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Two-dimensional quantum liquids from interacting non-Abelian anyons

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Abstract. A set of localized, non-Abelian anyons—such as vortices in a $p_x + \mathrm{i} p_y$ superconductor or quasiholes in certain quantum Hall states—gives rise to a macroscopic degeneracy. Such a degeneracy is split in the presence of interactions between the anyons. Here, we show that in two spatial dimensions this splitting selects a *unique* collective state as ground state of the interacting many-body system. This collective state can be a novel gapped quantum liquid nucleated inside the original parent liquid (of which the anyons are excitations). This physics is of relevance for any quantum Hall plateau realizing a non-Abelian quantum Hall state when moving off the center of the plateau.

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Contents

1. Introduction

While many conventional phases of matter appear as a consequence of spontaneous symmetry breaking, quantum systems may form more unusual ground states, so-called *quantum liquids*, which do not break any symmetry (see e.g. [1]). One particularly captivating case is given by two-dimensional (2D) *topological* quantum liquids, which exhibit a bulk gap and harbor exotic quasiparticle excitations, so-called anyons, that obey fractional statistics [2]. Time-reversal symmetry-breaking topological liquids with non-Abelian anyons have attracted considerable recent interest in the context of fractional quantum Hall states ([3]–[5] and references therein), heterostructures involving a novel class of materials, so-called topological insulators [6], rotating Bose–Einstein condensates [7] and unconventional $p_x + ip_y$ superconductors [8]. When populating such a topological quantum liquid with a set of anyons, their non-Abelian nature manifests itself in a degenerate *manifold of states* that grows exponentially with the number of anyons—even in the case of fixed particle positions. It is this degenerate space of states that is key to proposals for topological quantum computation [9].

However, this degeneracy will be *split by interactions* between the anyons. If the anyons are spaced far from each other, e.g. their distance is much larger than the coherence length, this interaction will be exponentially small and so will the splitting. In this paper, we address the question of how this degeneracy will be lifted and a new collective ground state is formed when interactions cannot be ignored. In particular, we determine the nature of this collective state for a 2D arrangement of non-Abelian anyons. Note the conceptual similarity of this question with the one asked in the context of fractional quantum Hall states, where the Coulomb interaction splits the degenerate manifold of electronic states of a partially filled Landau level and leads to the formation of a new collective ground state—the fractional quantum Hall state [10].

One example of where this physics is of relevance is given by non-Abelian quantum Hall states *off the center* of the quantum Hall plateau. The quantum Hall state at filling fraction $\nu = 5/2$ is considered the best candidate to date, with an intense experimental effort [11]–[13] aimed at demonstrating its non-Abelian nature [3]. By tuning the magnetic field away from the center of the plateau, a finite density of quasiholes or quasiparticles will populate the quantum Hall liquid, depending on whether the magnetic field is increased or decreased. It is natural
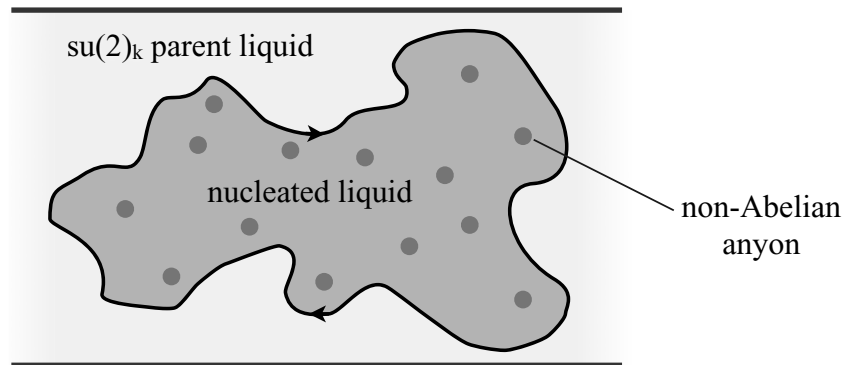
Figure 1. The collective state of a set of interacting, localized, non-Abelian anyons is a gapped quantum liquid, which is nucleated within the original parent liquid. The two liquids are separated by a neutral, chiral edge state.

that these quasiholes (or quasiparticles) would form a Wigner crystal [14], which renders them immobile. Remarkably, as we show in this paper, interactions between these anyons can then lead to the formation of a novel *gapped* quantum Hall liquid[6], which is separated from the original parent liquid by a neutral, chiral edge state, as illustrated in figure 1. These gapless edge modes contribute to heat transport, leading to a distinct experimental signature.

2. Degeneracy splitting for su(2)$_k$ anyons

To be specific, we consider here a setting where su(2)$_k$ anyons are pinned to static positions. These non-Abelian anyons are described by generalized angular momentum quantum numbers of so-called su(2)$_k$ Chern–Simons theories [15], which correspond to certain quantum deformations of SU(2) (see e.g. [16]). The case of $k = 2$ describes the *non-Abelian* statistics of anyonic excitations in the Moore–Read state, as well as the Majorana modes of half-vortices in $p_x + i p_y$ superconductors or vortex excitations in the gapped 'B-phase' of Kitaev's honeycomb model in a magnetic field. We note that the latter cases also possess an additional *Abelian* phase contributing to the overall statistics of the anyons, which however does not modify their non-Abelian properties. For instance, the Moore–Read state is a *fermionic* quantum Hall state, which is described by a more general $Z_2 \times U(1)$ theory, where the non-Abelian aspects are now incorporated in the Z_k factor[7], and the U(1) factor stands for an ordinary Abelian U(1) Chern–Simons theory[8] (familiar of the Laughlin states [10]).

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[6] Note that this quantum liquid forms 'on top' of the Wigner crystal of localized anyons, i.e. the liquid forms only in the topological degrees of freedom without affecting the underlying crystalline positional order. For the charged anyons of the quantum Hall state, this implies that this novel (nucleated) liquid has the same Hall conductivity σ_{xy}. (For an Abelian quantum Hall liquid, on the other hand, the state describing a set of anyons at *fixed* positions is unique and non-degenerate, and formation of a new quantum liquid 'on top' of the Wigner crystal cannot occur. The familiar Haldane hierarchy state arises from a liquid positional order of anyons, yielding a change of the Hall conductance σ_{xy}.)

[7] All non-Abelian aspects are incorporated in the Z_k factor. This factor also appears when we represent [17] su(2)$_k$ as $Z_k \times U(1)$.

[8] That is an ordinary Laughlin-type state, see e.g. [18].

	$1/2 \times 1/2 \to \mathbf{0}$	$1/2 \times 1/2 \to \mathbf{1}$
bosonic quantum Hall	$\mathrm{su}(2)_2$ $\downarrow$ $\mathrm{su}(2)_1 \times \mathrm{su}(2)_1$ 220 Halperin bilayer state	$\mathrm{su}(2)_2$ $\downarrow$ $U(1)$
fermionic quantum Hall	$Z_2 \times U(1)$ $\downarrow$ $U(1) \times U(1)$ 331 Halperin bilayer state	$Z_2 \times U(1)$ $\downarrow$ $U(1)$
p-wave superconductor	Z_2 $\downarrow$ $U(1)$	Z_2 $\downarrow$ $\emptyset$

Figure 2. Table of collective ground states of interacting Majorana zero modes ($k = 2$) for various anyonic theories.

The most elementary example of the lifting of a degeneracy occurs for a pair of non-Abelian anyons. An interaction mediated by topological charge tunneling [19, 20] results in a splitting of the energies of the possible pair states—similar to the singlet–triplet splitting of two ordinary spin-1/2s. We consider the most relevant case, where the anyonic degree of freedom corresponds to a generalized angular momentum $j = 1/2$ of these $\mathrm{su}(2)_k$ theories. They obey the fusion rule $1/2 \otimes 1/2 = 0 \oplus 1$ reminiscent of two ordinary spin-1/2s coupling into a singlet or triplet. We therefore denote a coupling favoring the generalized $j = 0$ (singlet) state as antiferromagnetic (AFM) coupling, while ferromagnetic (FM) coupling then corresponds to favoring the generalized $j = 1$ (triplet) state. The magnitude and sign of this splitting are not universal and in general depend on microscopic details such as the spatial separation between the anyons and the nature of the topological liquid. Explicit calculations of this pair splitting have recently been performed for two quasiholes in a Moore–Read quantum Hall state [21], a pair of vortices in a $p_x + \mathrm{i}p_y$ superconductor [22], and a pair of vortex excitations in the gapped 'B-phase' of Kitaev's honeycomb model in a magnetic field [23].

For more than two anyons this elementary pair interaction will result in a splitting of their manifold of macroscopically degenerate states and the formation of non-trivial collective quantum many-body states. As a generic model for the many-anyon problem, we consider a Hamiltonian that is a sum of local, pairwise projectors on every bond of the underlying lattice onto one of the two fusion channels, e.g. $\mathcal{H} = J \sum_{ij} \Pi_{ij}^0$, where the operator Π_{ij}^0 projects onto the singlet channel and J is an overall coupling constant. The main result of this paper is that the ground state of a 2D array of interacting non-Abelian anyons is a gapped topological liquid, whose precise character depends on the sign of the interactions and the nature of the parent liquid of which the anyons are excitations. In the absence of any additional Abelian phases, i.e. a bosonic quantum Hall state described by $\mathrm{su}(2)_k$, our result is that AFM couplings yield a non-Abelian $\mathrm{su}(2)_{k-1} \times \mathrm{su}(2)_1$ liquid, while FM couplings lead to a gapped Abelian Laughlin-type $U(1)$ liquid [18][9]. In figure 2, we rephrase our results in the language of interacting Majorana zero modes (corresponding to $k = 2$) and apply them in the context of the fermionic

[9] The original and the two nucleated liquids are described by $\mathrm{su}(2)_k$, $\mathrm{su}(2)_{k-1} \times \mathrm{U}(1)$ and $\mathrm{U}(1)_k$ Chern–Simons field theories [15], respectively.

a) antiferromagnetic coupling b) ferromagnetic coupling

parent liquid $su(2)_k$ edge state

$$\frac{su(2)_{k-1} \times su(2)_1}{su(2)_k}$$

nucleated liquid
$su(2)_{k-1} \times su(2)_1$

parent liquid $su(2)_k$ edge state

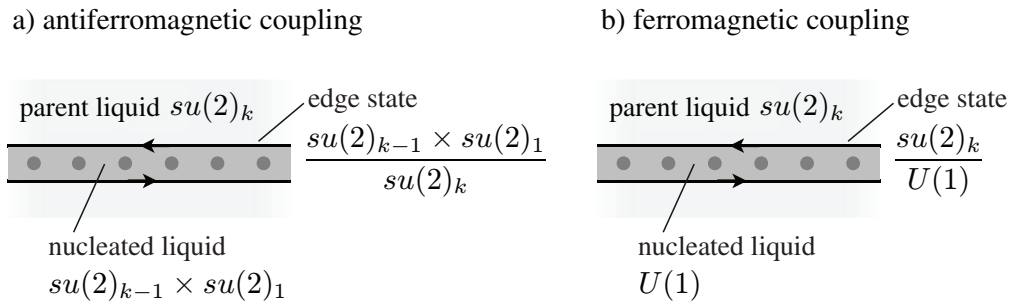$$\frac{su(2)_k}{U(1)}$$

nucleated liquid
$U(1)$

Figure 3. A linear array of interacting $su(2)_k$ anyons nucleates a sliver of a gapped liquid separated from the original parent liquid by a chiral edge state.

Moore–Read quantum Hall state and the $p_x + \mathrm{i}p_y$ superconductors, i.e. also taking into account the corresponding additional Abelian phases. We note that some of these special cases have been analyzed before and are indeed captured by our more general result, thereby further corroborating it. The first column gives results for AFM anyon interactions, whereas the second column gives results for FM interactions. Let us first focus on the first row '*bosonic quantum Hall*', which describes the non-Abelian physics of certain proposed rotating Bose–Einstein condensates [7] and literally corresponds to the $k = 2$ case of our result. In this case we find that for AFM interactions an Abelian topological liquid, described by $su(2)_1 \times su(2)_1$ Chern–Simons theory, is nucleated. This state is identical to the (bosonic) '220 Halperin bilayer state' [24]. For FM interactions, an Abelian U(1) liquid is nucleated and all non-Abelian aspects of the original $su(2)_2$ state disappear. The second row '*fermionic quantum Hall*' generalizes the above result to the fermionic version [25] of the $su(2)_k$ state, which can also be described by $Z_k \times U(1)$ non-Abelian Chern–Simons theory (see footnote 7). For $k = 2$ this state occurs e.g. in the Moore–Read quantum Hall state [3]. For AFM interactions between the anyons, the nucleated liquid is an Abelian $U(1) \times U(1)$ liquid, which corresponds precisely to the '331 Halperin bilayer state' [24]. For FM interactions the Z_2 degree of freedom simply disappears upon nucleating an Abelian U(1) liquid. This was first discovered using Majorana fermions in the context of random FM anyon interactions [26]. Both cases were recently also discussed using different techniques in [27]. The third row describes the case of $p_x + \mathrm{i}p_y$ superconductors. The non-Abelian nature of vortices in such superconductors is captured by a non-Abelian Z_2 theory (see footnote 7). For AFM pairwise interactions between vortices, an Abelian U(1) liquid is nucleated. This was observed for a triangular vortex lattice using free fermion calculations [28, 29] and the observation of a non-trivial Chern number [29]. For FM vortex pair interactions, the topological order of the $p_x + \mathrm{i}p_y$ superconductor is entirely destroyed.

The above case of $k = 2$ turns out to be rather special, since this problem can be mapped to non-interacting Majorana fermions, which are more tractable than the general $su(2)_k$ theories for $k > 2$. A first step towards understanding the collective states of interacting $su(2)_k$ anyons has been taken by analyzing 1D arrangements of localized anyons [19, 30, 31]. In this case, the gapless collective modes found for a chain of interacting anyons have been interpreted as *edge states* [31] between the original topological liquid and a sliver of a novel, gapped nucleated liquid, as illustrated in figure 3.
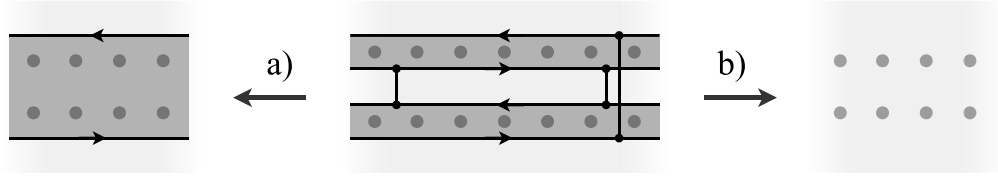
Figure 4. Interactions between the anyons in two slivers can (a) gap out the inner edges and merge the two liquids into one droplet or (b) gap out inner and outer edges, thereby destroying the nucleated liquid in the sliver.

3. Two-dimensional (2D) arrangements of interacting anyons

To approach the case of 2D arrangements of interacting anyons, we generalize this 'liquids picture' and consider a system with several such 1D slivers. We find that this picture allows us to precisely describe (i) the subtleties in the spectrum of collective edge states for a system of (multiple) decoupled slivers, and (ii) the effect of coupling the individual slivers, which originates from interactions between anyons in the different slivers. In particular, we find that the central charge of two decoupled chains (slivers) changes from $2c$ to c upon introducing inter-chain couplings J_{rung} or J_{diag}, as indicated in figure 5. The spectrum then changes to that of a single sliver, which indicates that interactions between the slivers can gap out the inner edges and merge the two slivers into one big droplet of nucleated liquid of identical character; see figure 4(a). We numerically find that such droplet nucleation occurs for a wide range of couplings indicated by the shaded wedges in the phase diagrams of figure 5 for different lattice geometries. Having established the merging of two slivers into a big droplet, this process can be iterated to yield macroscopically large droplets.

We first consider the case of two decoupled slivers, which gives rise to a sequence of five liquids and four intervening edges, as illustrated at the center of figure 4. It turns out that in this case the combined spectrum of gapless edge states is *not* that of the free tensor product of the edge modes of the individual slivers, but rather a subset of the latter. This subset can be understood when considering all possible tunneling processes between the various liquids, which give rise to certain constraints. For example, for AFM interactions $J_{\mathrm{leg}} > 0$, the spectrum of these four edges takes the form

$$(\psi_L)_{j_1}^{i_1}(\psi_R)_{j_1}^{i_2}(\psi_L)_{j_2}^{i_2}(\psi_R)_{j_2}^{i_1},$$

where $(\psi_L)_j^i$ is a primary field in the coset [32] $\mathrm{su}(2)_{k-1} \times \mathrm{su}(2)_1 / \mathrm{su}(2)_k$ and describes the tunneling in and out of the respective edge between the two topological liquids appearing in the coset[10]. Here i is a quantum number in $\mathrm{su}(2)_k$ and j in $\mathrm{su}(2)_{k-1}$. Consider for simplicity odd k, in which case we can choose i to run over integer values $0, 1, \ldots, (k-1)/2$ and j to run over values $0, 1/2, 1, .., (k-1)/2$. No topological charges can be ejected into the surrounding $\mathrm{su}(2)_k$ liquid in a physical tunneling process, which enforces $i_1 = 0$. With these constraints we can then derive the complete conformal spectrum of the four edges, including conformal weights, degeneracies and topological charge [19] assignments. Checking these results against numerical calculations of the spectra, we find perfect agreement. Analogous results apply to FM interactions $J_{\mathrm{leg}} < 0$.

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[10] The sequence $(i_1, j_1, i_2, j_2, i_1)$ is associated with the consecutive five liquids in the center of figure 4.
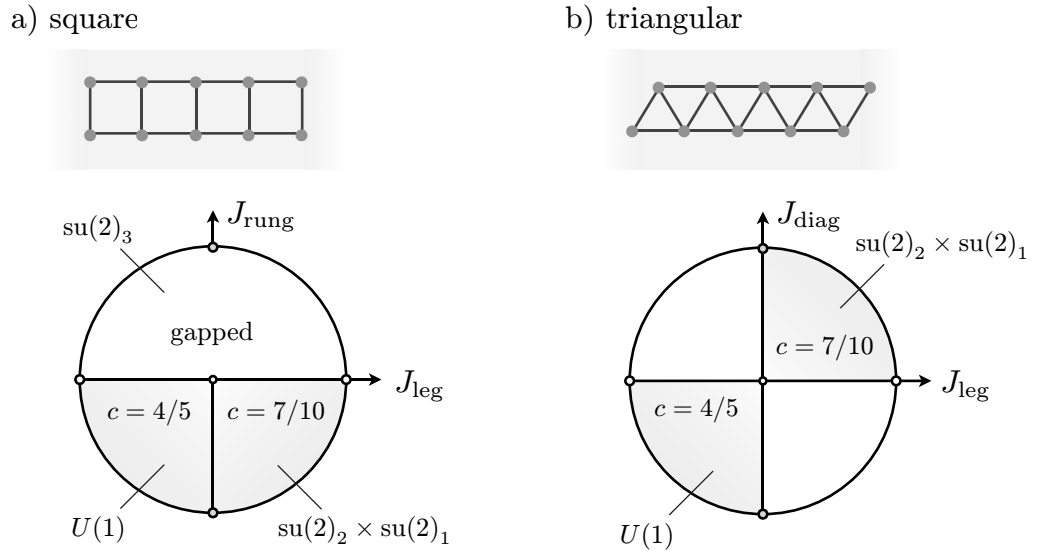
a) square

b) triangular

Figure 5. Phase diagrams for anyonic $su(2)_3$ ladder systems with (a) square lattice geometry (of width $W = 2$ or $W = 4$) and (b) triangular geometry (of width $W = 2$) versus their respective couplings. The shaded wedges indicate gapless phases, which are described by conformal field theories with central charges $c = 4/5$ or $c = 7/10$. These gapless theories correspond to edge modes between the original $su(2)_3$ anyonic liquid and a nucleated U(1) or $su(2)_2 \times su(2)_1$ liquid, respectively.

4. A microscopic model

We now turn to a microscopic lattice model of interacting anyons, which allows for a precise numerical characterization of the collective ground states. As a concrete realization we consider interacting $su(2)_3$ (Fibonacci) anyons arranged in a square lattice geometry, which we build up by coupling W chains of anyons into W-leg ladders[11]. We denote the strength of the anyon–anyon interaction as J_{leg} and J_{rung} for the coupling along and perpendicular to the chains, respectively, with positive (negative) coupling referring to AFM (FM) interactions favoring the generalized $j = 0$ ($j = 1$) state. We parametrize these couplings as $\tan \theta = J_{rung}/J_{leg}$.

4.1. Numerical approach

We have performed a numerical analysis of these ladder systems based on Lanczos exact diagonalization. This provides us with the low-energy spectra of finite $W \times l$ systems, where W is the width of the ladder and l the extent of the legs. We have been able to analyze systems of size $2 \times l$ (from $l = 8$ to $l = 21$), $3 \times l$ (from $l = 6$ to $l = 12$) and $4 \times l$ ($l = 4, 6, 8$), which allow for an accurate finite-size scaling analysis. We have typically used periodic boundary conditions along the leg direction, which then allows us to block-diagonalize the Hamiltonian into l blocks labeled by the total momentum $K = 2\pi \frac{p}{l}$ of the eigenstates due to the translational symmetry of the system along the legs. It should be noted that the Hilbert space (in each symmetry

[11] For a detailed discussion of the microscopic model, see [33].

sector) grows approximately as ϕ^N/l (ϕ is the Fibonacci number and $N = lW$), which is one of the limiting factors of our simulations. A CFT structure (if any) can then be unambiguously identified from a precise structure of the (gapless) low-energy spectrum. In such a case, we find that the lowest energies (per rung) e_n scale as

$$e_n(l) \simeq e_\infty + \pi v \left(-\frac{c}{12} + h_{L,n} + h_{R,n} \right) \frac{1}{l^2}, \tag{1}$$

where v is a zero-mode velocity, c is the central charge of the underlying CFT, and $h_{L,n}, h_{R,n}$ are the (holomorphic and anti-holomorphic) conformal weights of the CFT.

4.2. The strong rung-coupling limit

Following, at first, a similar route to that in the discussion of SU(2) quantum spin ladders (for a review, see [34] and references therein) we consider the anyonic ladders in their strong rung-coupling limit $|J_{\mathrm{rung}}| \gg |J_{\mathrm{leg}}|$. For an isolated rung the total spin of the ground state is 0 or 1/2,[12] depending on the sign of the coupling and the width of the ladder: it is a spin 1/2 for odd width ladders with $J_{\mathrm{rung}} > 0$, and for ladders with $J_{\mathrm{rung}} < 0$ whose width is not a multiple of 3. The low-energy effective model for weakly coupled rungs is that of a single spin-1/2 anyonic chain. Indeed, we find that the low-energy spectrum is gapless and can be described by a conformal field theory identical to that of a single chain [19]. In the particular case of $k = 3$, the gapless theories are those of the tricritical Ising model ($c = 7/10$), see also [35, 36], or three-state Potts model ($c = 4/5$) for AFM and FM couplings along the chains, respectively. We numerically find that these gapless phases extend all the way to the weak rung-coupling limit $|J_{\mathrm{rung}}| \ll |J_{\mathrm{leg}}|$, as shown in the phase diagram of figure 5(a). Interpreting the gapless modes in these phases as edge states, we conclude that AFM couplings along the chain result in the nucleation of an $\mathrm{su}(2)_2 \times \mathrm{su}(2)_1$ topological liquid, whereas for FM couplings we find that a U(1) topological liquid is nucleated. We note that for a triangular arrangement of anyons, as is characteristic for the positional ordering in a Wigner crystal, the nucleation of new liquids occurs for *equal* signs of the couplings; see figure 5(b). This is in line with the results obtained for $k = 2$ on an infinite triangular lattice [28]. Furthermore, the results obtained in [35] for the special case of the non-Abelian spin singlet state, using techniques entirely different from ours and available only for this state, can be seen to be in full agreement with our general result.

If, on the other hand, the total spin of the ground state of an isolated rung is 0, the elementary excitation on a rung is gapped and carries spin 1. Its gap remains finite when turning on a leg coupling J_{leg} and we find numerically that, as in ordinary SU(2) ladders, the gapped phase extends all the way to the weak rung-coupling limit $|J_{\mathrm{rung}}| \ll |J_{\mathrm{leg}}|$. The absence of gapless modes in this coupling regime indicates that the original $\mathrm{su}(2)_k$ liquid survives. We summarize the phase diagram for ladders of width $W = 2$ and $W = 4$ in figure 5(a).

5. Topology and ground-state degeneracies

A characteristic feature of a topological quantum liquid is that it is sensitive to the topology of the underlying manifold [37], which is reflected in the ground-state degeneracy and the occurrence of gapless edge modes for open boundaries. So far, we have considered ladder systems with open boundary conditions along the rung direction (an annulus) and have

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[12] In the $\mathrm{su}(2)_3$ theory, spin 1 can be identified with 1/2 and spin 3/2 with 0.
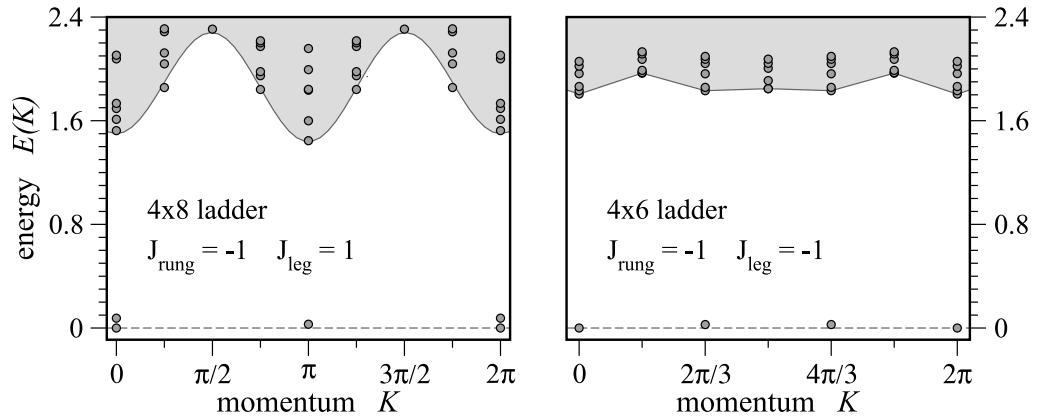
Figure 6. Gapped energy spectra of four-leg $su(2)_3$ anyonic ladders in a toroidal geometry versus the momentum along the legs.

understood the occurring gapless modes as edge states. Gluing the boundaries of this annulus to form a torus by adding a coupling between the first and last leg removes the boundary and is thus expected to also remove the edge states. Indeed, the spectrum of the four-leg ladder on a torus is found to be gapped, as shown in figure 6. Most importantly, the topological nature of the collective ground state is confirmed by the numerically observed degeneracy. For both signs of leg coupling this degeneracy is found to be threefold, which precisely corresponds to the expected number of different topological charges on the torus for the $su(2)_2 \times su(2)_1$ theory arising for $J_{\text{leg}} > 0$ and the U(1) theory arising for $J_{\text{leg}} < 0$.

6. Outlook

Our essential result is that anyon–anyon interactions split the degeneracy of a macroscopic number of non-Abelian anyons, pinned at fixed spatial locations, and select a *unique* gapped collective many-body ground state. For a wide range of anyon couplings, this collective state is found to be again a topological liquid, distinct from the original liquid of which the anyons are excitations. This physics should be of relevance for any quantum Hall plateau realizing a non-Abelian quantum Hall state: away from the center of the plateau, a finite density of quasiholes or quasiparticles is expected to be present. If these charged quasiparticles(-holes) are pinned at fixed spatial positions (e.g. by the formation of a Wigner crystal), our results predict the appearance of a *different* quantum Hall state with the *same* electrical but different thermal transport properties. The latter arise solely from the splitting of the topological degeneracy due to inter-anyon interactions. Furthermore, our results give a general many-body perspective of how topological quantum computing schemes will fail at a temperature scale comparable to the interactions between the anyons. While the pairwise interaction between the anyons has been expected to be a cause of decoherence for a topological quantum computer [20, 22], our results demonstrate that at temperatures below the gap scale of the nucleated liquid—which we find to be of the same order as the bare two-anyon interaction—all anyonic quasiparticles effectively disappear by forming a new collective state, thereby rendering computational schemes based on the individual manipulation of anyons impracticable.

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Acknowledgments

References

[1] Wen X-G 2004 *Quantum Field Theory of Many-Body Systems* (Oxford: Oxford University Press)
[2] Leinaas J M and Myrheim J 1977 *Nuovo Cimento* B **37** 1
[3] Moore G and Read N 1991 *Nucl. Phys.* B **360** 362
[4] Bonderson P and Slingerland J K 2008 *Phys. Rev.* B **78** 125323
[5] Ilan R, Grosfeld E, Schoutens K and Stern A 2009 *Phys. Rev.* B **79** 245305
[6] Fu L and Kane C L 2008 *Phys. Rev. Lett.* **100** 096407
[7] See e.g. Cooper N R, Wilkin N K and Gunn J M F 2001 *Phys. Rev. Lett.* **87** 120405
[8] Read N and Green D 2000 *Phys. Rev.* B **61** 10267
[9] Nayak C, Simon S H, Stern A, Freedman M and Das Sarma S 2008 *Rev. Mod. Phys.* **80** 1083
[10] Laughlin R B 1983 *Phys. Rev. Lett.* **50** 1395
[11] Dolev M, Heiblum M, Umansky V, Stern A and Mahalu D 2008 *Nature* **452** 829
[12] Radu I P, Miller J B, Marcus C M, Kastner M A, Pfeiffer L N and West K W 2008 *Science* **320** 899
[13] Willett R L, Pfeiffer L N and West K W 2009 *Proc. Natl Acad. Sci. USA* **106** 8853
[14] Zhu H, Chen Y P, Jiang P, Engel L W, Tsui D C, Pfeiffer L N and West K W 2010 *Phys. Rev. Lett.* **105** 126803
[15] Witten E 1989 *Commun. Math. Phys.* **121** 351
[16] Kassel C *Quantum Groups* (New York: Springer)
[17] Zamolodchikov A B and Fateev V A 1986 *Sov. J. Nucl. Phys.* **43** 654
[18] Wen X G and Zee A 1992 *Phys. Rev.* B **46** 2290
[19] Feiguin A, Trebst S, Ludwig A W W, Troyer M, Kitaev A, Wang Z and Freedman M 2007 *Phys. Rev. Lett.* **98** 160409
[20] Bonderson P 2009 *Phys. Rev. Lett.* **103** 110403
[21] Baraban M, Zikos G, Bonesteel N and Simon S H 2009 *Phys. Rev. Lett.* **103** 076801
[22] Cheng M, Lutchyn R M, Galitski V and Das Sarma S 2009 *Phys. Rev. Lett.* **103** 107001
[23] Lathinen V, Kells G, Carollo A, Stitt T, Vala J and Pachos J K 2008 *Ann. Phys.* **323** 2286
[24] Halperin B I 1983 *Helv. Phys. Acta* **56** 75
[25] Read N and Rezayi E 1999 *Phys. Rev.* B **59** 8084
[26] Read N and Ludwig A W W 2000 *Phys. Rev.* B **63** 024404
[27] Levin M and Halperin B I 2009 *Phys. Rev.* B **79** 205301
[28] Grosfeld E and Stern A 2006 *Phys. Rev.* B **73** 201303
[29] Kitaev A 2006 unpublished
     Lahtinen V and Pachos J 2010 *Phys. Rev.* B **81** 245132
[30] Trebst S, Ardonne E, Feiguin A, Huse D A, Ludwig A W W and Troyer M 2008 *Phys. Rev. Lett.* **101** 050401
     Trebst S, Troyer M, Wang Z and Ludwig A W W 2008 *Prog. Theor. Phys. Suppl.* **176** 384
[31] Gils C, Ardonne E, Trebst S, Ludwig A W W, Troyer M and Wang Z 2009 *Phys. Rev. Lett.* **103** 070401
[32] Goddard P *et al* 1986 *Commun. Math. Phys.* **103** 105
[33] Poilblanc D *et al* 2011 *Phys. Rev.* B at press (arXiv:1101.1186)
[34] Dagotto E and Rice T M 1996 *Science* **271** 618 and references therein
[35] Grosfeld E and Schoutens K 2009 *Phys. Rev. Lett.* **103** 076803
[36] Bais F A, Slingerland J K and Haaker S M 2009 *Phys. Rev. Lett.* **102** 220403
[37] Wen X-G 1989 *Phys. Rev.* B **40** 7387