Breakdown of a topological phase

Quantum phase transition(s) in a loop gas with tension

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Outline

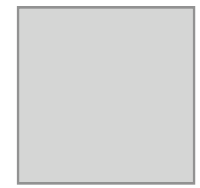
- **Topological phases**
  - Fault-tolerant quantum computing
  - A simple, soluble spin model - the toric code
- **Breakdown of topological order**
  - Perturbing the toric code around soluble point
  - Numerical simulations
- **Outlook**

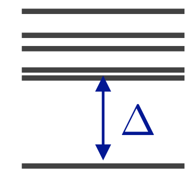
Introduction

- **Spontaneous symmetry breaking**
  - ground state has **less** symmetry than Hamiltonian
  - Landau-Ginzburg-Wilson theory
  - **local** order parameter
- **Topological order**
  - ground state has **more** symmetry more than Hamiltonian
  - degenerate ground states
  - **non-local** order parameter

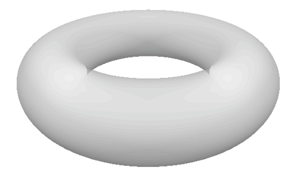
Topological quantum liquids

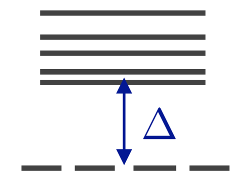
- Gapped spectrum
- **No** broken symmetry



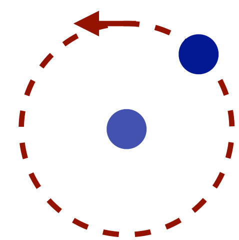


- Degenerate ground state on torus



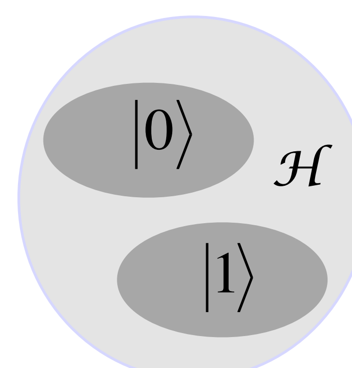


- Fractional statistics of excitations



$$e^{i\theta}$$

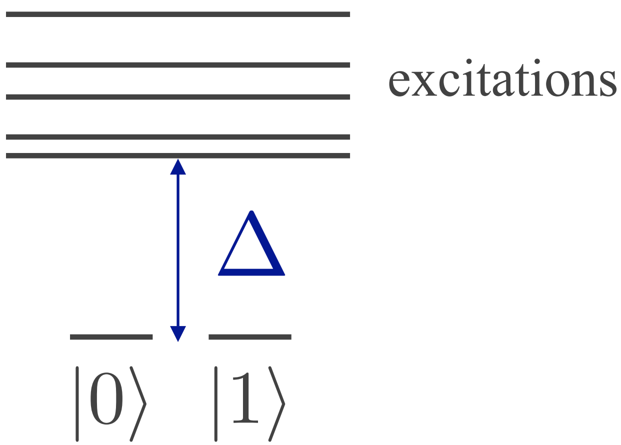
- Hilbert space split into topological sectors
  No **local** matrix element mixes the sectors



Fault-tolerant quantum computing

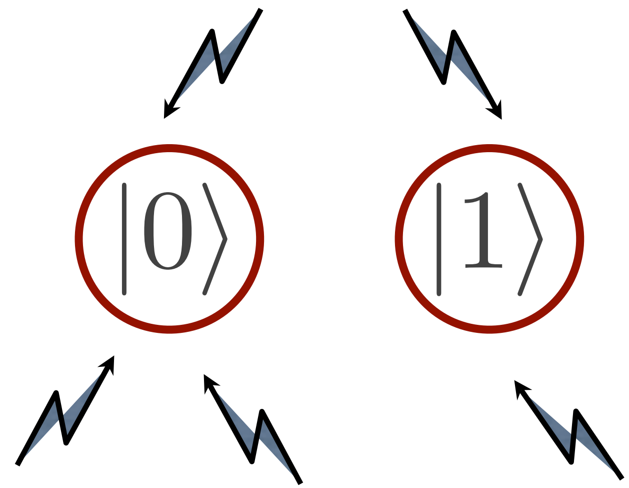
A. Kitaev, Ann. Phys. **303**, 2 (2003).

Idea: Use degenerate groundstates as qubits



Energy gap Δ
protects from
thermal excitations

Topological order makes it **robust** with respect to local perturbations



States are locally indistinguishable $\longrightarrow$ no phase errors

States do not couple $\longrightarrow$ no bit flip errors

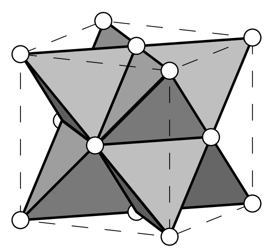
Liquid is required since environment couples to broken symmetries

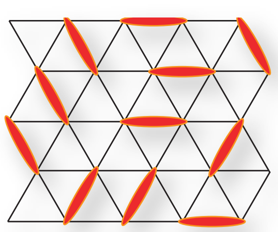
Example systems

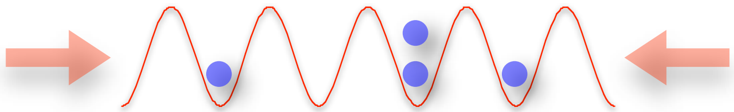
- **Experimental realization** of topological quantum liquid
  - Fractional quantum Hall effect (FQHE)
    but gapless edge states are a problem

- **First implementation proposal** in Josephson junction arrays

- Other **candidate systems**
  - Frustrated magnets
  - Quantum dimer models
  - Ultra-cold atoms







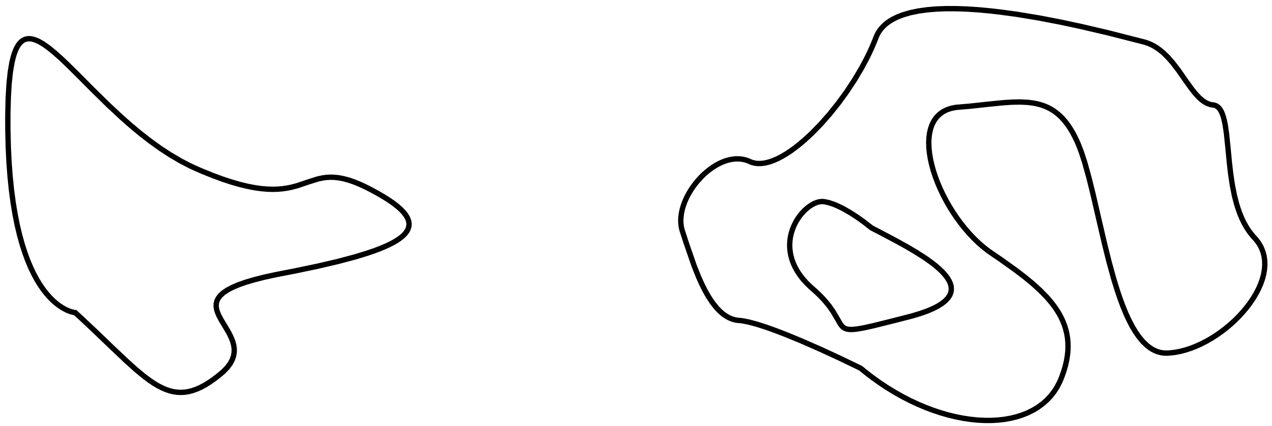
Stability of the topological phase

- Topological phases have **not** been **directly observed** in experiment (beyond FQHE), because
  - they are unstable?
  - exist only in small regions of phase space?
  - we miss the appropriate tools?

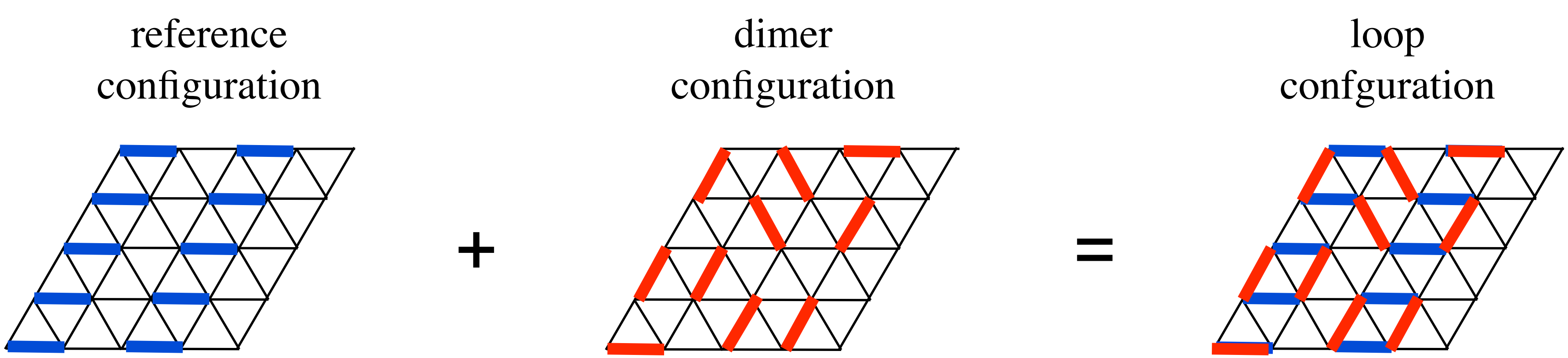
- We will look at the simplest model of an (abelian) topological phase: **the quantum loop gas**

Quantum loop gases

- are the "Ising models" of topological phases



- are hidden also in the quantum dimer model

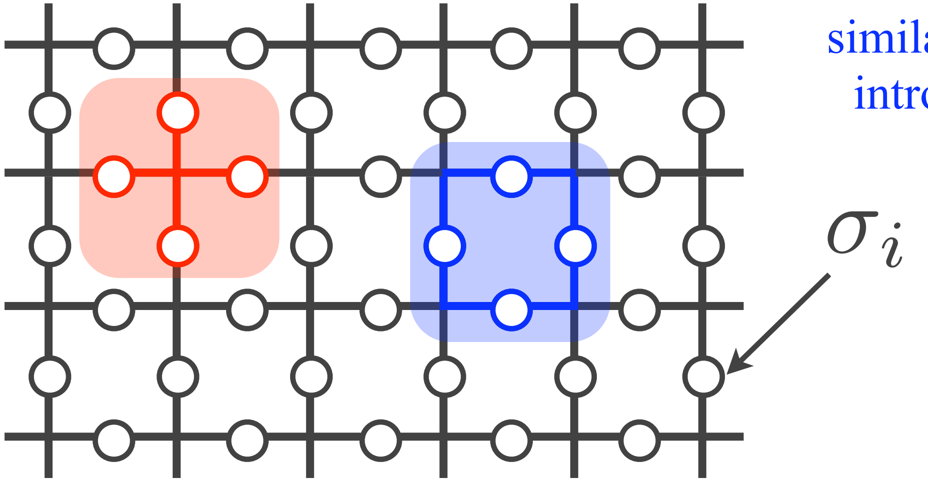


The toric code: the simplest loop gas

A. Kitaev, Ann. Phys. **303**, 2 (2003).

$$H_{\mathrm{TC}} = -A\sum_{v}\prod_{j\in\mathrm{vertex}(v)}\sigma_j^z - B\sum_{p}\prod_{j\in\mathrm{plaquette}(p)}\sigma_j^x$$

similar to ring exchange
introduces frustration



Hamiltonian has only local terms.

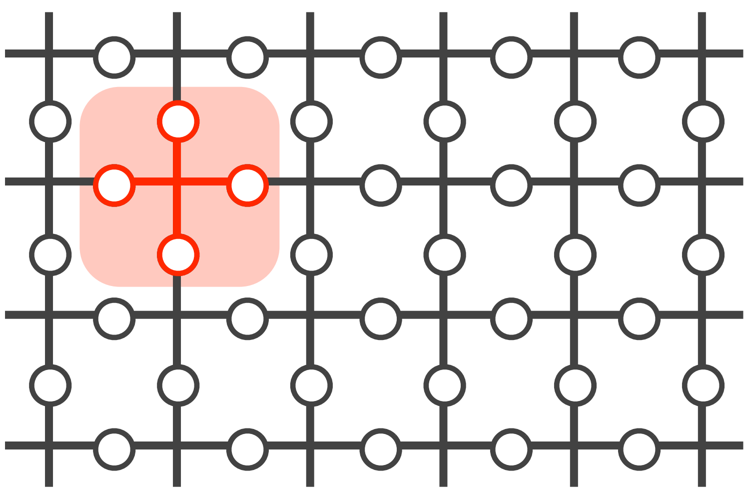
All terms commute → **exact solution!**

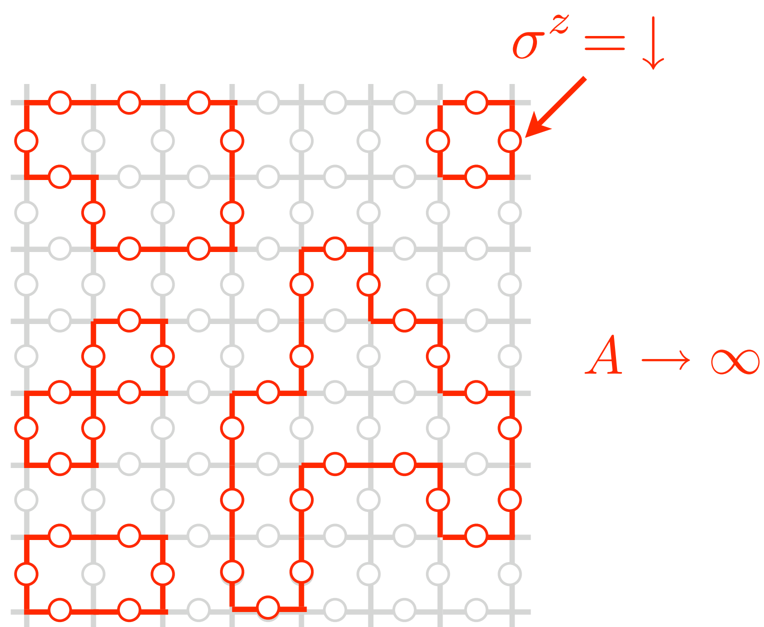
The vertex term

A. Kitaev, Ann. Phys. **303**, 2 (2003).

$$H_{\mathrm{TC}} = -A\sum_{v}\prod_{j\in\mathrm{vertex}(v)}\sigma_j^z - B\sum_{p}\prod_{j\in\mathrm{plaquette}(p)}\sigma_j^x$$

- is minimized by an **even** number of down-spins around a vertex.
- Replacing down-spins by loop segments maps ground state to closed loops.
- Open ends are (charge) excitations costing energy $2A$.



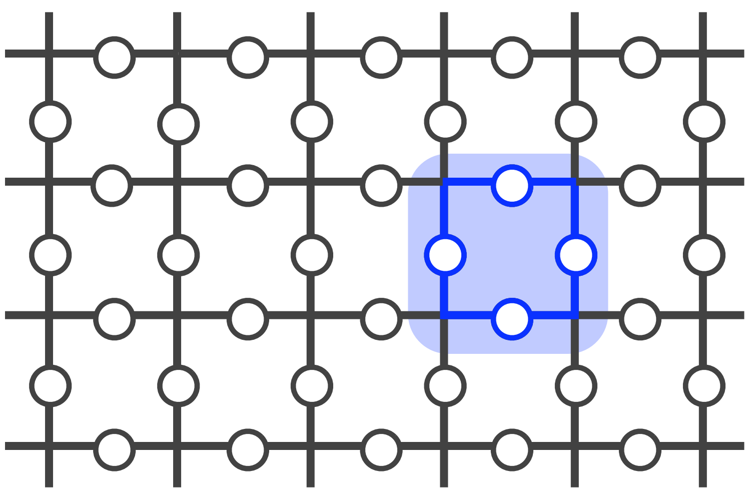


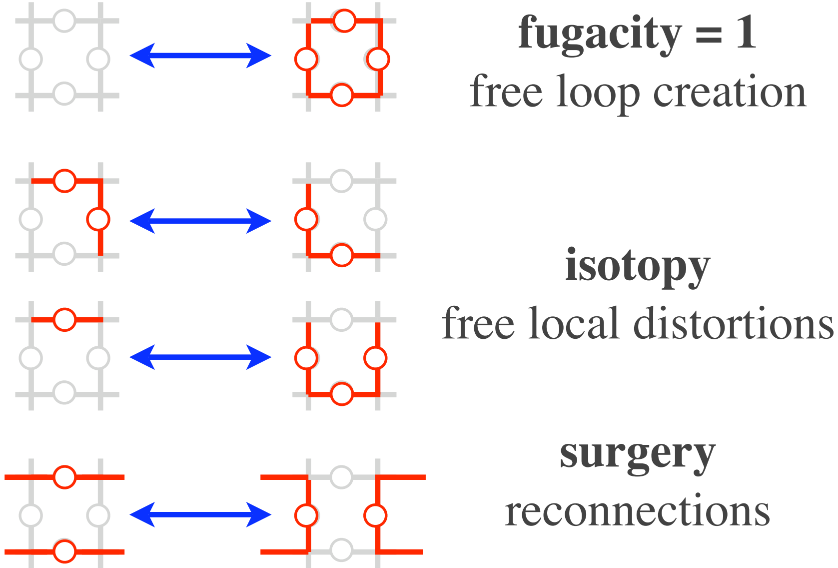
The plaquette term

A. Kitaev, Ann. Phys. **303**, 2 (2003).

$$H_{\rm TC} = -A\sum_v \prod_{j\in \text{vertex}(v)} \sigma_j^z - B\sum_p \prod_{j\in \text{plaquette}(p)} \sigma_j^x$$

- flips all spins on a plaquette.
- favors equal amplitude superposition of all loop configurations.
- Sign changes upon flip (vortices) cost energy $2B$.

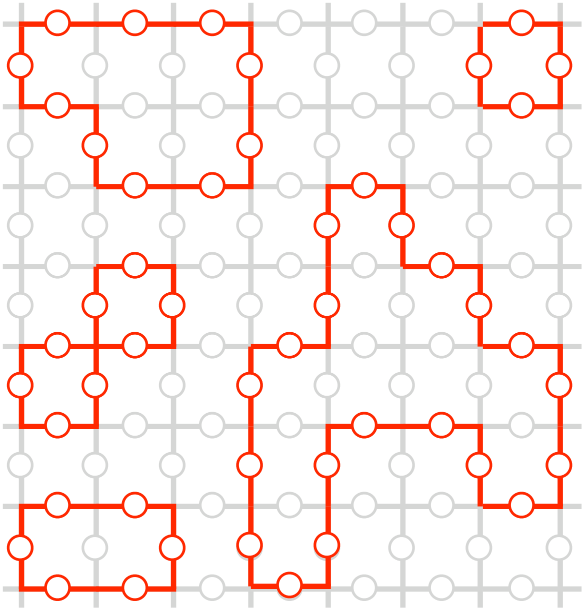


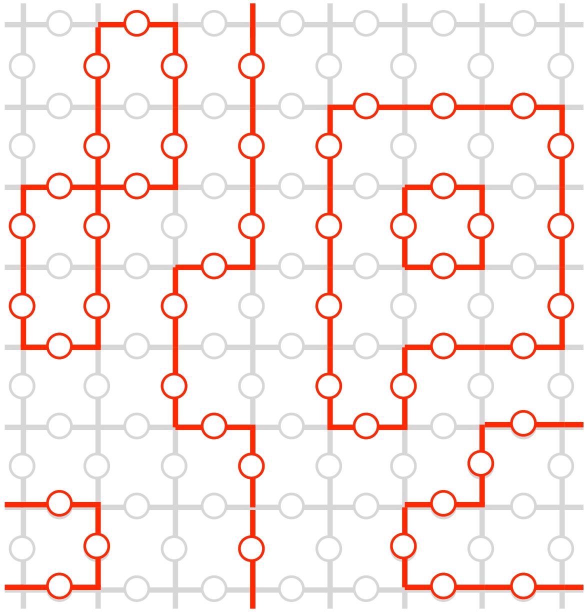


The toric code

A. Kitaev, Ann. Phys. **303**, 2 (2003).

Ground-state manifold is a **quantum loop gas**.





Ground-state wavefunction is **equal superposition** of loop configurations.

The toric code

A. Kitaev, Ann. Phys. **303**, 2 (2003).

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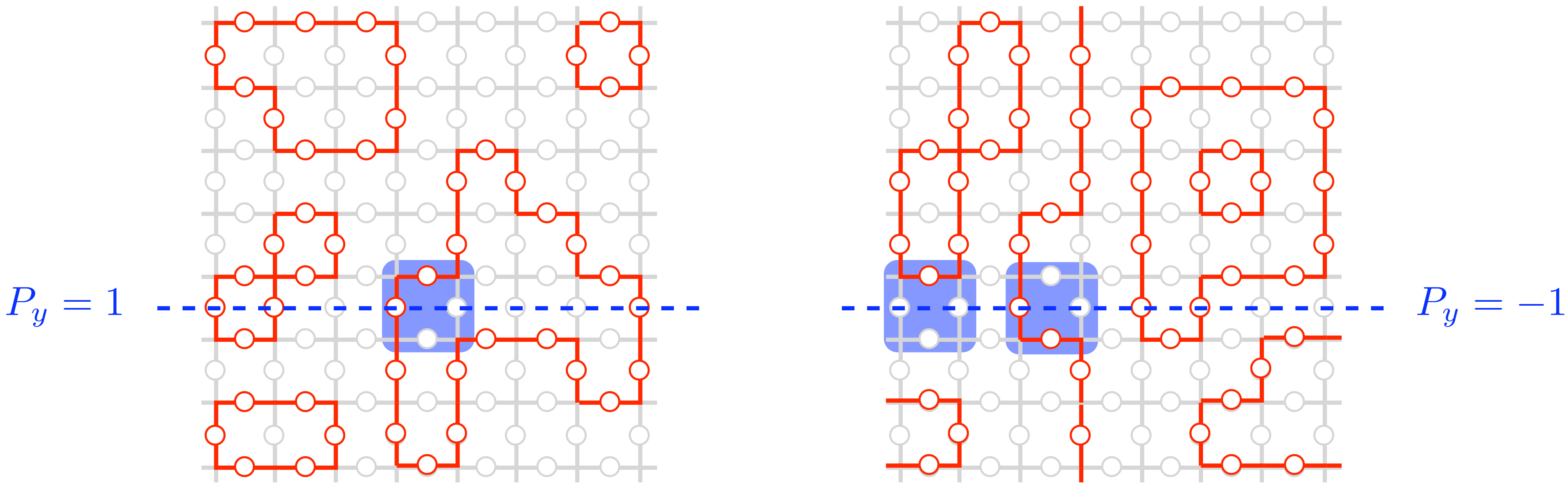

Topological sectors defined by winding number parity $P_{y/x} = \prod_{i \in c_{x/y}} \sigma_i^z$

The toric code

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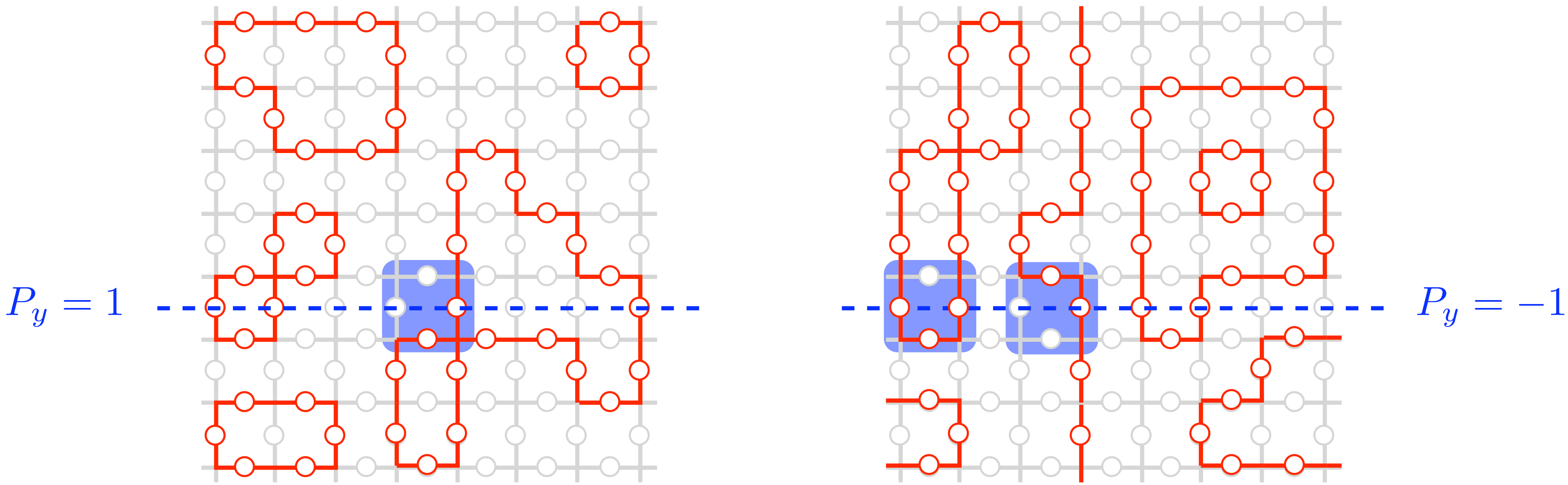


Topological sectors defined by winding number parity $P_{y/x} = \prod_{i \in c_{x/y}} \sigma_i^z$

Perturbing the toric code

$$H = H_{\mathrm{TC}} - h \sum_i \sigma_i^z - J \sum_{\langle ij \rangle} \sigma_i^z \sigma_j^z$$

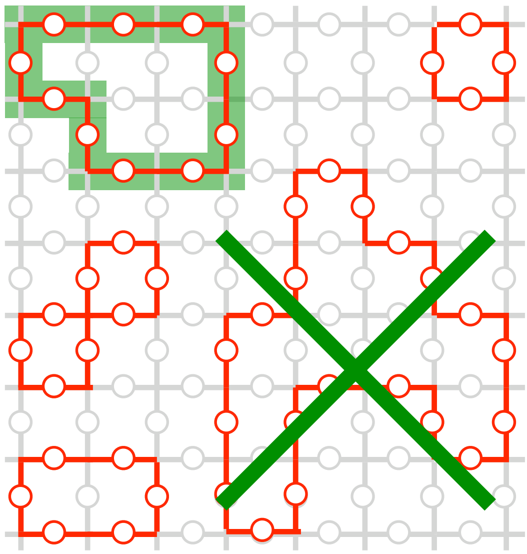
magnetic field

Ising interaction

local perturbations

chemical potential for loops

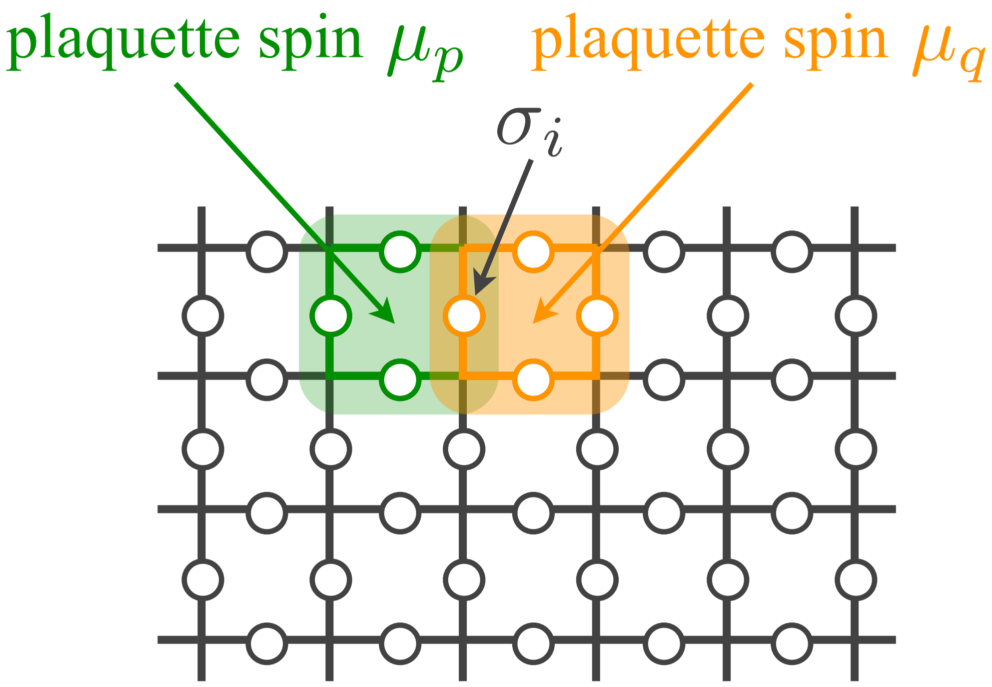
small loops are favored



Magnetic field / Ising interaction introduce **bare loop tension**.

Mapping to plaquette variables

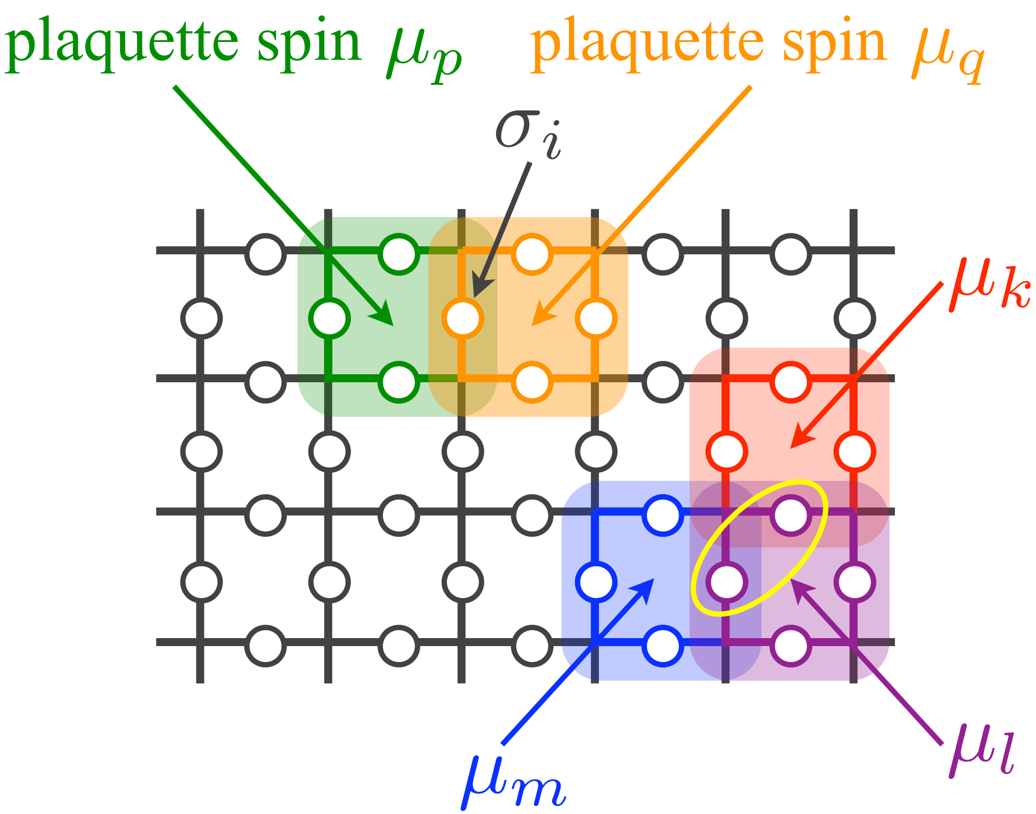
Express bond spins by product of two **plaquette spins.**



$$\sigma_i^z = \pm\frac{1}{2}\,\mu_p^z\,\mu_q^z$$

Choice of local signs is gauge-invariant
up to fixing the topological sector.

Mapping of interaction terms



$$\sigma_i^z = \frac{1}{2}\,\mu_p^z\,\mu_q^z$$

$$\sigma_i^z\sigma_j^z = \left(\frac{1}{2}\,\mu_k^z\,\mu_l^z\right)\left(\frac{1}{2}\,\mu_l^z\,\mu_m^z\right) = \frac{1}{4}\,\mu_k^z\,\mu_m^z$$

Both magnetic field and Ising interaction give **Ising coupling** between **plaquette spins**.

Mapping to 3D Ising model

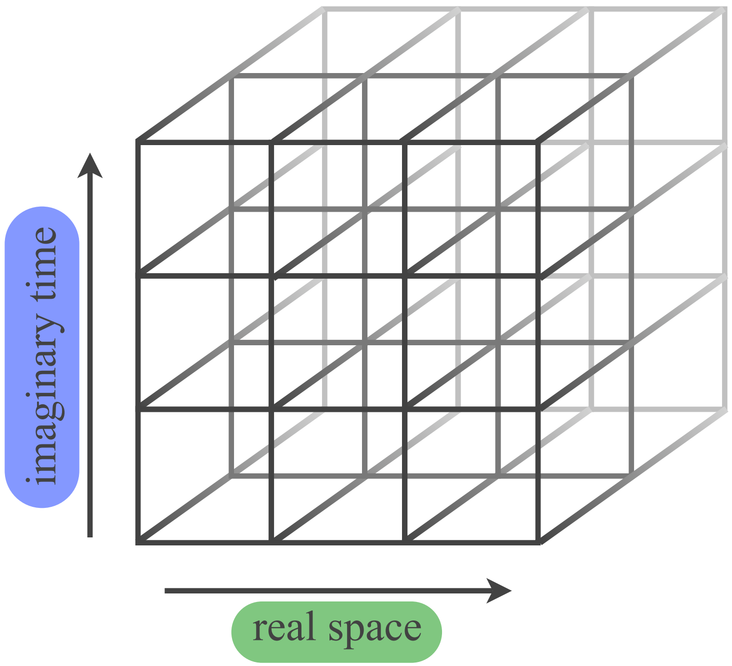
$$\mathcal{H}_{\rm cl} = -K_\tau \sum_{\tau,p} S_p(\tau) S_p(\tau + \Delta\tau) - K \sum_{\tau,\langle p,q\rangle} S_p(\tau) S_q(\tau)$$

$$K_\tau = -\frac{1}{2} \ln\left[\tanh(\Delta\tau \cdot B)\right]$$

plaquette flips

$$K = \frac{1}{2} \Delta\tau \cdot h$$

magnetic field



$$K_c = 0.2216595(26)$$

A. M. Ferrenberg and D. P. Landau,
Phys. Rev. B **44**, 5081 (1991).

$$B = 1, \quad K_\tau = K \quad \text{gives} \quad \Delta\tau = 0.761403, \quad h_c = 0.58224$$

→ numerical simulation

Lattice gauge theories

J. B. Kogut, Rev. Mod. Phys. **51**, 659 (1979).

Without dynamical electric charges ($A \to \infty$),
the toric code becomes equivalent to the **"even" Ising gauge theory**.

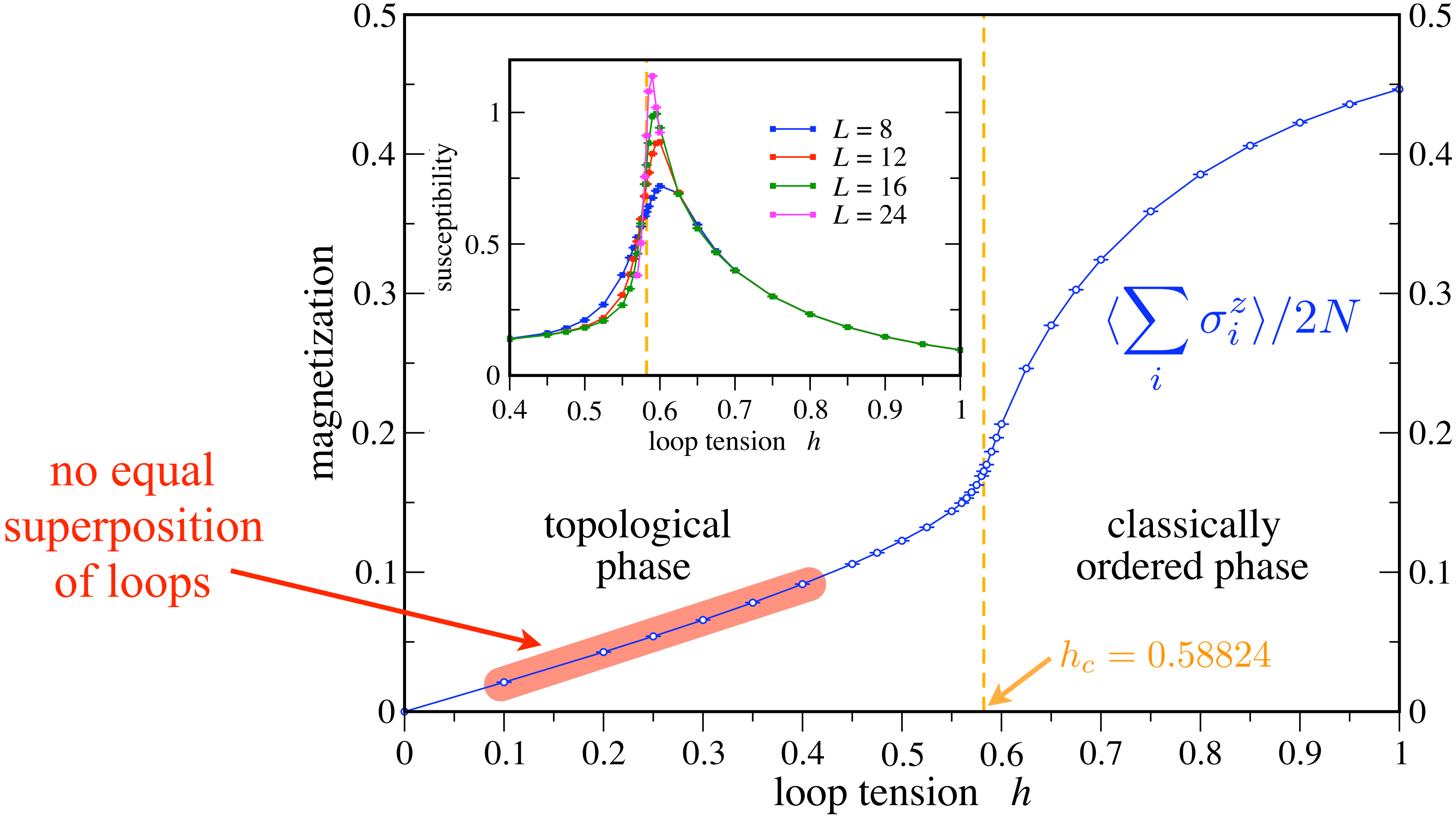
$$\mathcal{S} = -\beta_\tau \sum_{\{P_\tau\}} \sigma^z\sigma^z\sigma^z\sigma^z - \beta \sum_{\{P_s\}} \sigma^z\sigma^z\sigma^z\sigma^z$$

imaginary time   real space

In the context of QCD, the **finite-temperature** physics
of such gauge theories has been studied extensively.

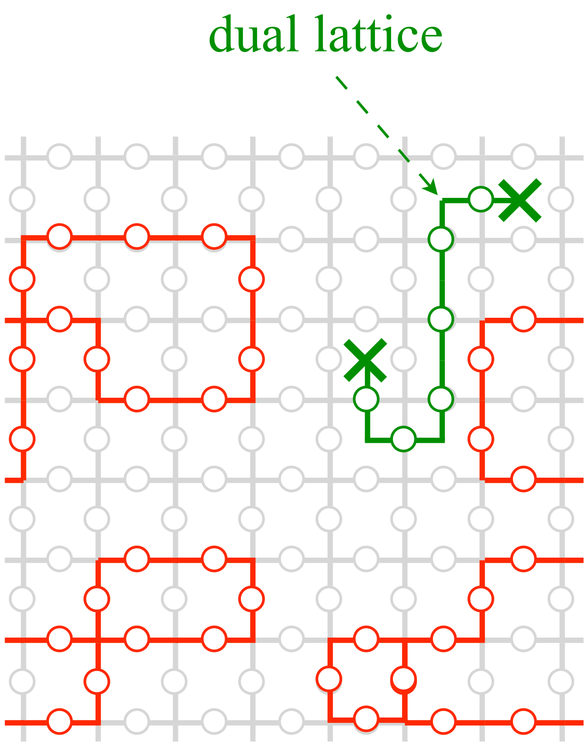
We are interested in the **zero-temperature** physics of this gauge theory.

Magnetic phase transition



Continuous quantum phase transition between topological phase and classically ordered phase (3D Ising universality class).

Magnetic vortices



Magnetic vortices are plaquettes with

$$\prod_j \sigma_j^x = -1$$

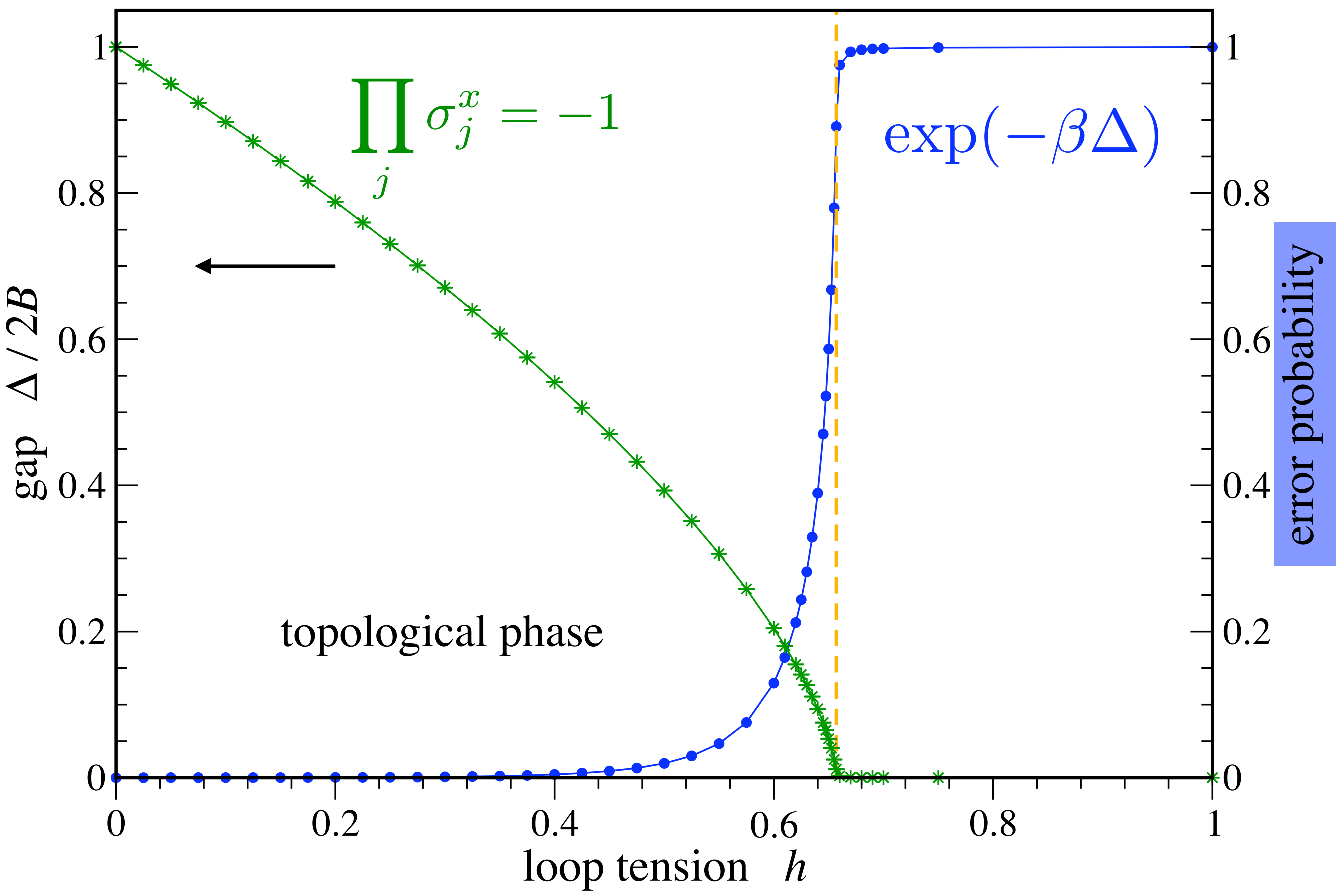
Massive in topological phase.

Correlation function for two vortices is given by product of bond spins along path

$$\langle \prod_i \sigma_i^z \rangle = \langle \mu_p^z \mu_q^z \rangle$$

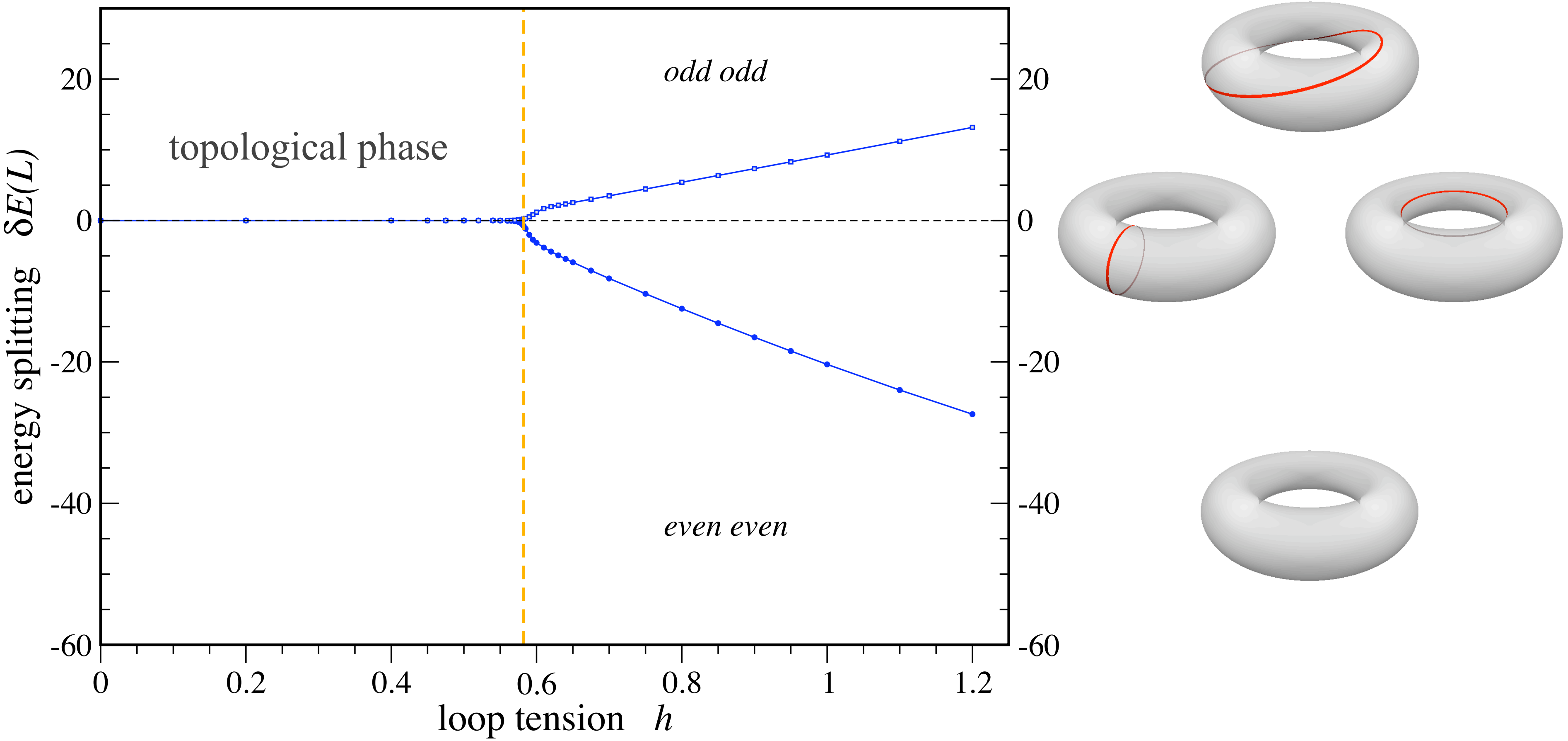
which is equivalent to spin-spin correlation function of plaquette spins.

Vortex condensation



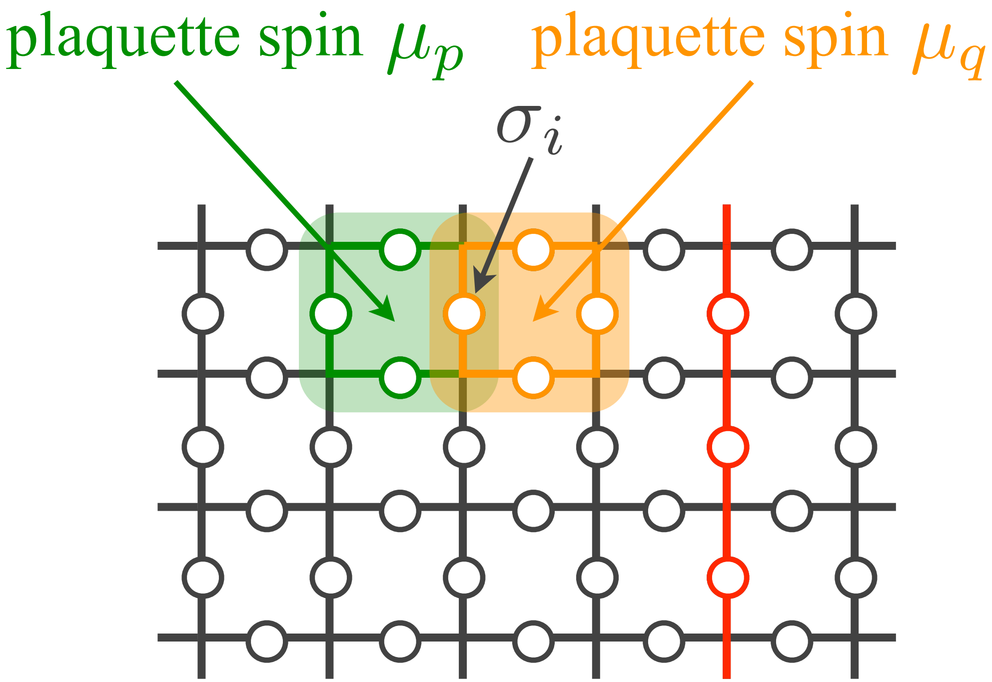
Gap estimated from imaginary time correlation function $\Delta \propto 1/\xi_\tau$.

Degeneracy splitting



Back to the Ising model

- Where are the topological sectors in the Ising model?



$$\sigma_i^z = \frac{1}{2}\,\mu_p^z\,\mu_q^z$$

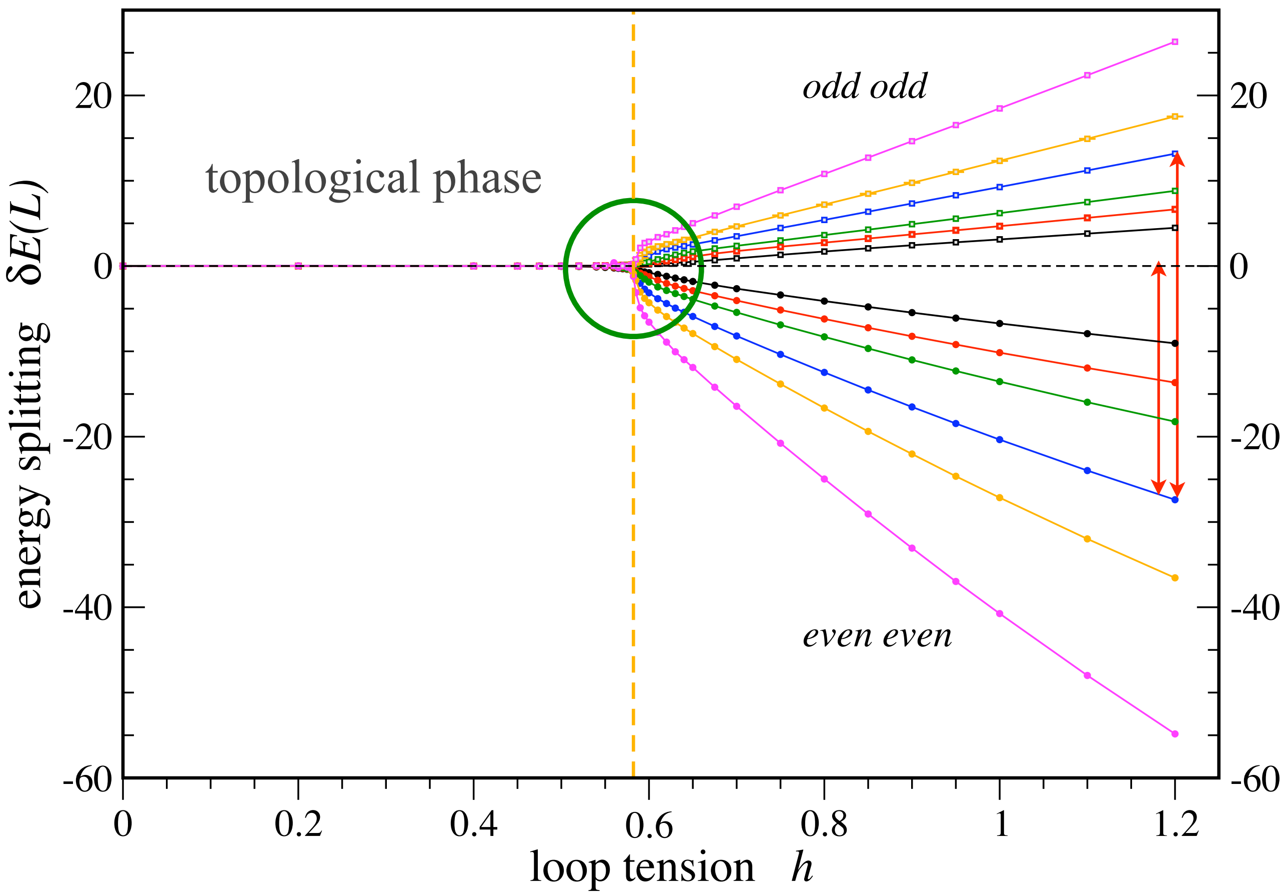
spin flip changes **sign** of Ising interaction along cut

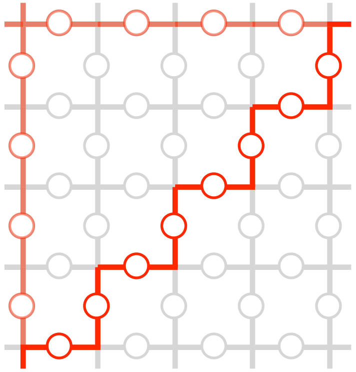
↓

change of boundary condition

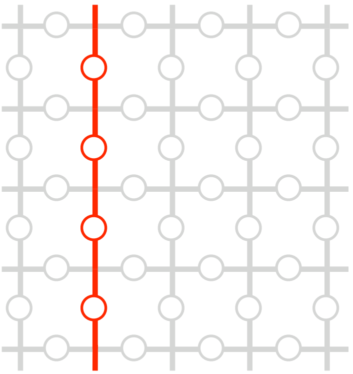
"Topological sectors" correspond to variations of the **boundary conditions**.

Degeneracy splitting





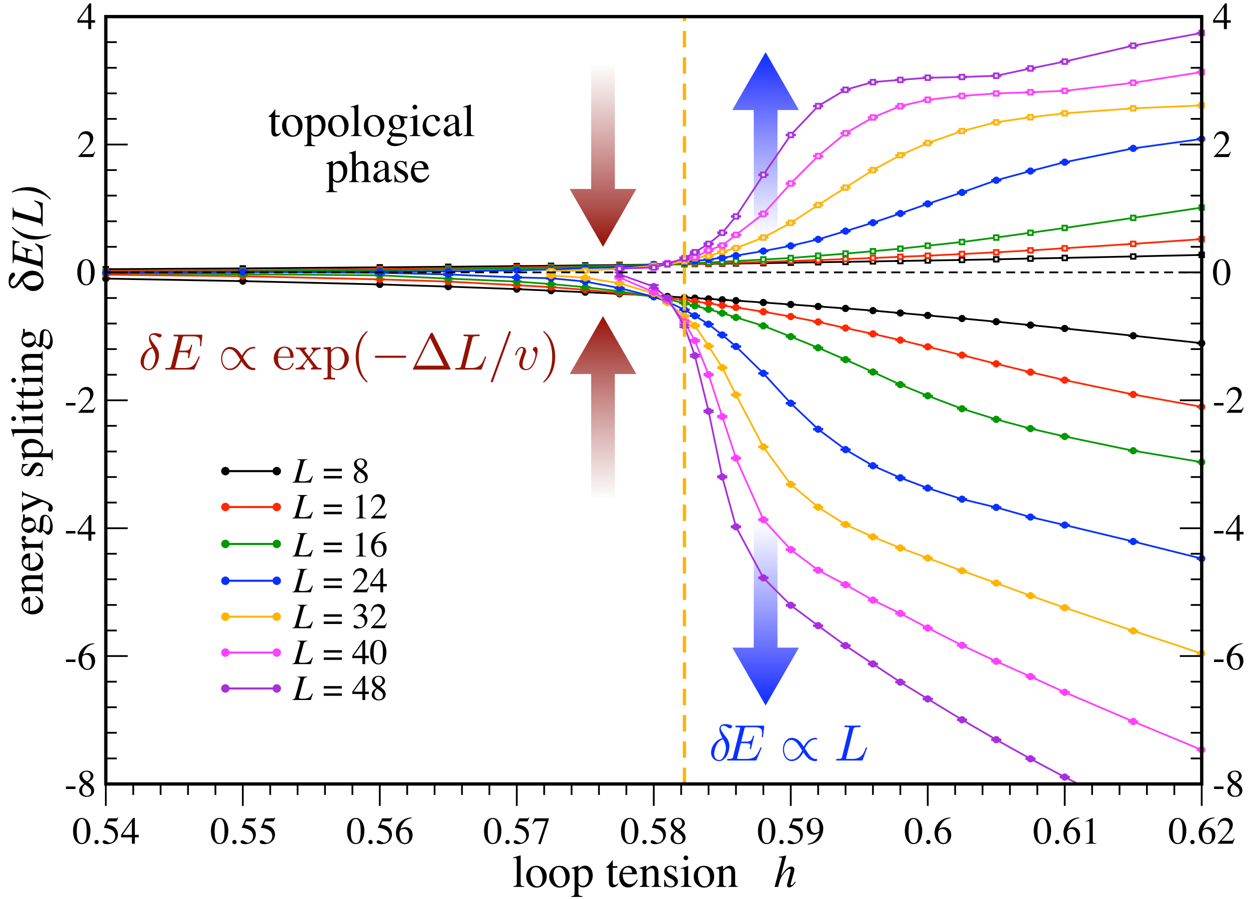
$\delta E \propto L\sqrt{2}$



$\delta E \propto L$

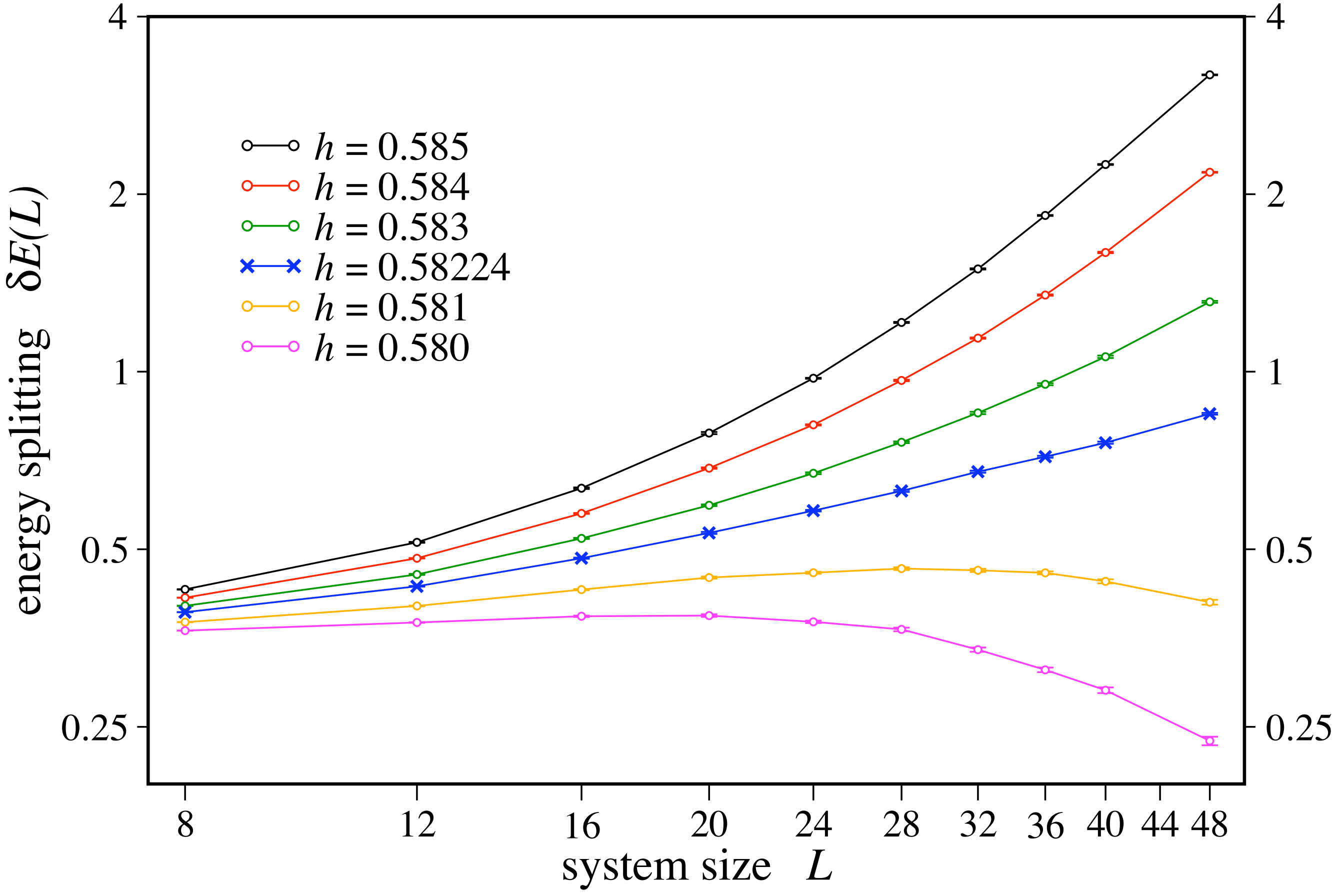
Strong tension limit well understood.

Degeneracy splitting



Transition from **exponential suppression** to **power-law growth**.

Finite-size scaling



critical exponent

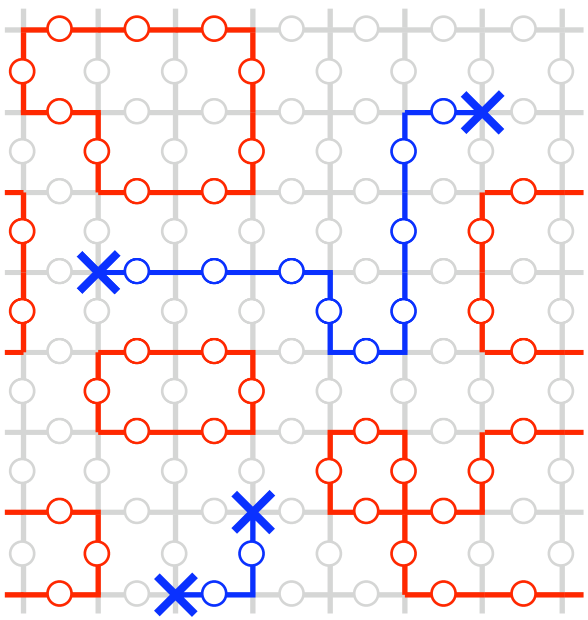
$z = 1.42 \pm 0.02$

exponential accuracy
for larger systems

100,000 cpu hours

Transition from power-law to exponential scaling.

Charge confinement



Charge excitations are open loop ends.

Deconfined in topological phase

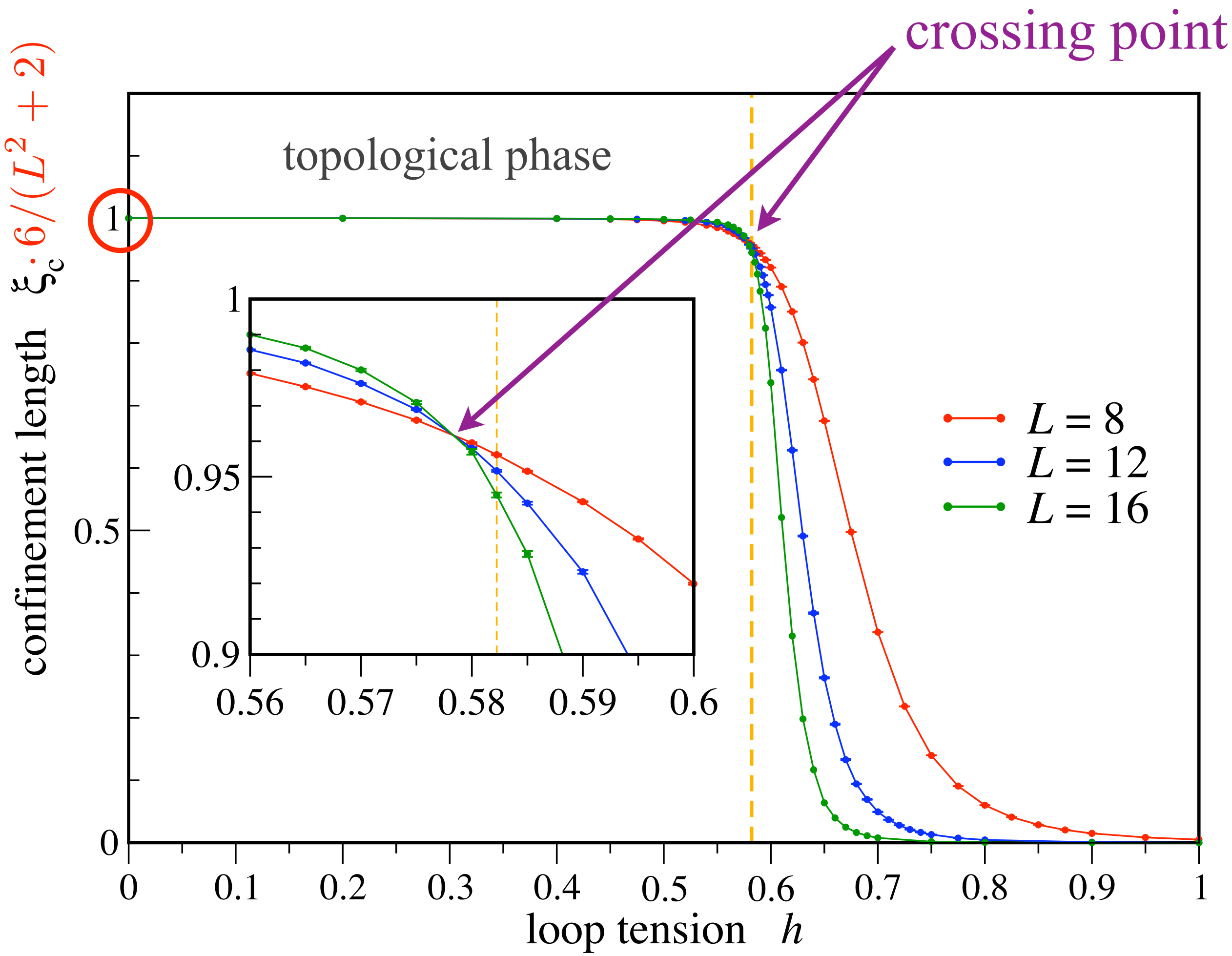
We can define a confinement length ξ_c : distance between loop ends.

At soluble point:

$$\langle \xi_c^2 \rangle = \frac{1}{L^2} \sum_{\Delta x=-L/2}^{L/2} \sum_{\Delta y=-L/2}^{L/2} \left(\Delta x^2 + \Delta y^2\right) = \frac{L^2+2}{6}$$

What happens for finite loop tension?

Charge confinement



Magnetic and confinement transitions occur **simultaneously**.

There is only **one length scale** $\xi = \xi_c$.

Dissipation

Coupling the environment to the **classical state** of the system

$$H = -2B\sum_p \mu_p^x + \sum_{p,k}\left\{C_k(b_{p,k}^\dagger + b_{p,k})\mu_p^z\right\} + \sum_{i,k}\omega_{i,k} b_{i,k}^\dagger b_{i,k}$$

plaquette spin flips (toric code)

coupling to environment

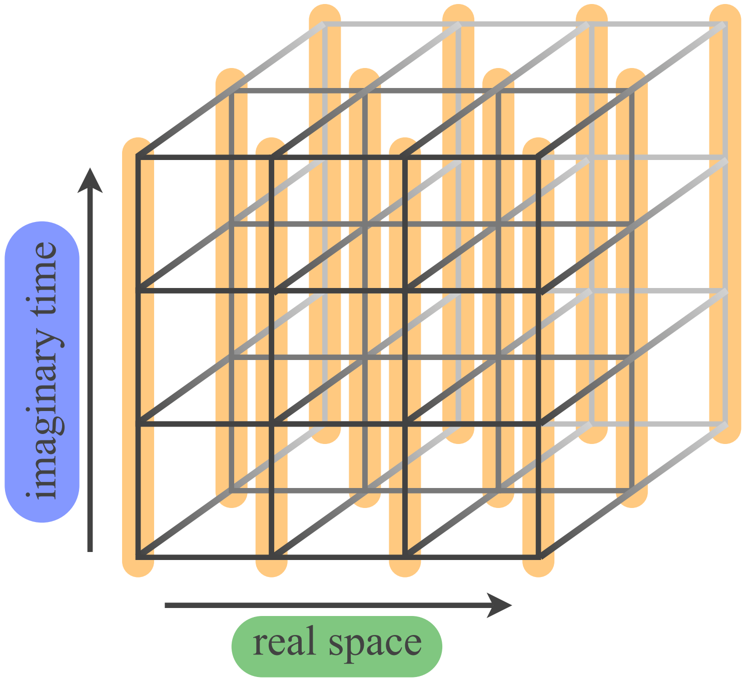
heat bath

Integrate out bath degrees of freedom (Ohmic dissipation)

$$\mathcal{H}_{\rm cl} = -K_\tau \sum_{\tau,p} S_p(\tau)S_p(\tau+\Delta\tau) - \frac{\alpha}{2}\sum_{\tau<\tau',p}\left(\frac{\pi}{N_\tau}\right)^2 \frac{S_p(\tau)S_p(\tau')}{\sin^2\left(\frac{\pi}{N_\tau}|\tau-\tau'|\right)}$$

Dissipation

$$\mathcal{H}_{\mathrm{cl}} = -K_\tau \sum_{\tau,p} S_p(\tau) S_p(\tau+\Delta\tau) - \frac{\alpha}{2} \sum_{\tau<\tau',p} \left(\frac{\pi}{N_\tau}\right)^2 \frac{S_p(\tau)S_p(\tau')}{\sin^2(\frac{\pi}{N_\tau}|\tau-\tau'|)}$$



spins couple along imaginary time only

no real space coupling

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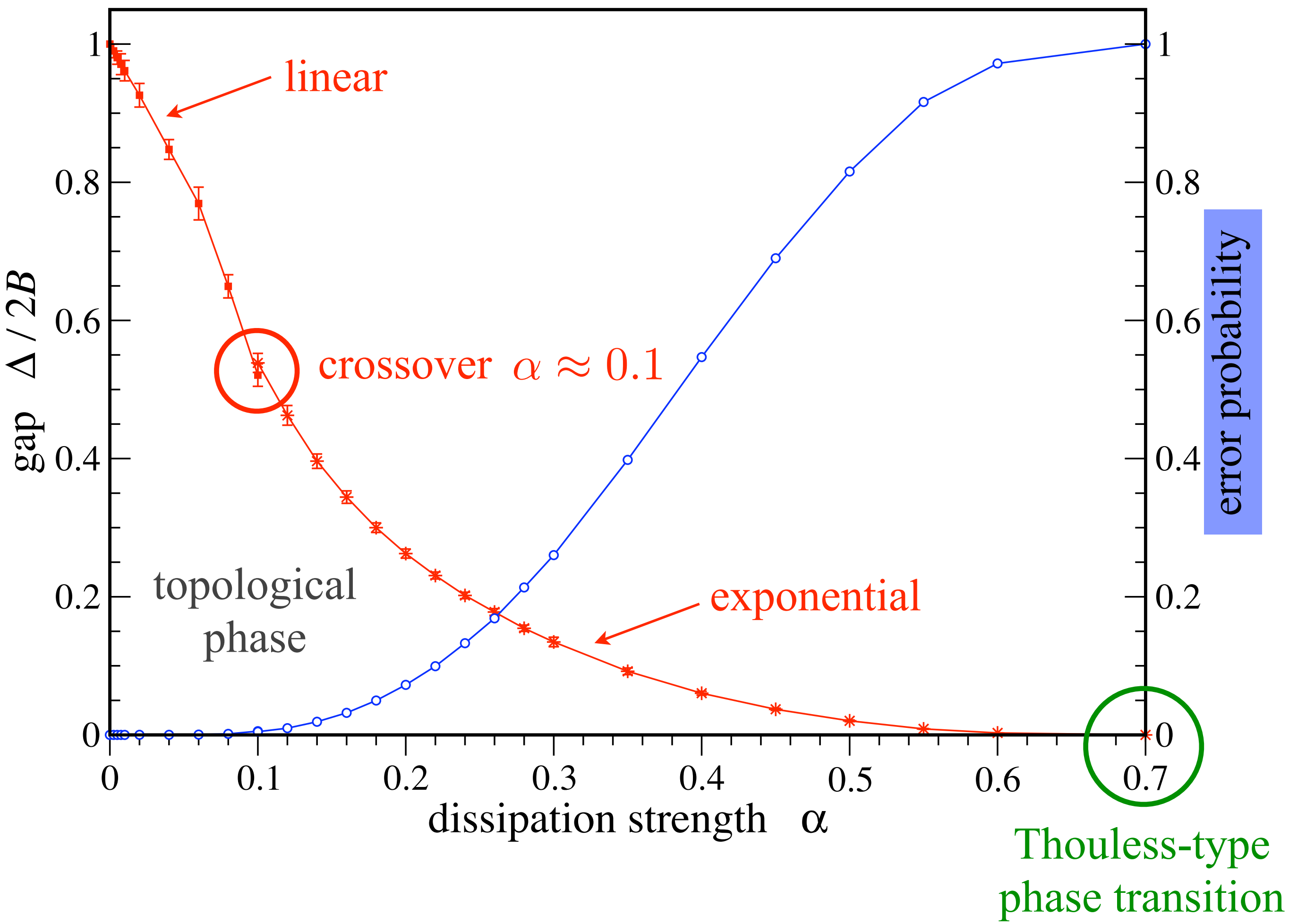
system decouples into 1-dimensional chains (with long-range interactions)

is just the Caldeira-Legget model

Dissipation



Topological phase is stable for small dissipation strength $\alpha < \alpha_c$

Summary and Outlook

- The topological phase in the toric code exists for an **extended range** around the soluble point.
- Local perturbations can drive a **continuous** quantum phase transition to a classically ordered phase.
- No need to fine-tune system to have topological order.
- Paucity of experimental observations not due to intrinsic delicateness of such phases.
- Does this picture hold for **non-abelian** phases?

S. Trebst, P. Werner, M. Troyer, K. Shtengel, and C. Nayak   cond-mat/0609048

Non-abelian topological quantum computing

- Topological phases with **non-abelian braiding statistics** of the excitations.
- **Universal** quantum computation can be done by braiding quasi-particles.
- **No** need for **fine control** of quantum gates.