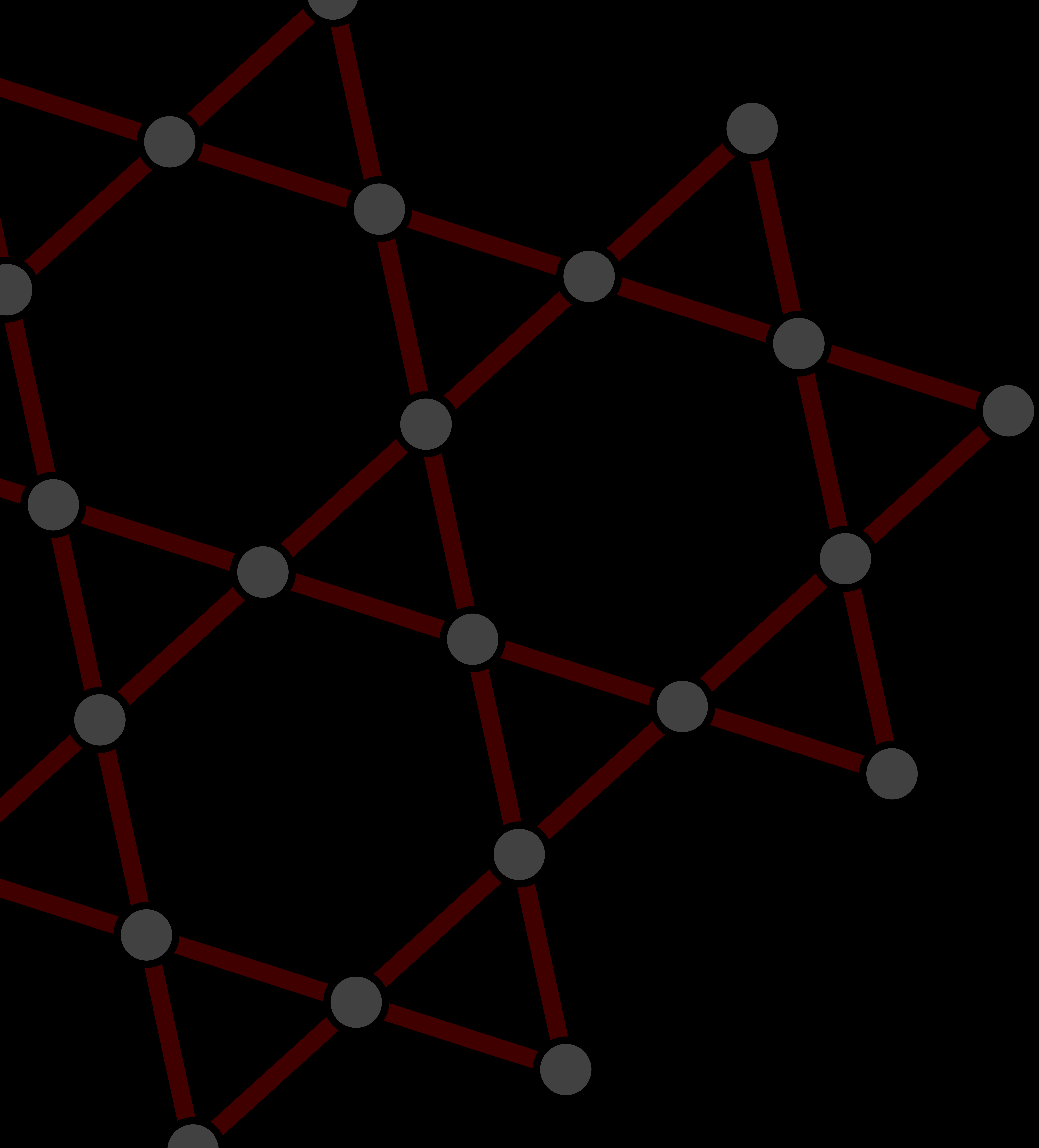
$$|u_m(\mathbf{k})\rangle = \frac{\mathbf{R}(\mathbf{k})}{\sqrt{|\omega_m(\mathbf{k})|}} |v_m(\mathbf{k})\rangle \equiv \tilde{\mathbf{R}}(\mathbf{k}) |v_m(\mathbf{k})\rangle$$

bosonic eigenstates

SUSY

bosonic Berry curvature

$$\begin{aligned} \mathcal{A}_{\mathrm{SUSY}} &= \langle v_m(\mathbf{k}) | i \tilde{\mathbf{R}}^\dagger \nabla_k \left(\tilde{\mathbf{R}} | v_n(\mathbf{k}) \rangle \right) \\ &= \boxed{\langle v_m(\mathbf{k}) | i \sigma_2 \left(\nabla_k + \underbrace{\sigma_2 \tilde{\mathbf{R}}^\dagger \nabla_k \tilde{\mathbf{R}}}_{\text{additional covariant derivative}} \right) | v_n(\mathbf{k}) \rangle} \end{aligned}$$



summary

summary

arXiv:2207.09475

Take-away messages

- versatile connections of **supersymmetry, locality,** and **topology**

bosons & fermions
(complex or real)

SUSY **graph correspondence**
graph squares & square roots

SUSY **bosonic invariants**
classification of **nexus points**

- **Unifying framework** for frustrated magnets & topological mechanics

ground-state manifolds
magnon / parton spectra
Maxwell counting

Maxwell counting
mechanical spin liquids

Phys. Rev. B 96, 085145 (2017)

Phys. Rev. Res. 1, 032047(R) (2019)

Institute for Theoretical Physics, Cologne / Germany