# Interactions and disorder in topological quantum matter

January 2012

**Simon Trebst**
University of Cologne

# Overview

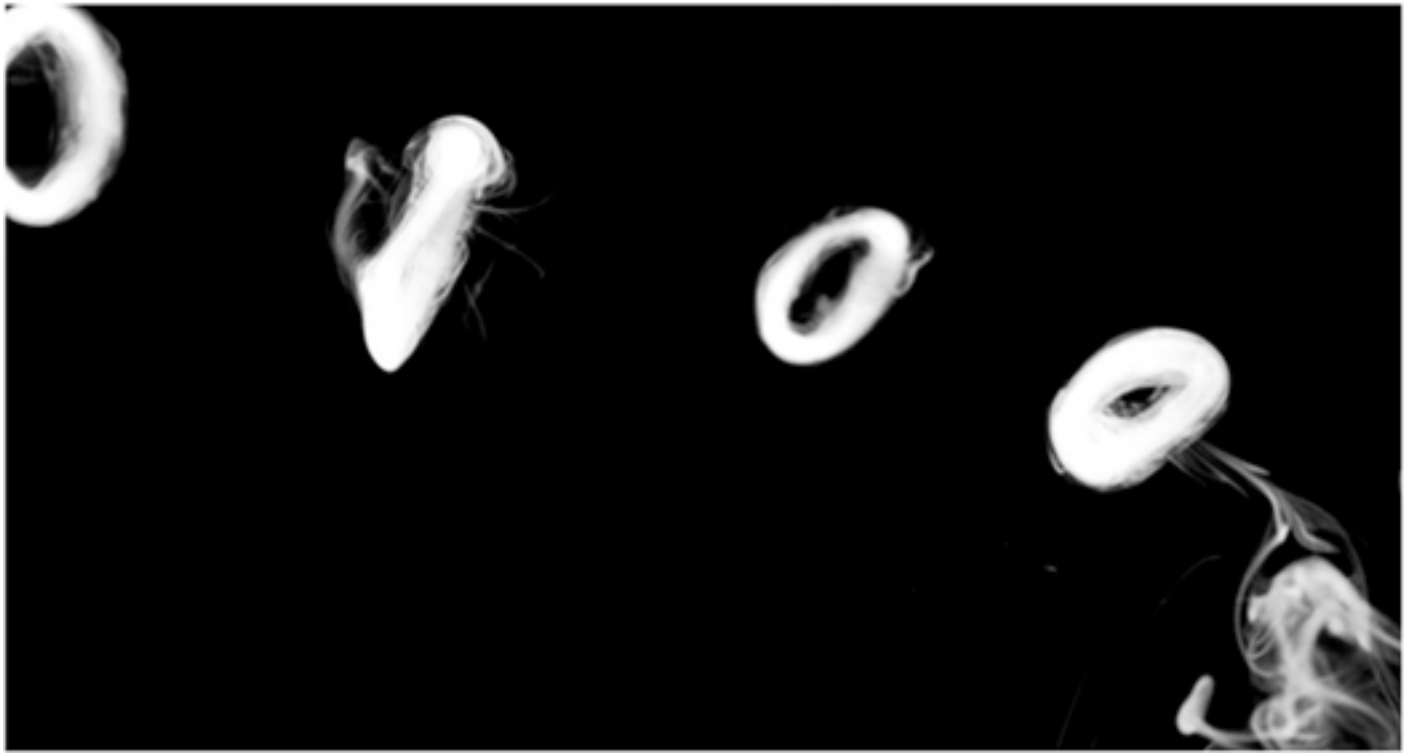
Topological quantum matter

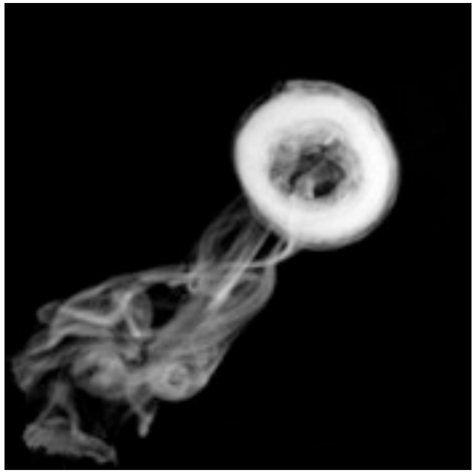
Interactions

Disorder

Experimental signatures

# Topological quantum matter



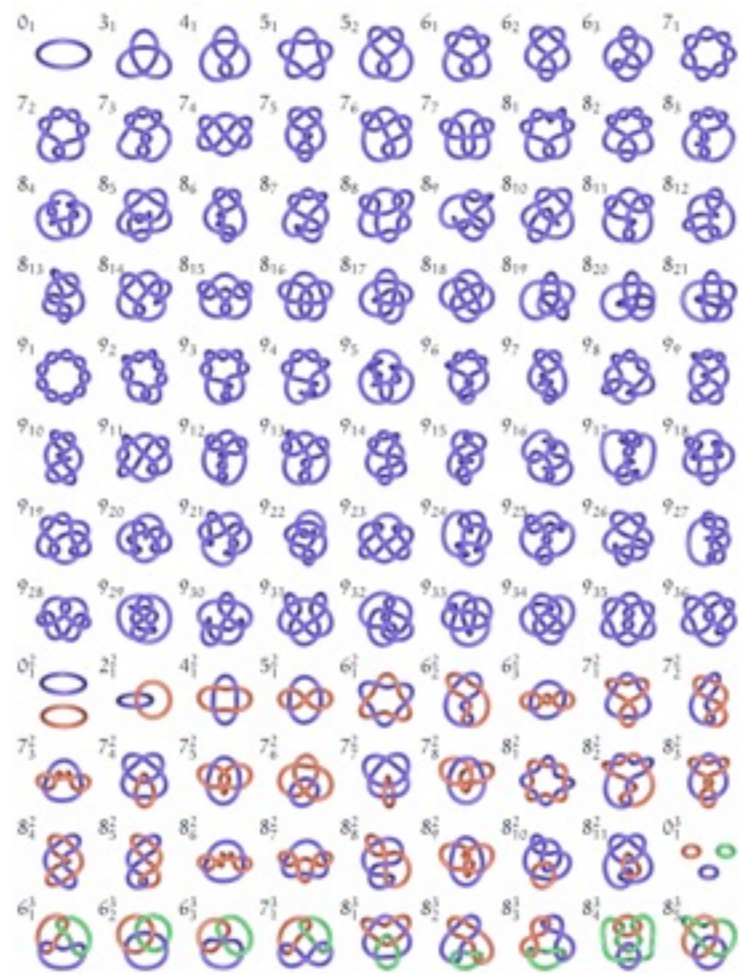


# Topological quantum matter

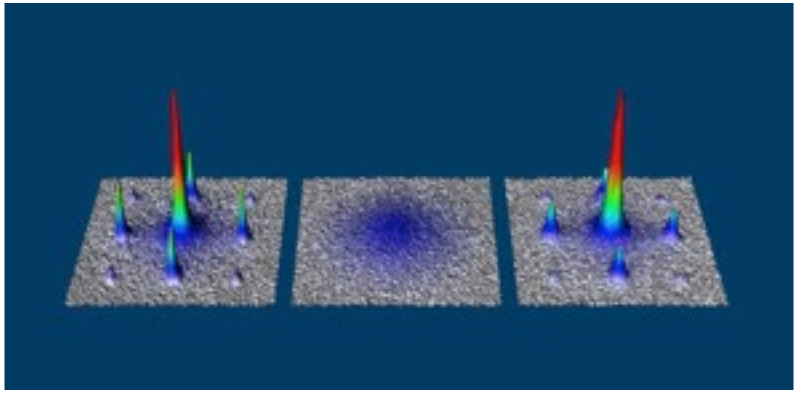
- 1867: **Lord Kelvin**
  atoms = knotted tubes of ether
  Knots might explain stability, variety, vibrations, ...

- **Maxwell**: “It satisfies more of the conditions than any atom hitherto imagined.”

- This inspires the mathematician **Tait** to classify knots.



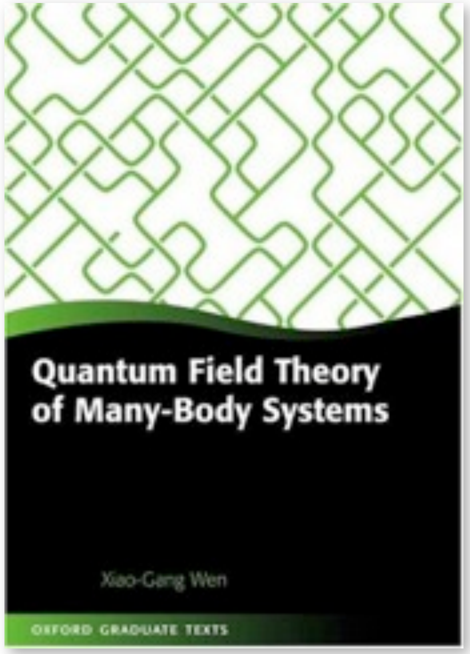
# Non-topological (quantum) matter



# Topological quantum matter

- **Xiao-Gang Wen:** A ground state of a many-body system that *cannot* be fully characterized by a *local* order parameter.

- A ground state of a many-body system that *cannot be transformed* to “simple phases” via local perturbations without going through phase transitions.




- Often characterized by a variety of “*topological properties*”.

# Topological matter / classification (rough)

## Topological order

**inherent**

Gapped phases that cannot be transformed
– without closing the bulk gap –
to "simple phases"
via any "paths"

quantum Hall states
spin liquids

**symmetry protected**

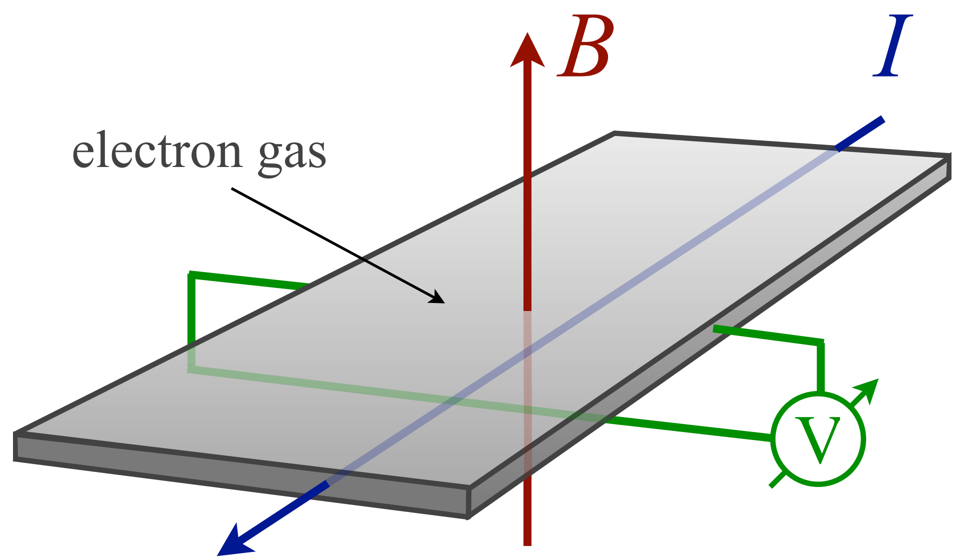
Gapped phases that cannot be transformed
– without closing the bulk gap –
to "simple phases"
via any *symmetry preserving* "paths".

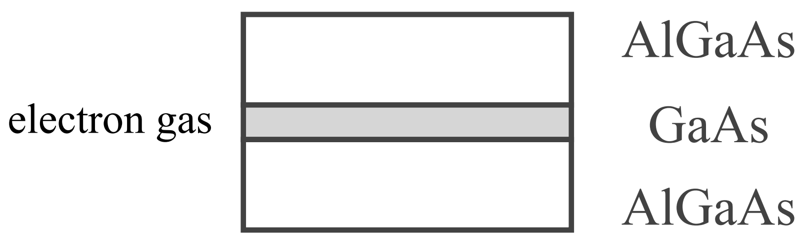
topological band insulators

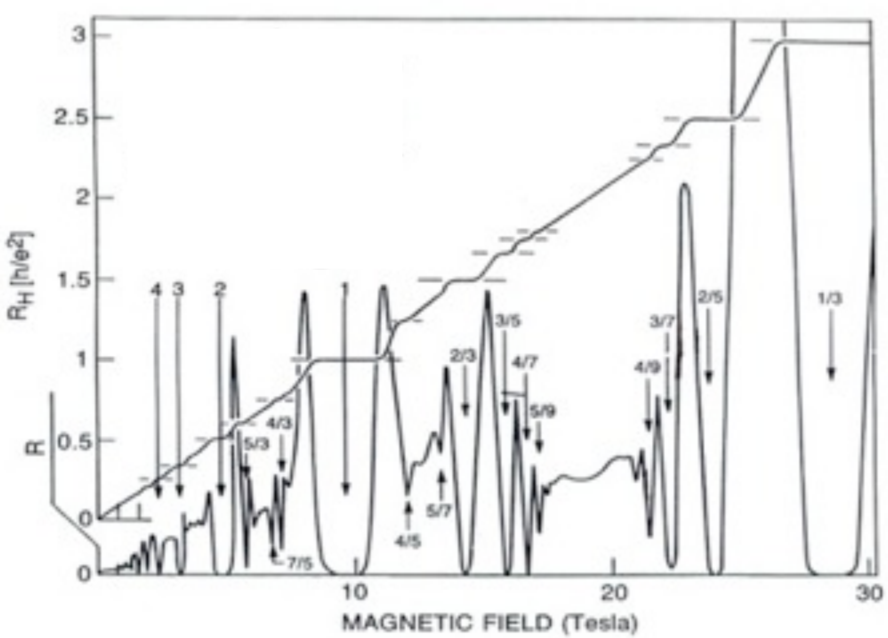
# Quantum Hall effect

**Quantization of conductivity**
for a two-dimensional electron gas
at very low temperatures
in a high magnetic field.

$$\sigma = \nu \frac{e^2}{h}$$



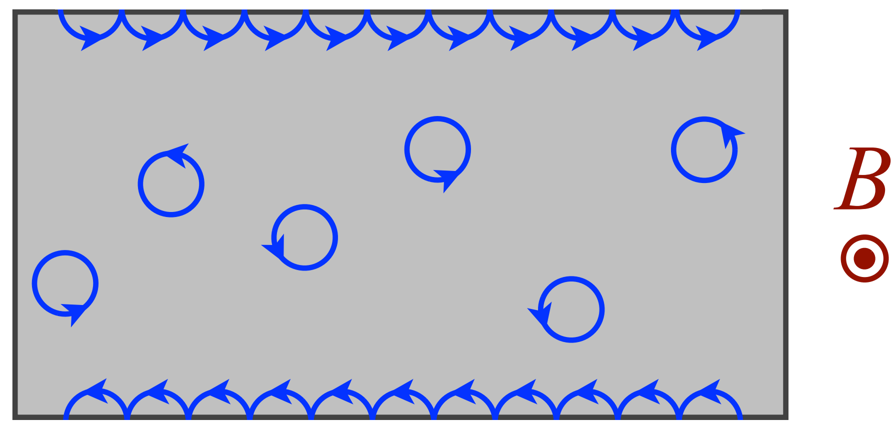




Semiconductor heterostructure confines
electron gas to two spatial dimensions.

# Quantum Hall states

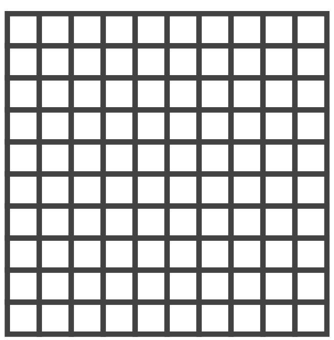
**Landau levels**

$$E_n = h\frac{eB}{m}\left(n + \frac{1}{2}\right)$$




edge states

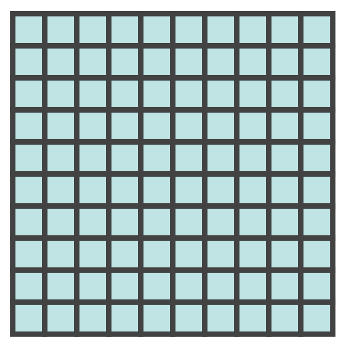
Landau level **degeneracy**



$$2\Phi/\Phi_0$$

orbital states

**integer** quantum Hall

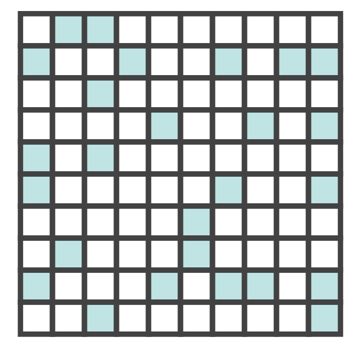


**filled** level

↓

**incompressible** liquid

**fractional** quantum Hall

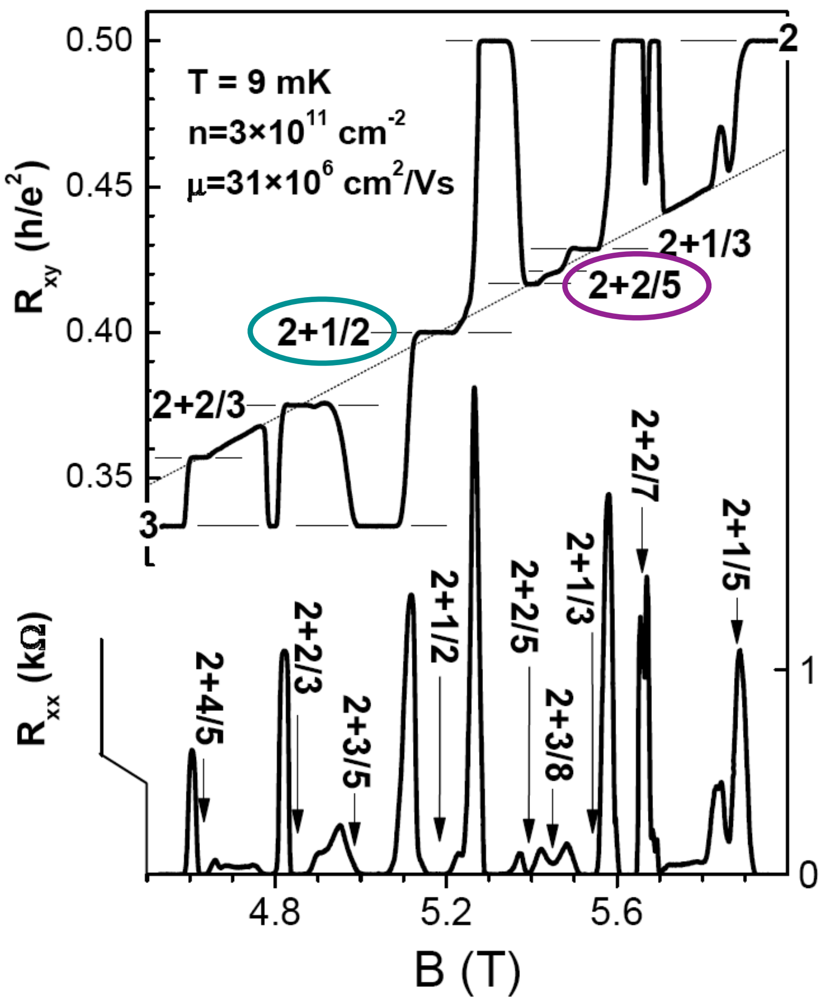


**partially filled** level

↓ Coulomb repulsion

**incompressible** liquid

# Fractional quantum Hall states




J.S. Xia *et al.*, PRL (2004)

**"Pfaffian" state**

Moore & Read (1994)

Charge e/4 quasiparticles
**Ising anyons**

$SU(2)_2$

Nayak & Wilczek (1996)

**"Parafermion" state**

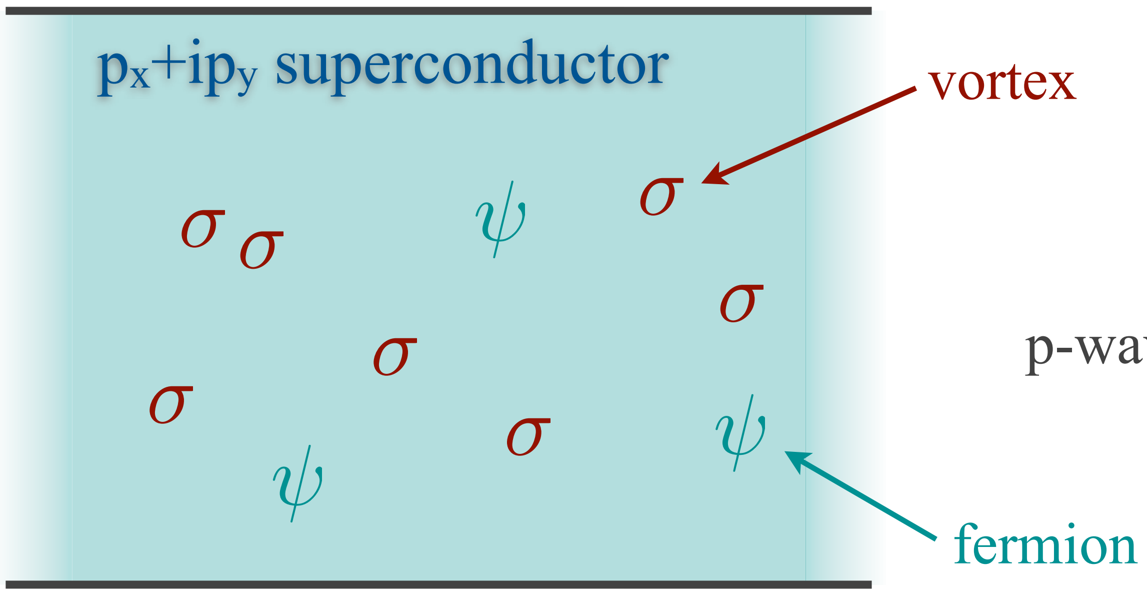
Read & Rezayi (1999)

Charge e/5 quasiparticles
**Fibonacci anyons**

$SU(2)_3$

Slingerland & Bais (2001)

# $p_x + ip_y$ superconductors




**possible realizations**

$Sr_2RuO_4$

p-wave superfluid of cold atoms

$A_1$ phase of $^3He$ films

**Topological properties of $p_x+ip_y$ superconductors**

Read & Green (2000)

$SU(2)_2$

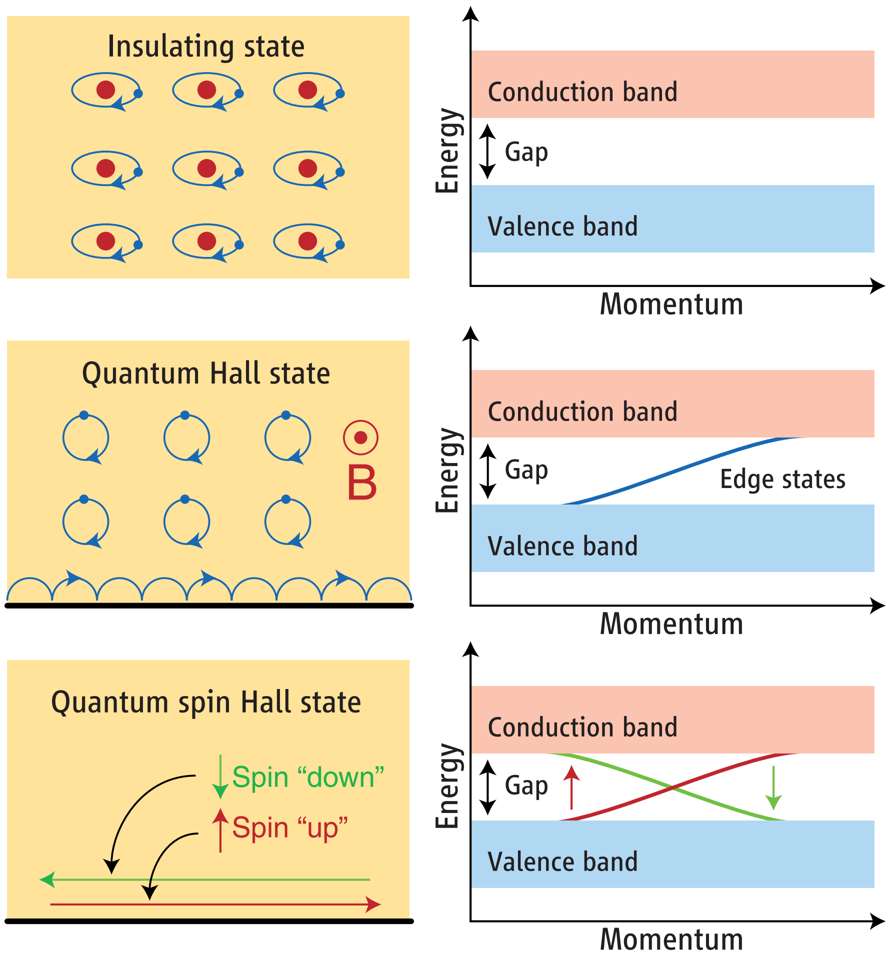
Vortices carry characteristic "zero mode"

2N vortices give degeneracy of $2^N$.

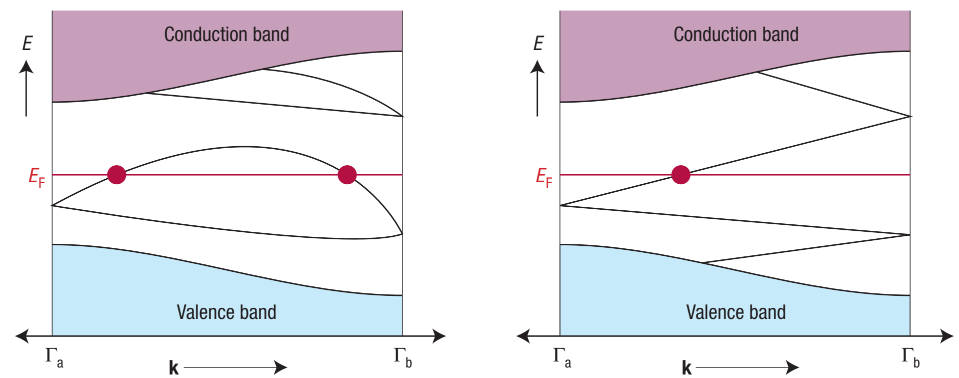
$$\sigma \times \sigma = 1 + \psi$$

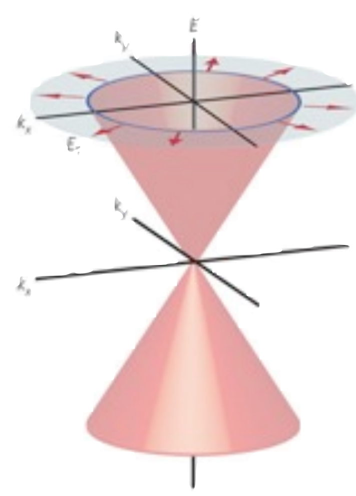
# Topological Insulators

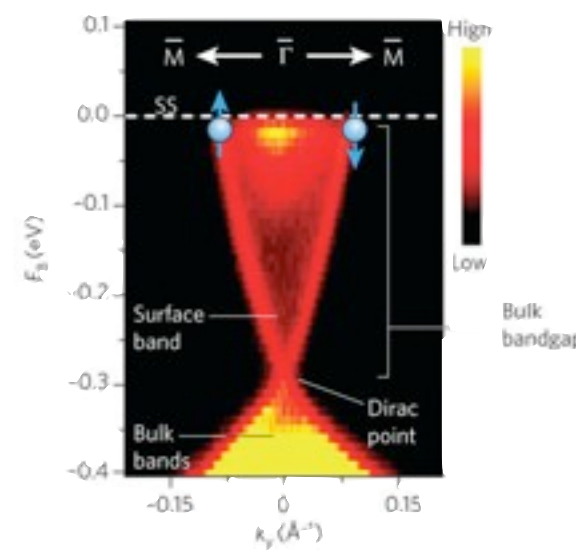
## 2D – HgTe quantum wells



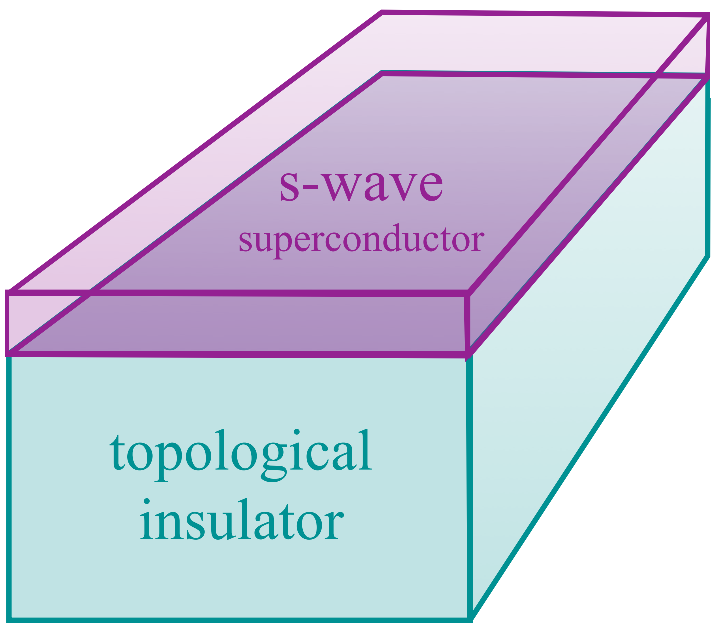

## 3D – $Bi_2Se_3$



# Proximity effects / heterostructures




**Proximity effect**

Proximity effect between an s-wave superconductor and the surface states of a (strong) topological insulator induces exotic vortex statistics in the superconductor.

Spinless $p_x+ip_y$ superconductor where vortices bind a zero mode.

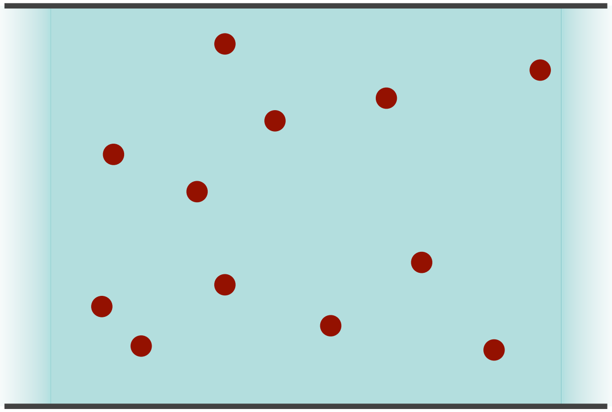
# Vortices, quasiholes, anyons, ...

# Interactions

# Abelian vs. non-Abelian vortices

Consider a set of '**pinned**' vortices at fixed positions.

**Abelian**

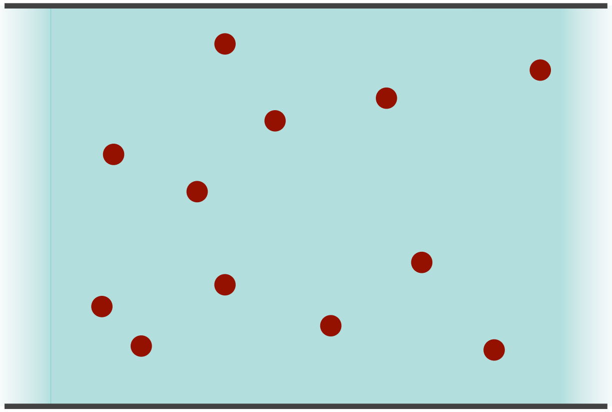


**single** state

Example:

Laughlin-wavefunction + quasiholes

**non-Abelian**



(degenerate) **manifold** of states

Manifold of states grows **exponentially** with the number of vortices.

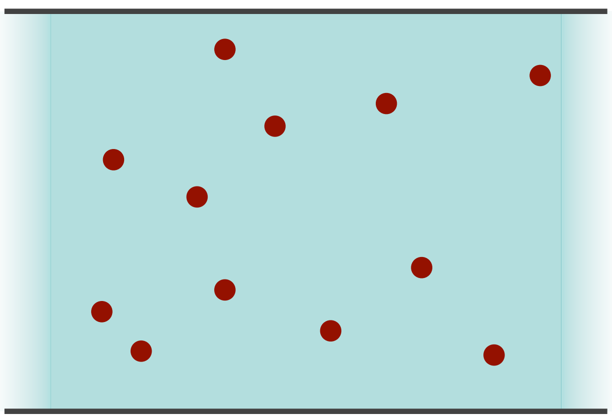
Ising anyons (Majorana fermions) $\sqrt{2}^N$

Fibonacci anyons $\phi^N$

# Abelian vs. non-Abelian vortices

Consider a set of '**pinned**' vortices at fixed positions.

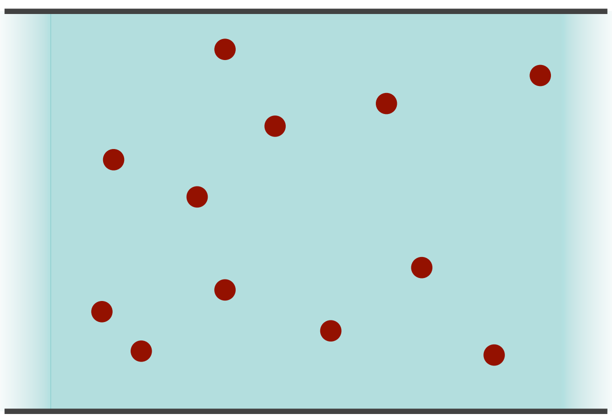
## Abelian



**single** state

$$\psi(x_2, x_1) = \underbrace{e^{i\pi\theta}}_{\text{fractional phase}} \cdot \psi(x_1, x_2)$$

## non-Abelian



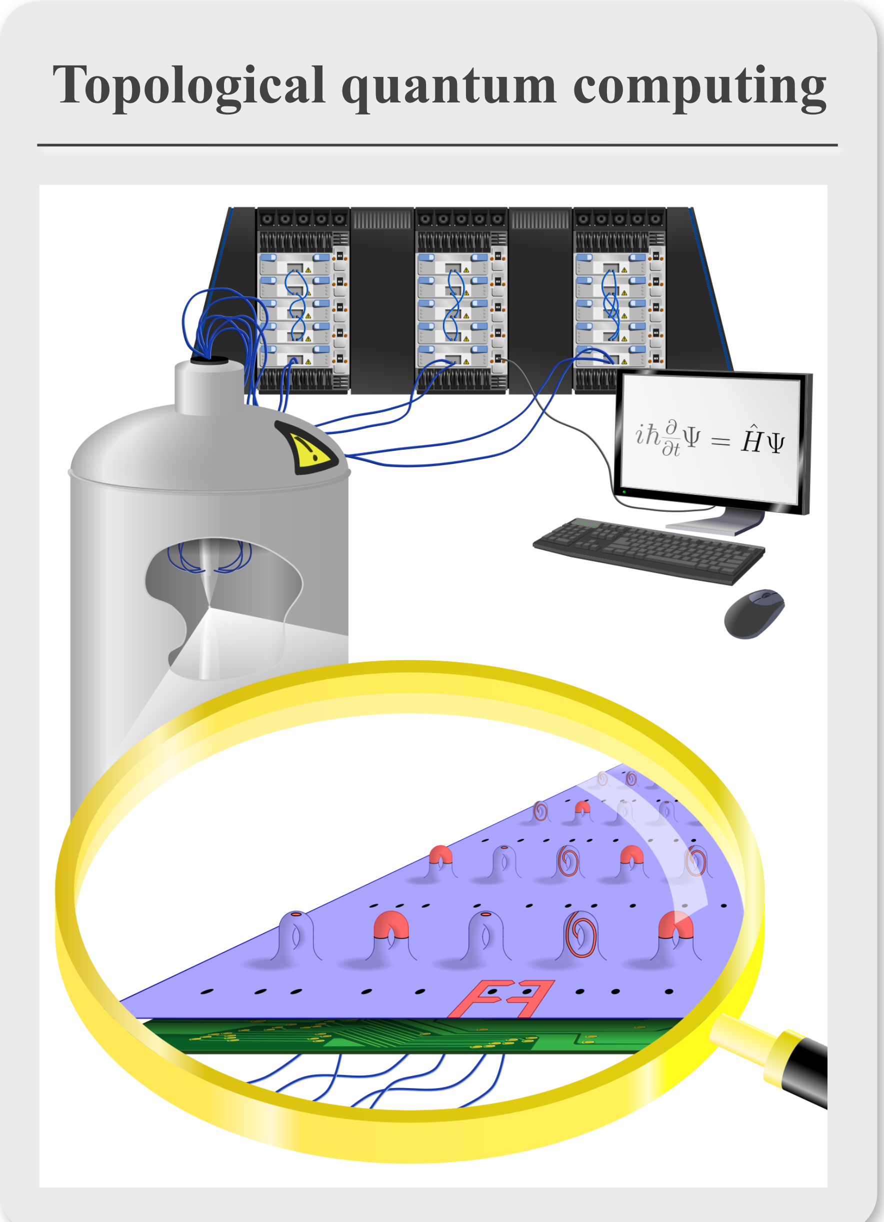
(degenerate) **manifold** of states

$$\psi(x_1 \leftrightarrow x_3) = \overbrace{M}^{\text{matrix}} \cdot \psi(x_1, \ldots, x_n)$$

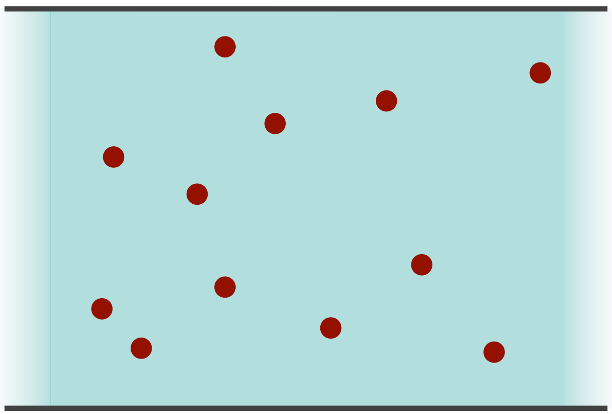
$$\psi(x_2 \leftrightarrow x_3) = N \cdot \psi(x_1, \ldots, x_n)$$

In general *M* and *N* do not commute!

# Topological quantum computing




non-Abelian



(degenerate) **manifold** of states

matrix

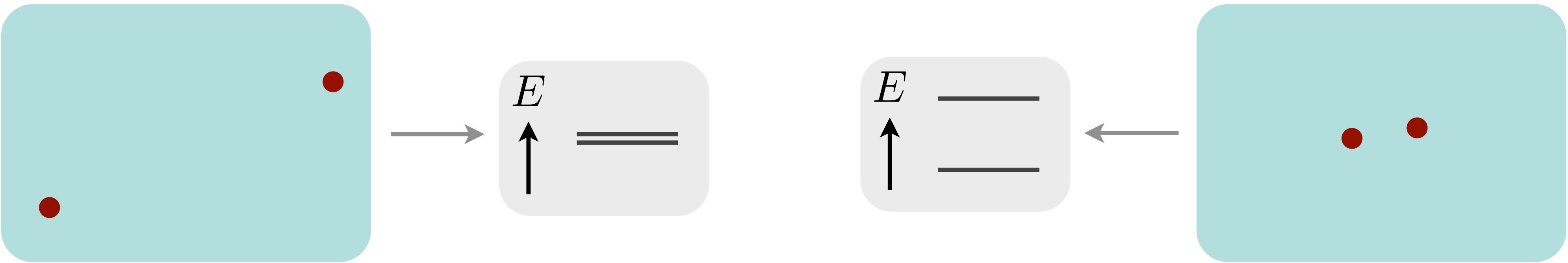
$$\psi(x_1 \leftrightarrow x_3) = M \cdot \psi(x_1, \ldots, x_n)$$

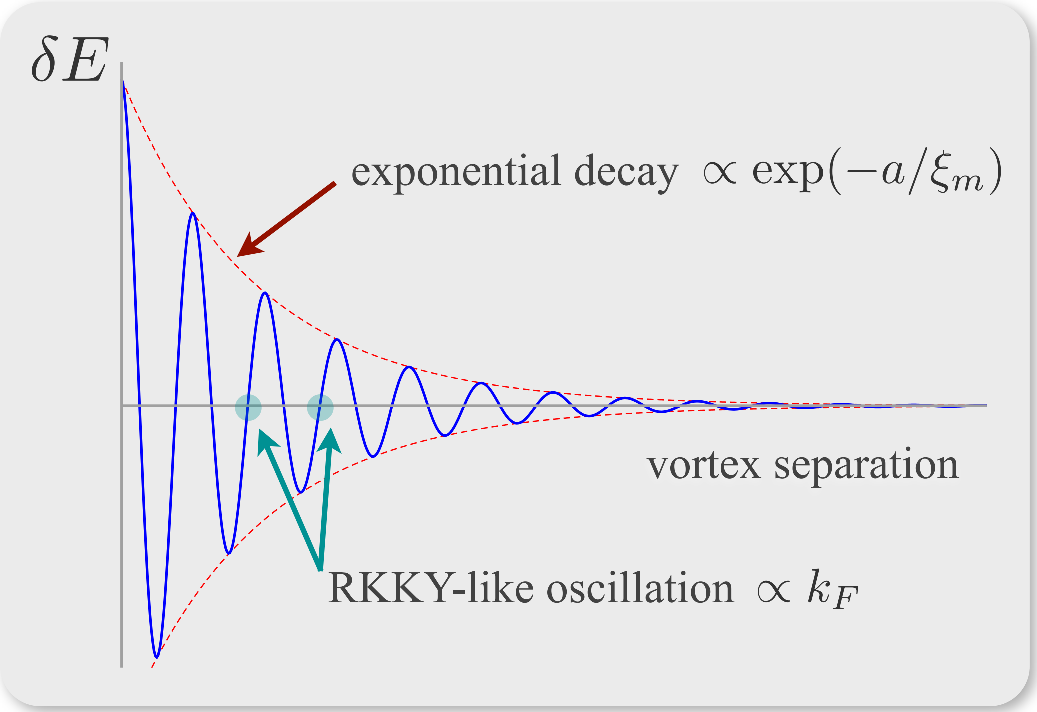
$$\psi(x_2 \leftrightarrow x_3) = N \cdot \psi(x_1, \ldots, x_n)$$

In general *M* and *N* do not commute!

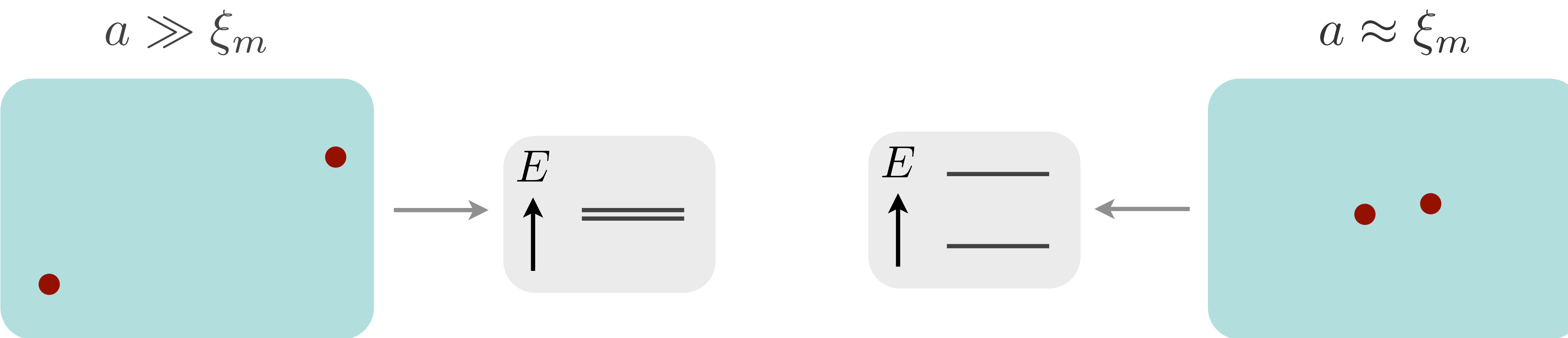
# Vortex-vortex interactions

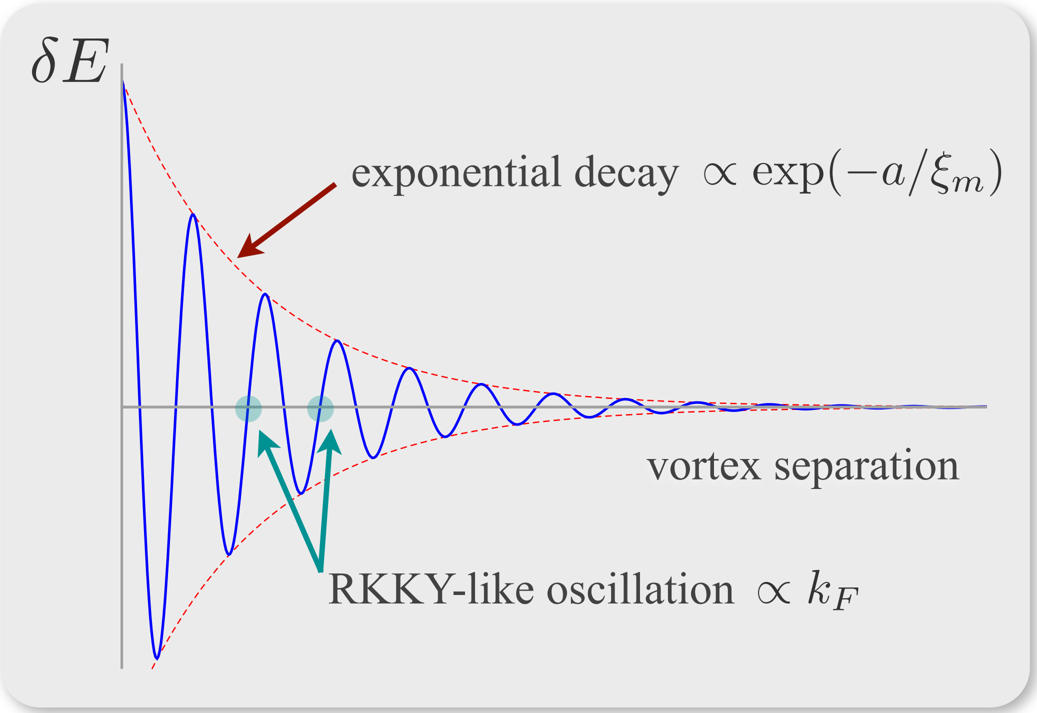
$a \gg \xi_m$ $\qquad$ $a \approx \xi_m$

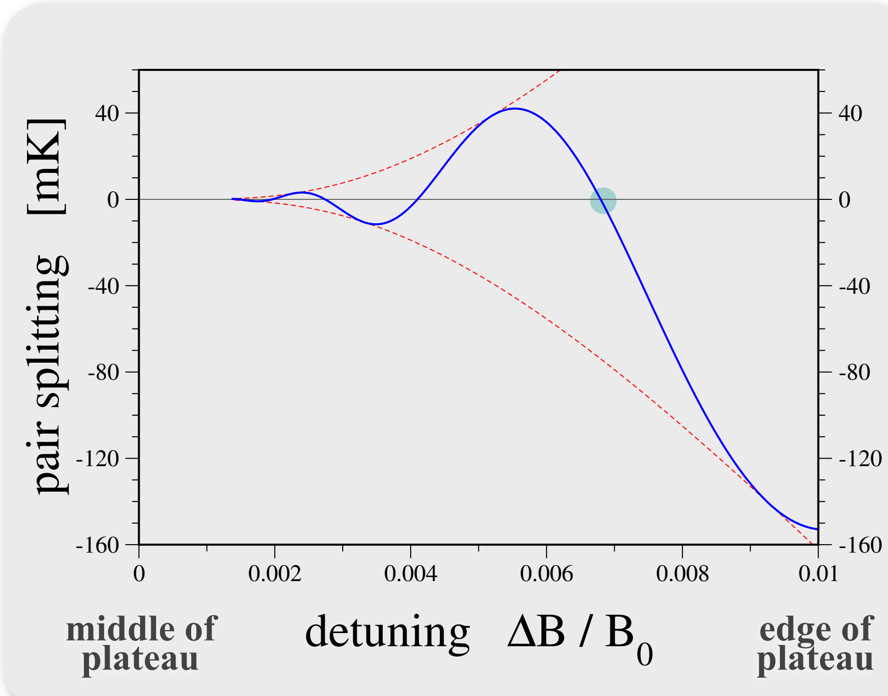


# Vortex-vortex interactions



# Vortex-vortex interactions

**Vortex quantum numbers**

SU(2)$_k$ = 'deformation' of SU(2)

with finite set of representations

$$0, \frac{1}{2}, 1, \frac{3}{2}, 2, \ldots, \frac{k}{2}$$

fusion rules

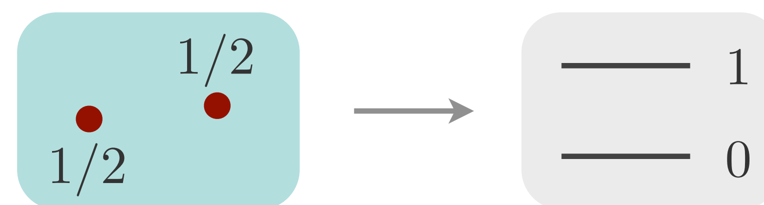
$$j_1 \times j_2 = |j_1 - j_2| + (|j_1 - j_2| + 1) + \ldots + \min(j_1 + j_2, k - j_1 - j_2)$$

example k = 2

$$1/2 \times 1/2 = 0 + 1$$

**Vortex pair**

$$1/2 \times 1/2 = 0 + 1$$




**Energetics for many vortices**

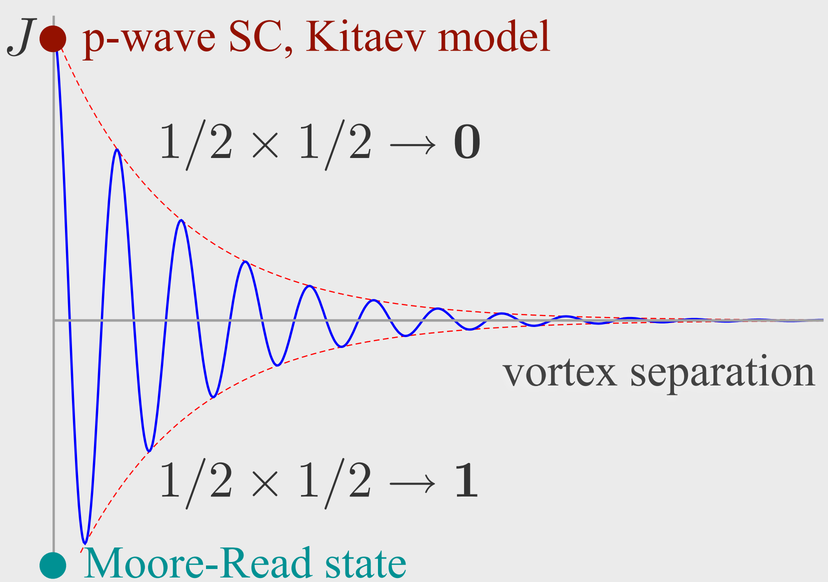
$$H = J \sum_{\langle ij \rangle} \prod_{ij}^{0}$$

"Heisenberg Hamiltonian"
for vortices

# Vortex-vortex interactions

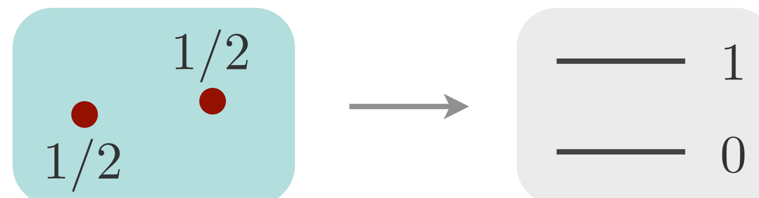
### Microscopics

Which channel is favored is
not universal, but microscopic detail.




### Vortex pair

$$1/2 \times 1/2 = 0 + 1$$

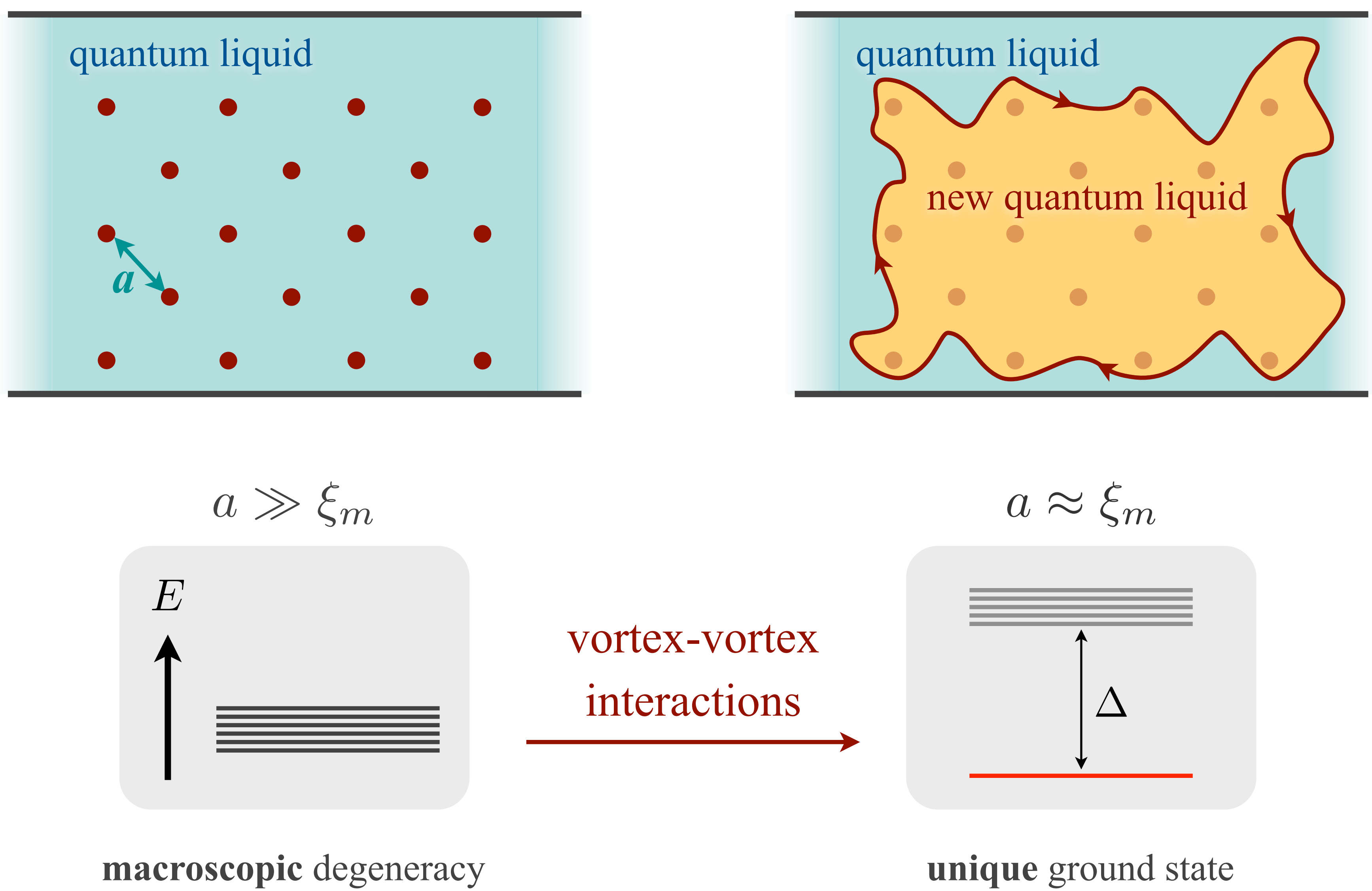



### Energetics for many vortices

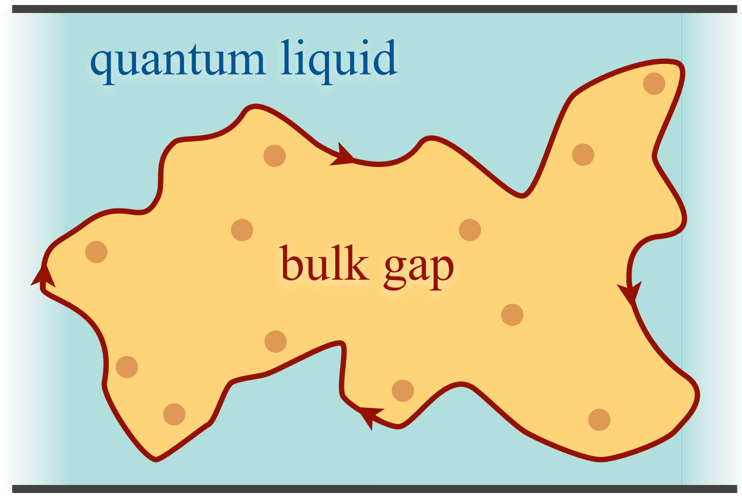
$$H = J \sum_{\langle ij \rangle} \prod_{ij}^{0}$$

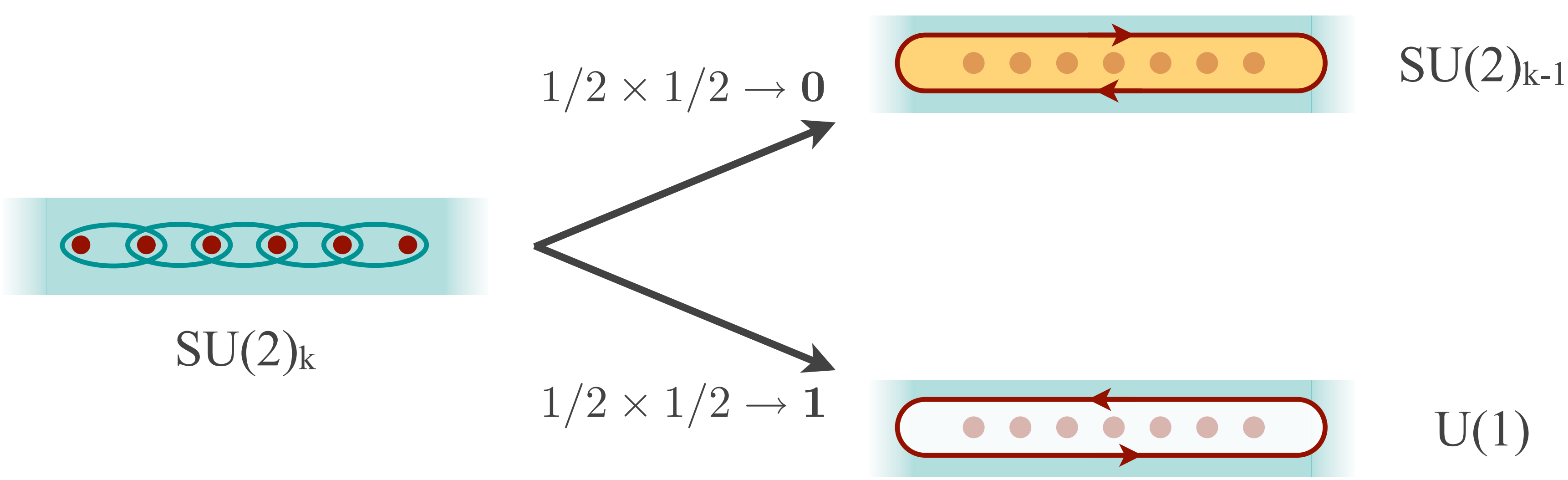
"Heisenberg Hamiltonian"
for vortices

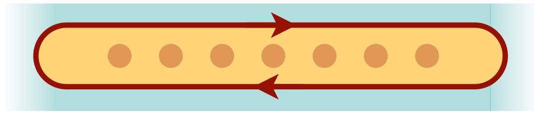
# The many-vortex problem



# The collective state



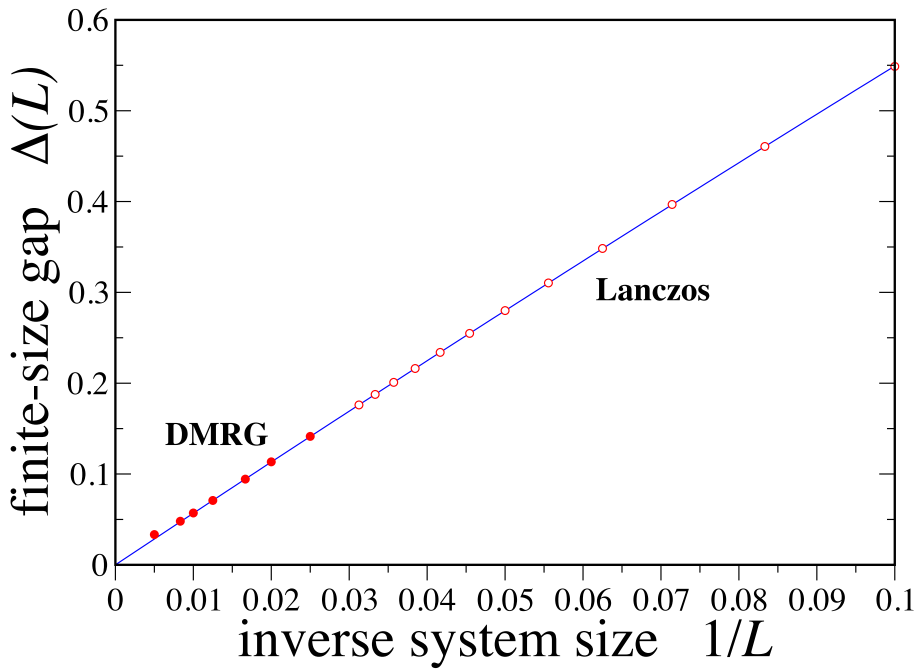


# Edge states

## Finite-size gap

$$\Delta(L) \propto (1/L)^{\boxed{z=1}}$$

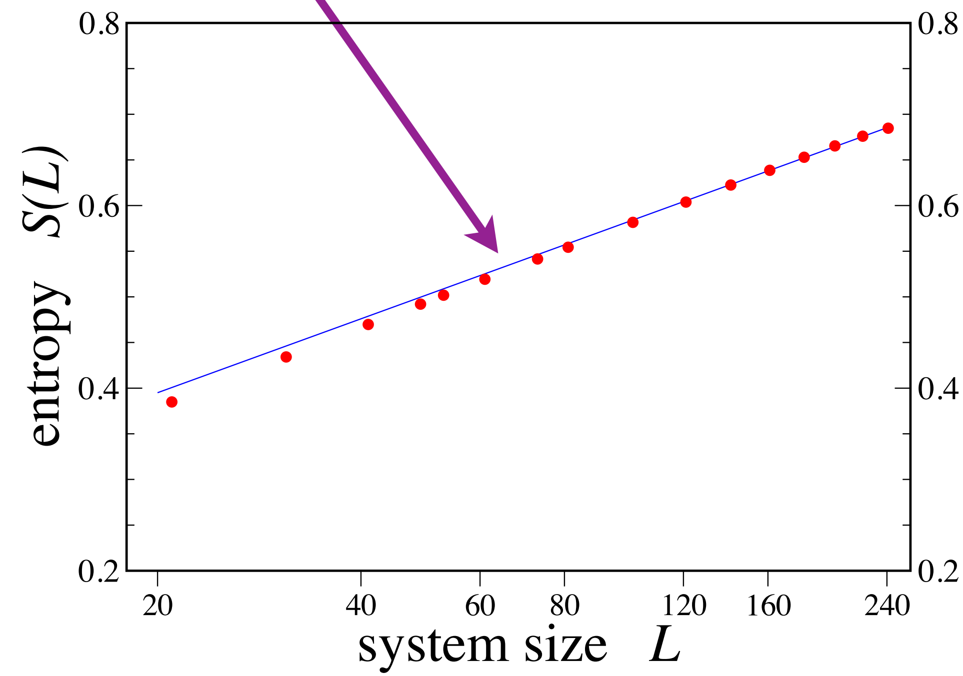
conformal field theory description




## Entanglement entropy

$$S(L) \propto \frac{\boxed{c}}{3} \log L$$

central charge
$c = 7/10$



# Mapping & exact solution

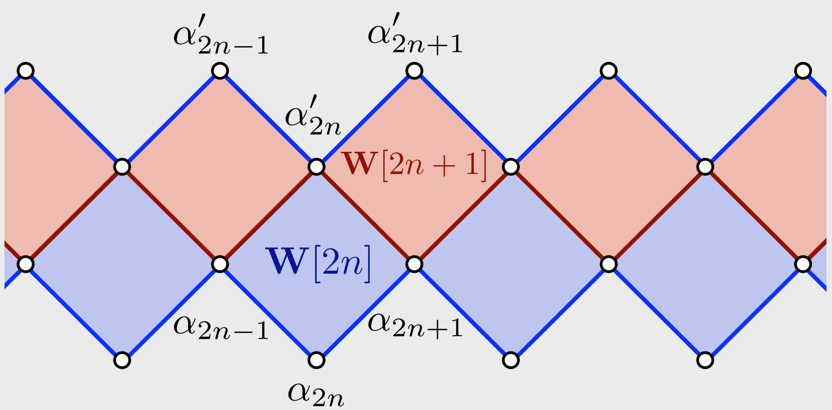
The operators $X_i = -d\, H_i$ form a representation of the **Temperley-Lieb algebra**

$$(X_i)^2 = \boxed{d} \cdot X_i \qquad X_i X_{i\pm1} X_i = X_i \qquad [X_i, X_j] = 0$$

for $|i-j| \geq 2$

$$d = 2\cos\left(\frac{\pi}{k+2}\right)$$

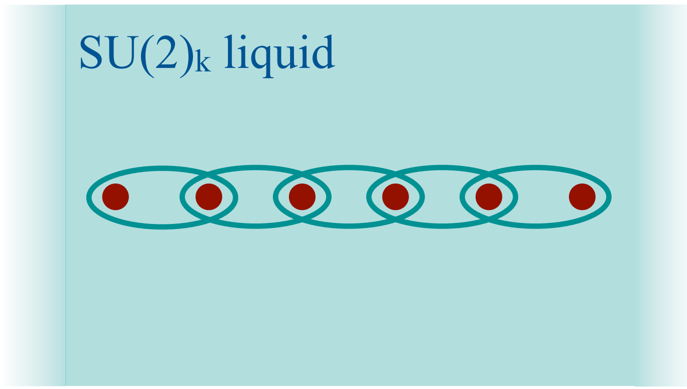
The transfer matrix is an integrable representation of the RSOS model.



# Gapless theories

| level $k$ | $1/2 \times 1/2 \to \mathbf{0}$ 'antiferromagnetic' | $1/2 \times 1/2 \to \mathbf{1}$ 'ferromagnetic' |
|---|---|---|
| 2 | Ising<br>c = 1/2 | Ising<br>c = 1/2 |
| 3 | tricritical Ising<br>c = 7/10 | 3-state Potts<br>c = 4/5 |
| 4 | $\frac{SU(2)_{k-1} \times SU(2)_1}{SU(2)_k}$ | $\frac{SU(2)_k}{U(1)}$ |
| 5 | | |
| k | **k-critical Ising**<br>c = 1-6/(k+1)(k+2) | **$Z_k$-parafermions**<br>c = 2(k-1)/(k+2) |
| $\infty$ | Heisenberg AFM<br>c = 1 | Heisenberg FM<br>c = 2 |

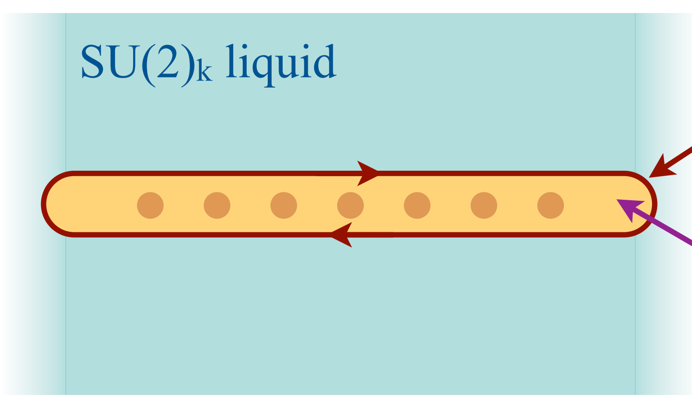
# Gapless modes & edge states




critical theory
(AFM couplings)

$$\frac{SU(2)_{k-1} \times SU(2)_1}{SU(2)_k}$$

interactions




gapless modes = edge states

$$\frac{SU(2)_{k-1} \times SU(2)_1}{SU(2)_k}$$

nucleated liquid

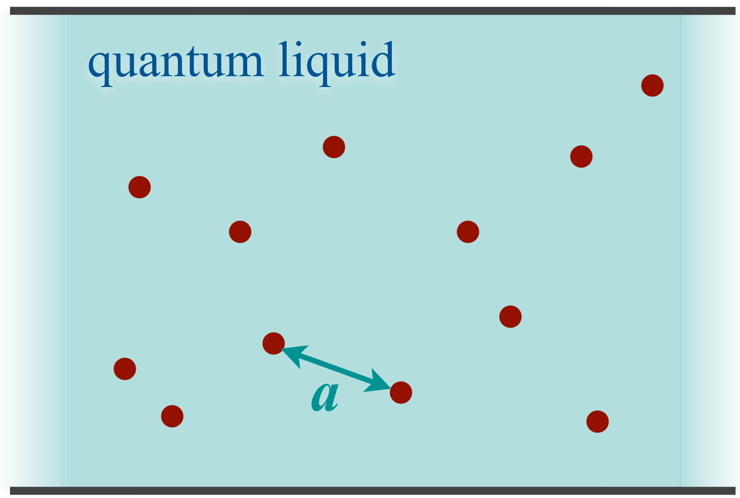
$$SU(2)_{k-1} \times SU(2)_1$$

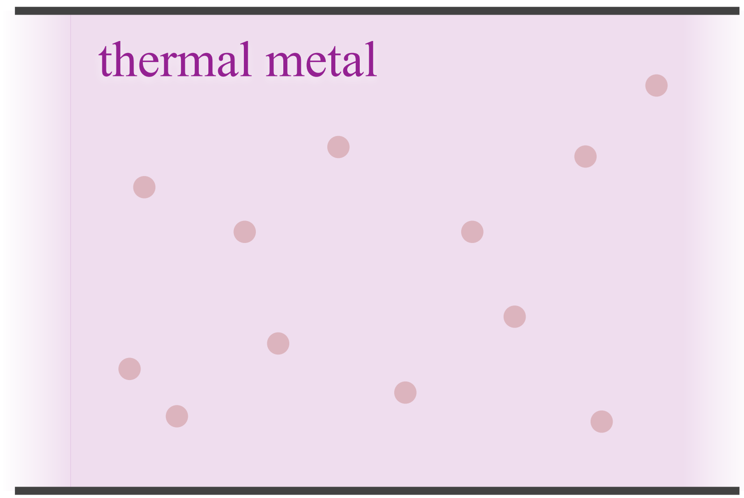
# Interactions + disorder

Work done with
**Chris Laumann** (Harvard)
**David Huse** (Princeton)
**Andreas Ludwig** (UCSB)

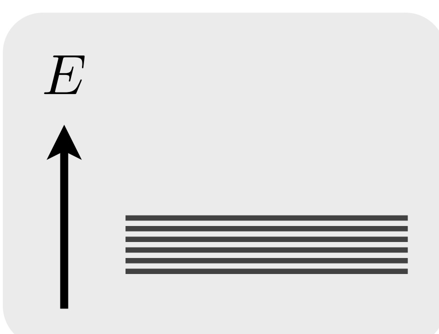
arXiv:1106.6265

# Disorder induced phase transition




$$a \gg \xi_m$$



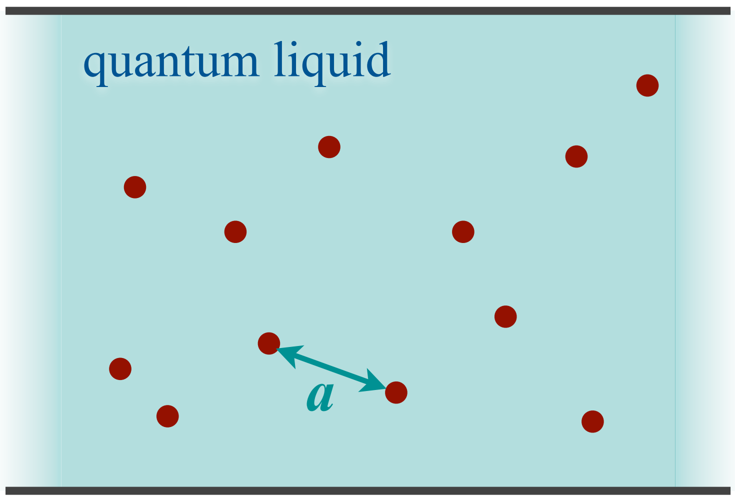

**macroscopic** degeneracy

disorder + vortex-vortex interactions

$$a \approx \xi_m$$

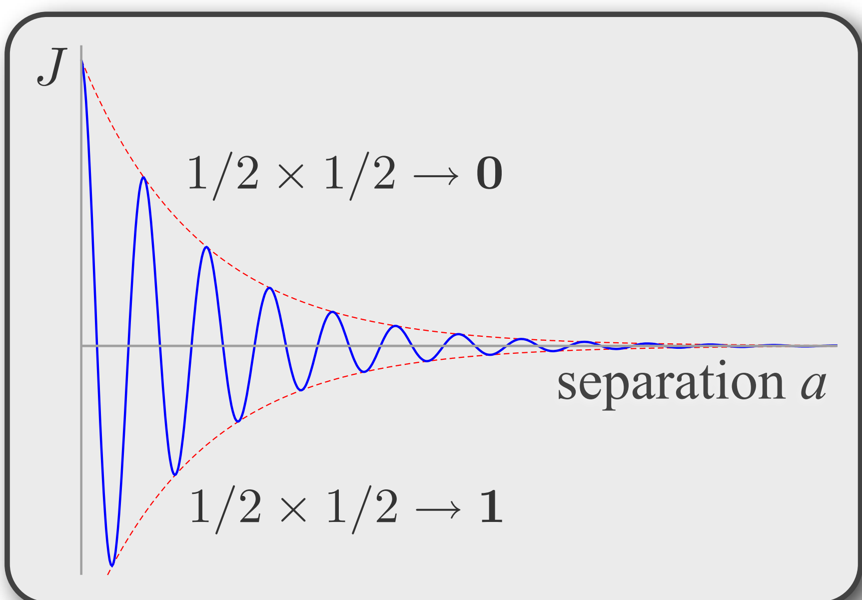
degeneracy is **split**

# Interactions and disorder




$$H = \sum_{\langle jk \rangle} J_{jk} \Pi_{jk}$$

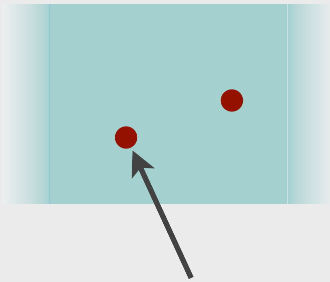
sign disorder
+ strong amplitude modulation




Natural analytical tool:
**strong-randomness RG**

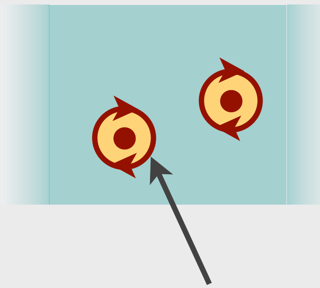
Unfortunately, this does not work.
The system flows *away* from strong disorder under the RG.
No infinite randomness fixed point.

# From Ising anyons to Majorana fermions



Ising anyon

$SU(2)_2$
quantum number



Majorana fermion
zero mode

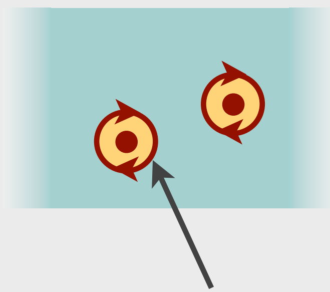
Majorana operator $\gamma_i$

$$H = \sum_{\langle jk \rangle} J_{jk} \Pi_{jk} \longrightarrow \mathcal{H} = -\sum_{\langle jk \rangle} i \mathcal{J}_{jk} \gamma_j \gamma_k$$

**interacting** Ising anyons
"anyonic Heisenberg model"

**free** Majorana fermion
hopping model

# From Ising anyons to Majorana fermions



Majorana fermion
zero mode

Majorana operator $\gamma_i$

Majorana operators

$$\{\gamma_i, \gamma_j\} = \delta_{ij}$$

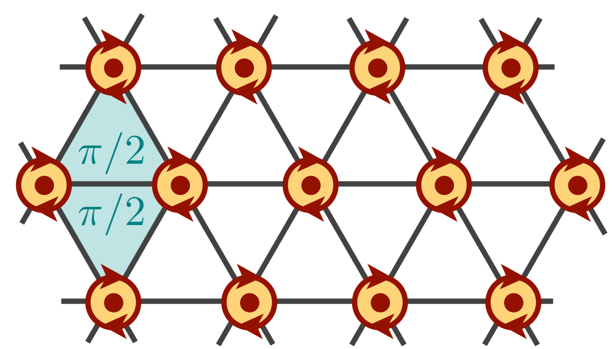
$$\gamma_i^\dagger = \gamma_i$$

$$\gamma_{i1} = (c_i^\dagger + c_i)/2$$

$$\gamma_{i2} = (c_i^\dagger - c_i)/2i$$

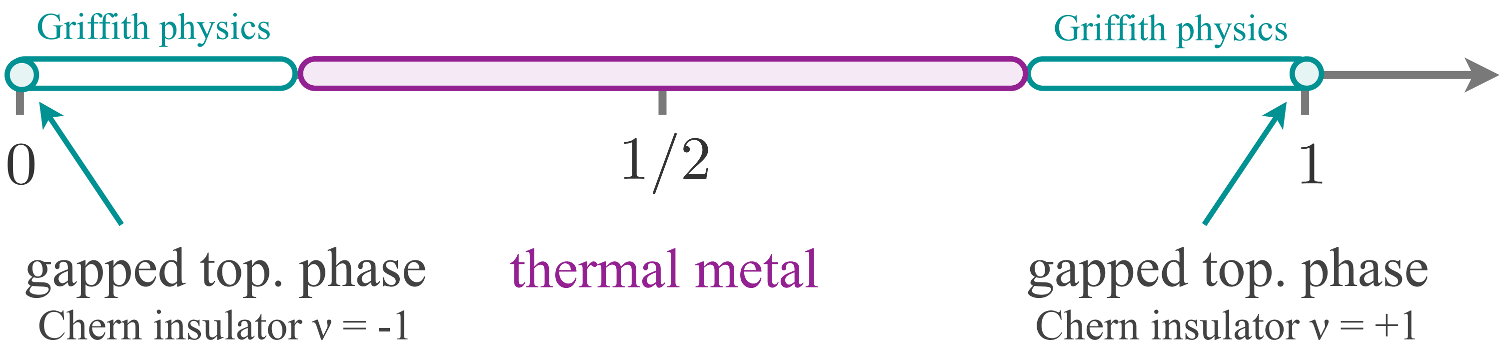
$$\mathcal{H} = -\sum_{\langle jk \rangle} i\mathcal{J}_{jk}\gamma_j\gamma_k$$

**free** Majorana fermion
hopping model

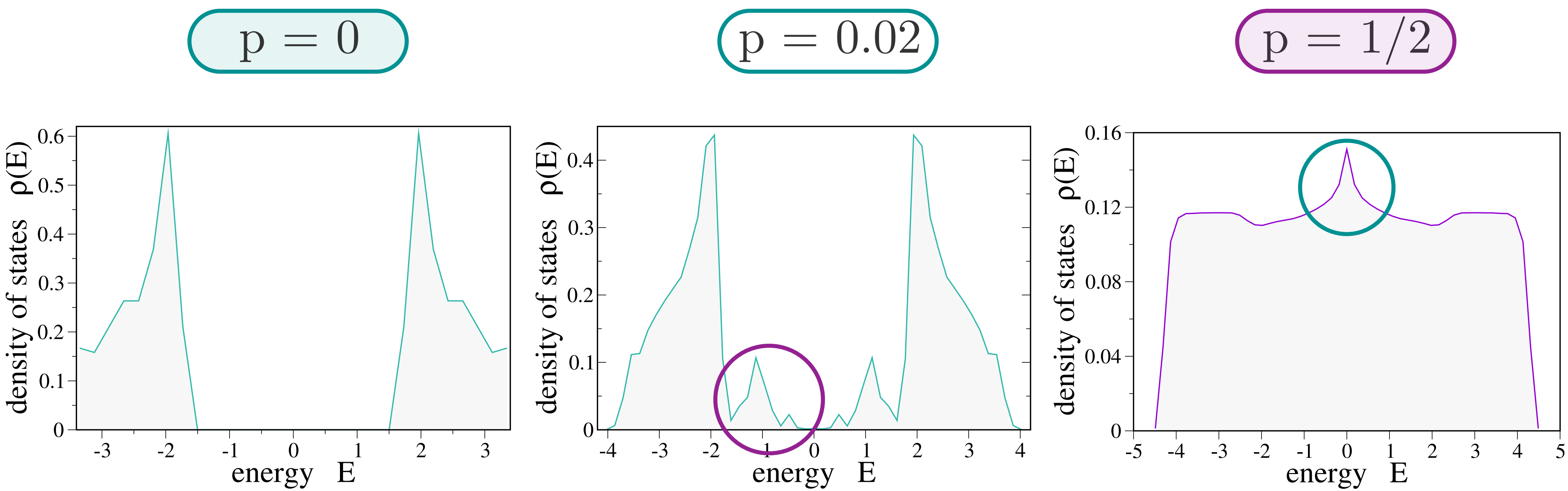



$$\left.\begin{array}{l}\text{particle-hole symmetry } \checkmark \\ \text{time-reversal symmetry } ✗\end{array}\right\} \begin{array}{c}\text{symmetry} \\ \text{class D}\end{array}$$

# A disorder-driven metal-insulator transition

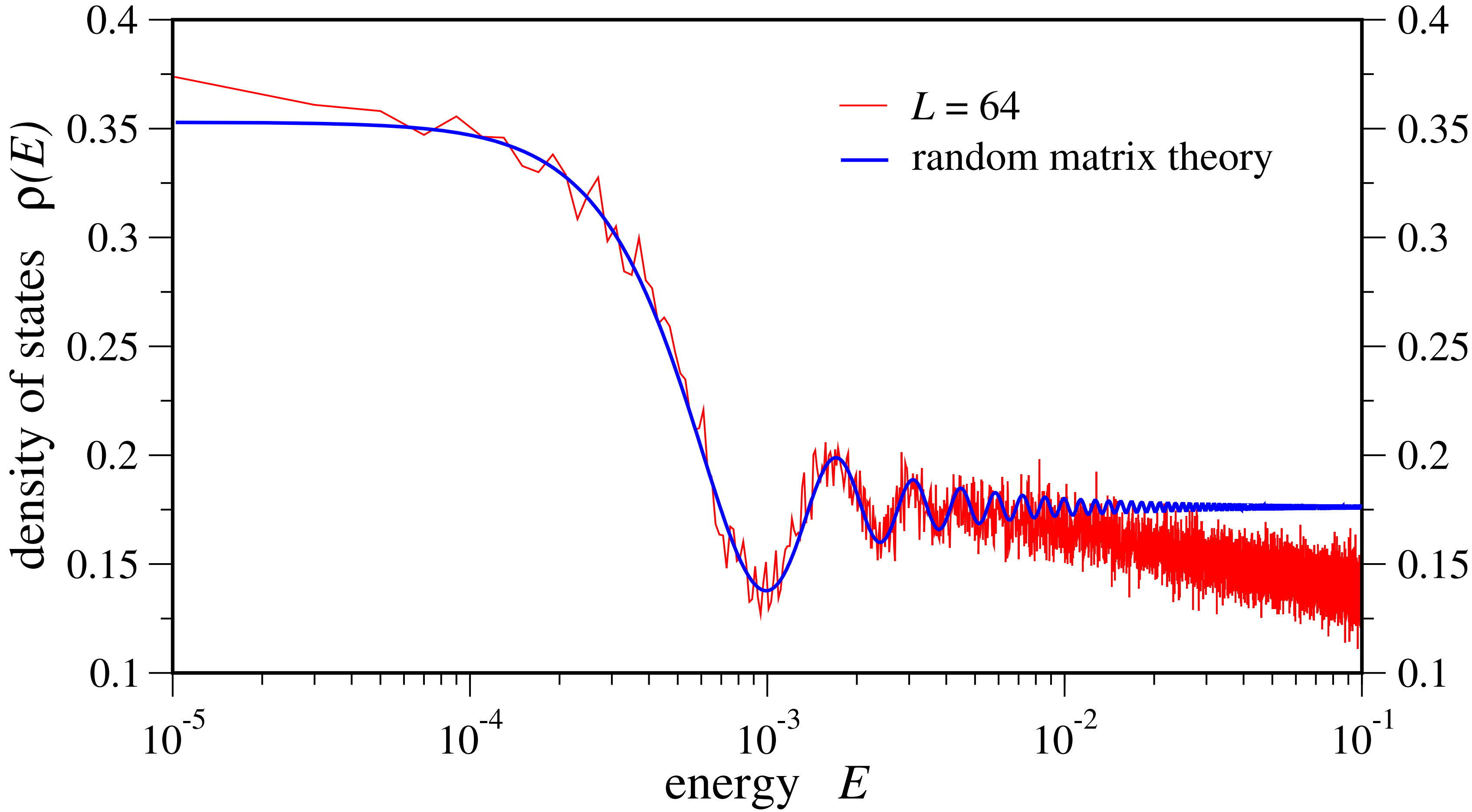



$$J = \begin{cases} +1, & p \\ -1, & 1-p \end{cases}$$




Density of states indicates phase transition.

# Density of states

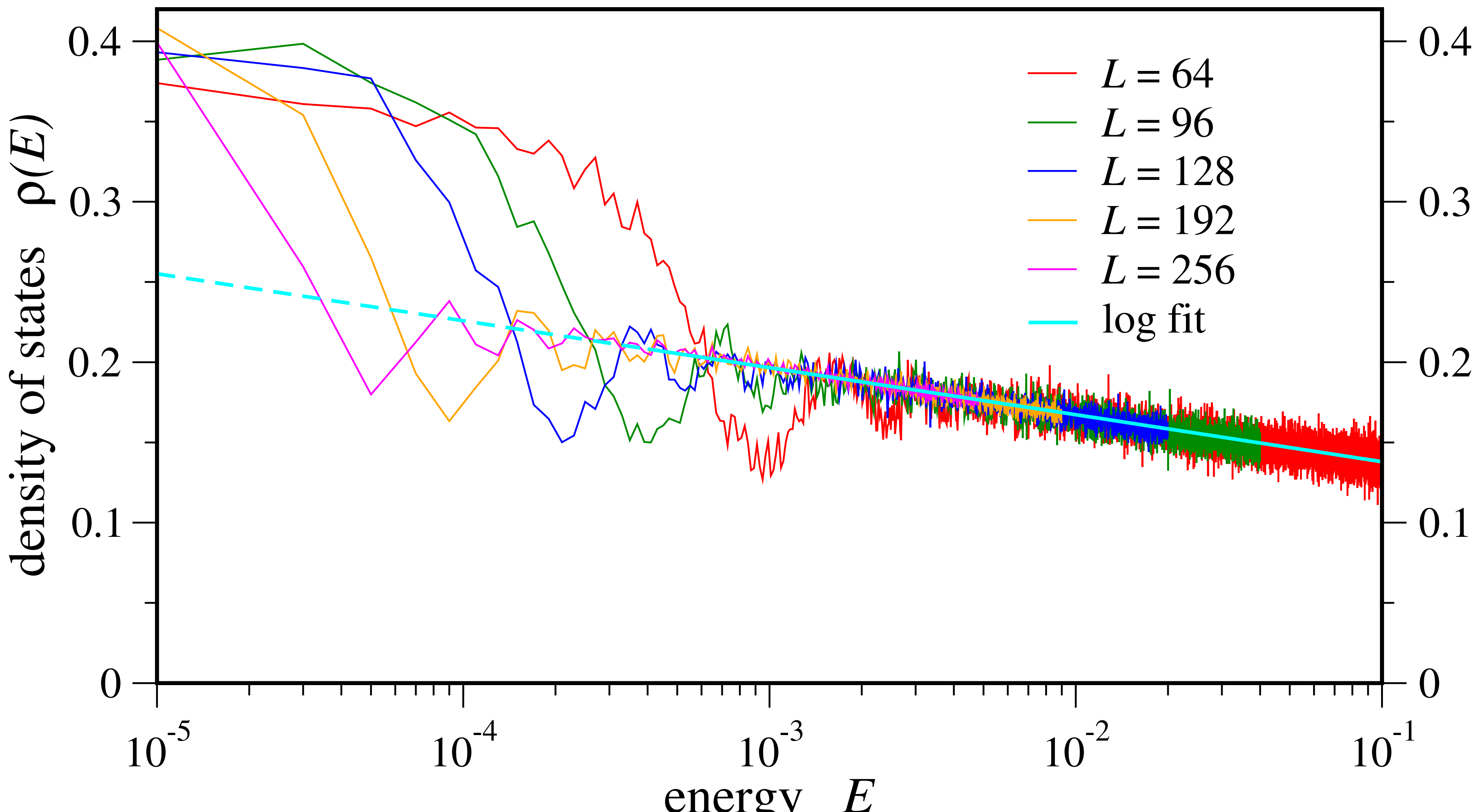



**Oscillations** in the DOS fit the prediction
from **random matrix theory** for symmetry class D

$$\rho(E) = \alpha + \frac{\sin(2\pi\alpha E L^2)}{2\pi E L^2}$$

# The thermal metal

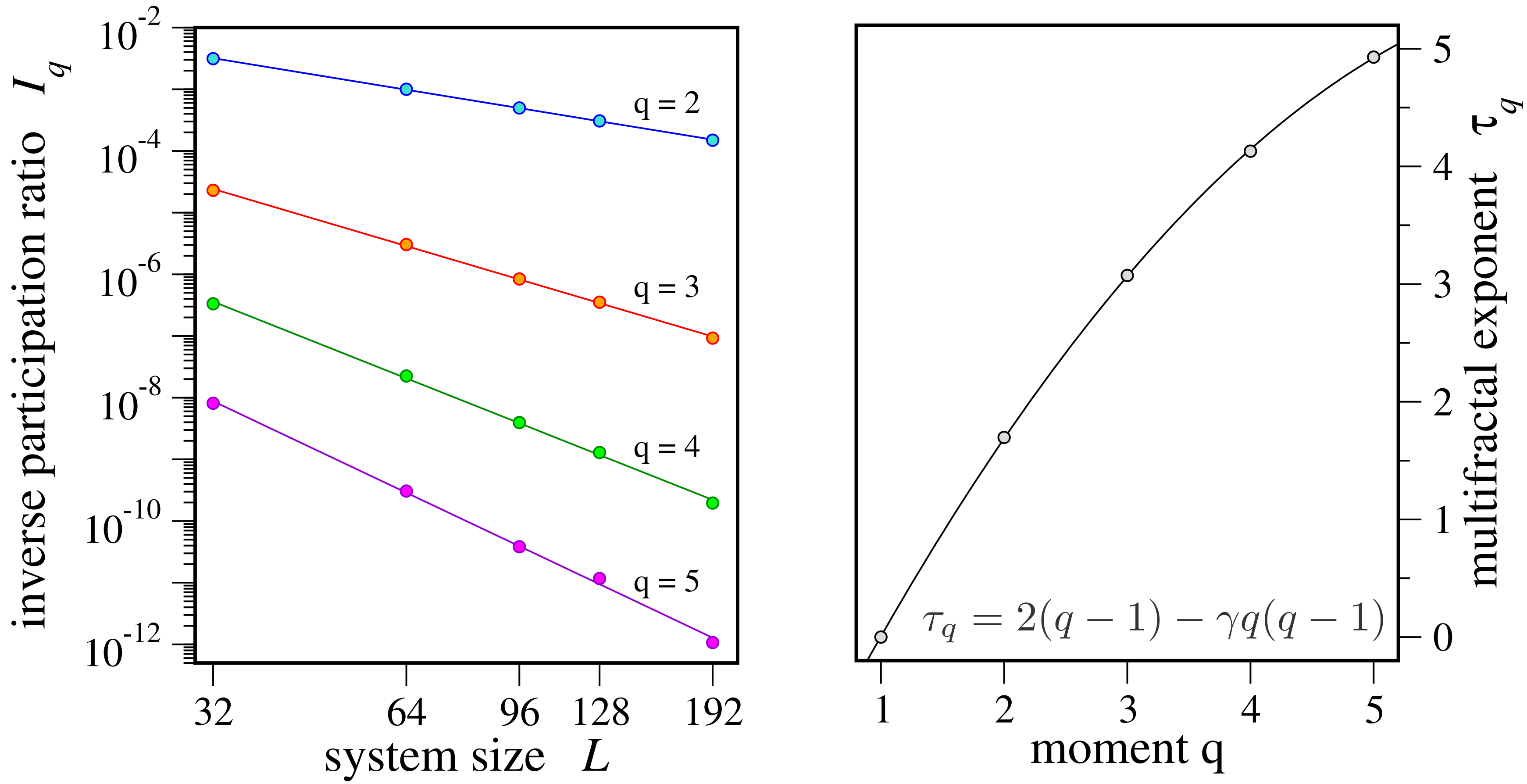
Density of states **diverges** logarithmically at zero energy.



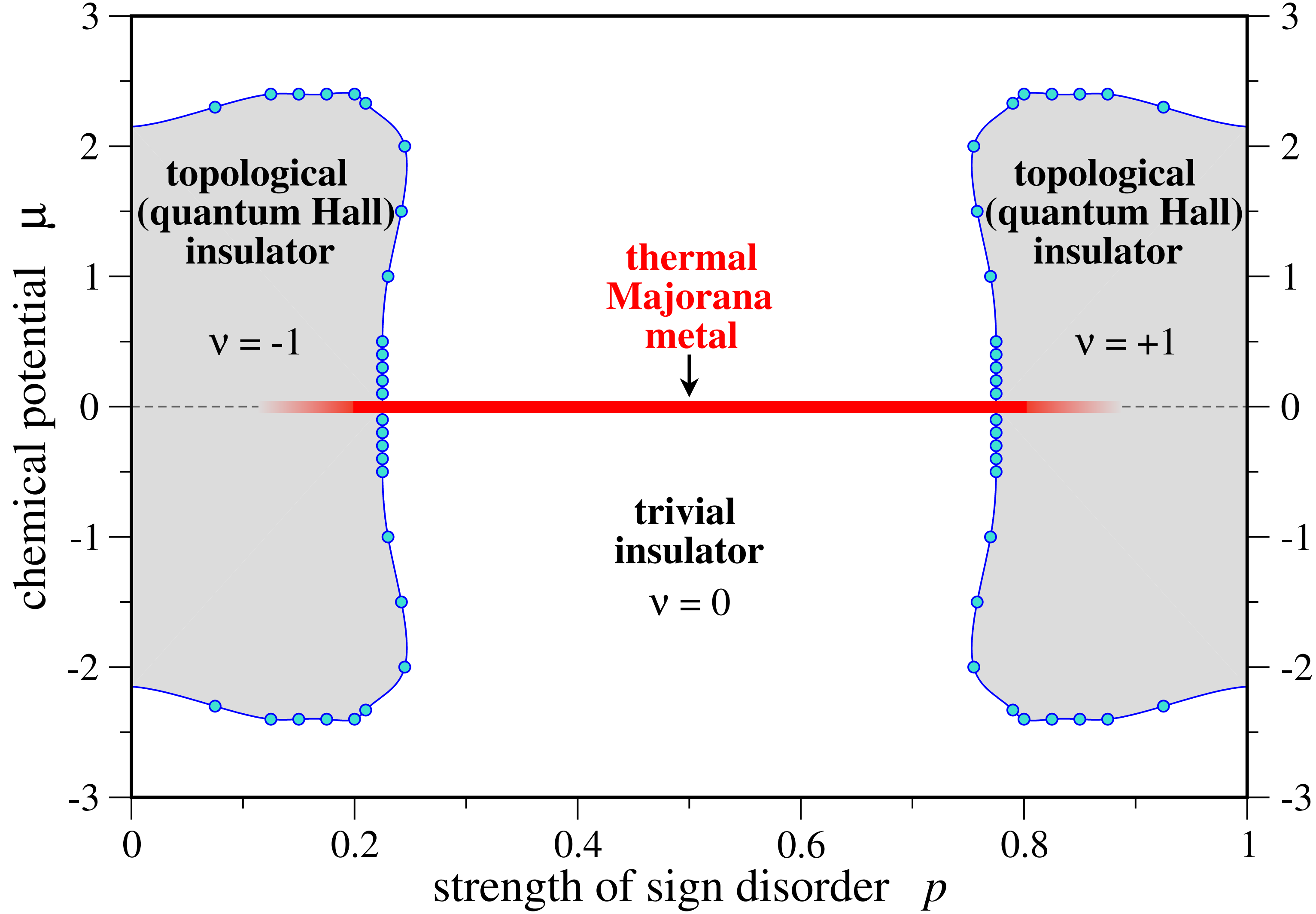
# The thermal metal

**Inverse participation ratios** (moments of the GS wavefunction) indicate **multifractal** structure characteristic of a metallic state.

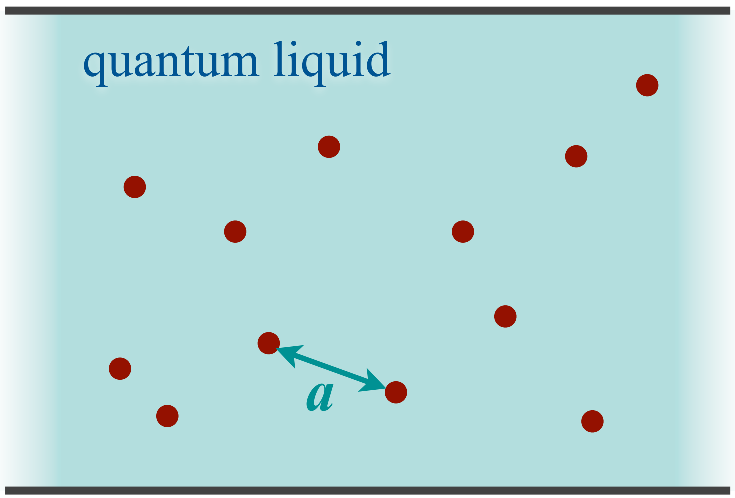
$$I_q = \int d^2r \, |\psi(r)|^{2q} \sim \frac{1}{L^{\tau_q}}$$

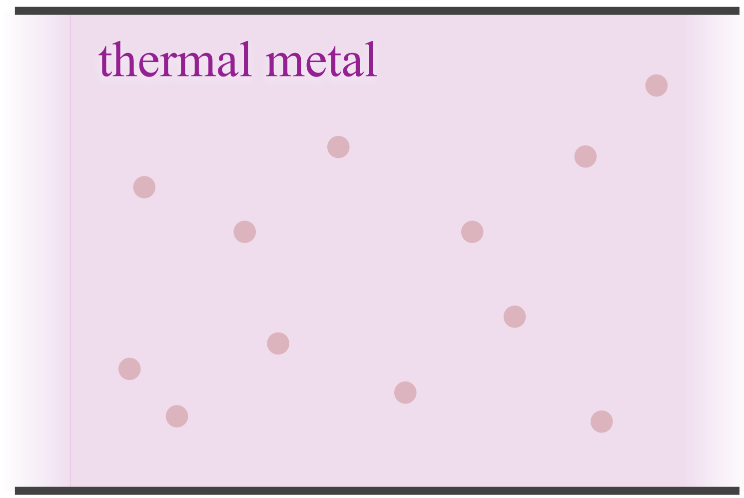


# A disorder-driven metal-insulator transition

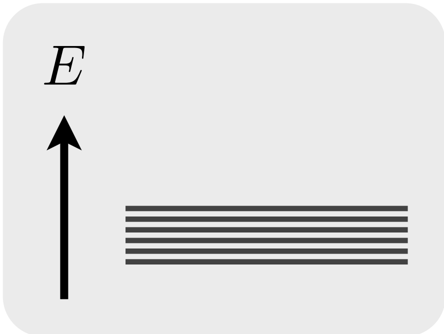


# Disorder induced phase transition




$a \gg \xi_m$




**macroscopic** degeneracy

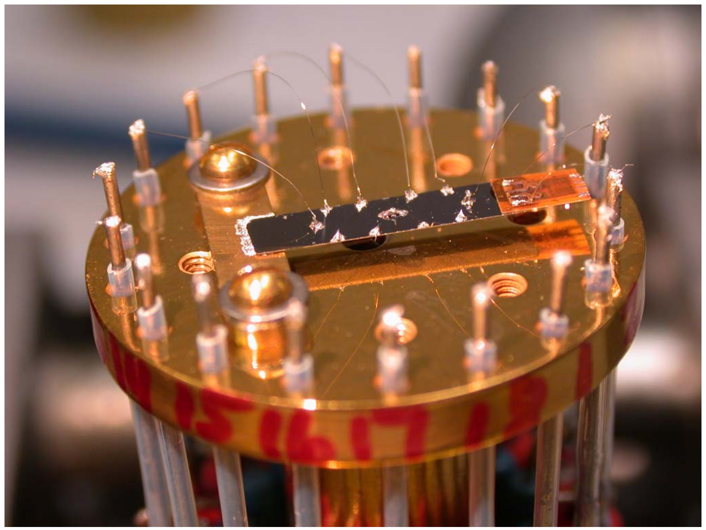
disorder +
vortex-vortex
interactions

$a \approx \xi_m$

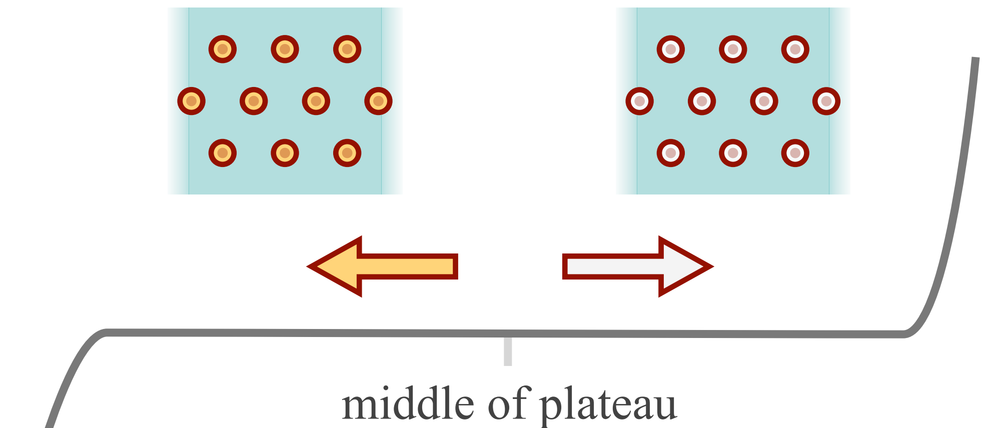
degeneracy is **split**

# Experimental consequences

# Heat transport



*Caltech thermopower experiment*

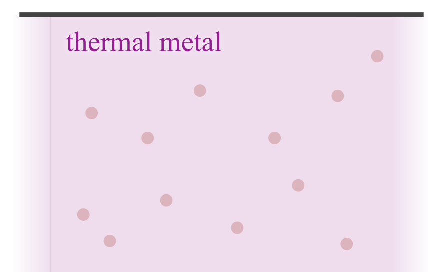




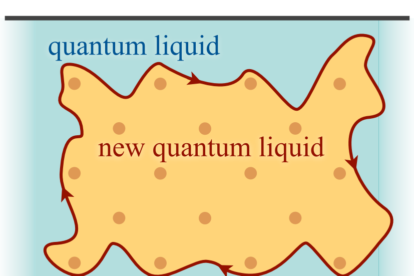
**electrical** transport remains unchanged

**Bulk heat** transport
diverges logarithmically
as T → 0.

$$\kappa_{xx}/T \propto \log T$$









**Heat** transport
along the sample **edges**
changes quantitatively

# Collective states – a good thing?

The interaction induced splitting
of the degenerate manifold = qubit states
is yet another obstacle to overcome.

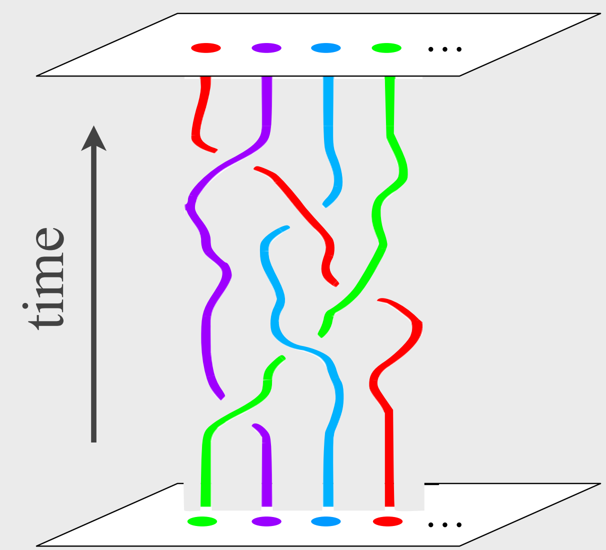
The formation of collective states renders
all ideas of manipulating individual anyons
inapplicable.

Probably, a topological quantum computer
works best at **finite** temperatures.

**Topological quantum computing**

Degenerate manifold = qubit

Employ **braiding** of non-Abelian
vortices to perform computing
(unitary transformations).



# Summary

- Lord Kelvin was way ahead of his time.
- Topology has re-entered physics in many ways.
- Topological excitations + interactions + disorder can give rise to a plethora of collective phenomena.
  - Topological liquid nucleation
  - Thermal metal
  - Distinct experimental bulk observable (heat transport) in search for Majorana fermions.