# **Robust Teleportation of a Surface Code and Cascade of Topological Quantum Phase Transitions**

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Teleportation is a facet where quantum measurements can act as a powerful resource in quantum physics, as local measurements allow us to steer quantum information in a nonlocal way. While this has long been established for a single Bell pair, the teleportation of a many-qubit entangled state using nonmaximally entangled resources presents a fundamentally different challenge. Here, we investigate a tangible protocol for teleporting a long-range entangled surface-code state using elementary Bell measurements and its stability in the presence of coherent errors that weaken the Bell entanglement. We relate the underlying threshold problem to the physics of anyon condensation under weak measurements and map it to a variant of the Ashkin-Teller model of statistical mechanics with Nishimori-type disorder, which gives rise to a cascade of phase transitions. Tuning the angle of the local Bell measurements, we find a continuously varying threshold. Notably, the threshold moves to infinity for the X + Z angle along the self-dual line—indicating that infinitesimally weak entanglement is sufficient in teleporting a self-dual topological surface code. Our teleportation protocol, which can be readily implemented in dynamically configurable Rydberg-atom arrays, thereby gives guidance for a practical demonstration of the power of quantum measurements.

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### I. INTRODUCTION

The basis for *fault-tolerant* quantum computation platforms are logical qubits that, built from many physical qubits, leverage long-range entanglement and topological protection to store quantum information [1,2]. One widely adopted blueprint for their implementation is the surface code [1,3], which, like the toric code, employs two commuting stabilizer measurements to induce a topological state of matter [4]. Its fault tolerance arises from the ability to perform quantum error correction based on the measurement outcomes of the stabilizers (the so-called syndromes) and is embodied in a finite error threshold against incoherent noise such as nondeterministic Pauli errors [5]. Going beyond a protected quantum memory, one of the most elementary building blocks for quantum information processing will be the *teleportation* of a logical qubit, e.g., the spatial transfer of quantum information nonlocally such that it can be employed in a quantum circuit, akin to loading a classical bit into a processor register. But while the teleportation of a single physical qubit is well studied both theoretically [6] and experimentally [7–9], the teleportation of a logical state [10-15] supported through many qubits is a nontrivial challenge. This raises a fundamental question: How much entanglement is needed to teleport an entire topological surface code, including its long-range entanglement pattern that organizes the physical qubits [16], and the logical qubits that they represent. If this task were to be confined to the teleportation of a pristine manyqubit wave function, any source of decoherence would immediately make it unattainable. So the real question should be how one can preserve not the wave function but the associated (topological) phase, such that the quantum information of a logical qubit remains protected during teleportation even in the presence of decoherence.

In this paper, we address these questions by introducing a protocol for the teleportation of a many-qubit surface-code state and demonstrate its ability to transfer a logical qubit. Introducing a source of coherent errors [17] (by weakening the Bell measurements), we determine its robustness, threshold behavior, and optimal performance.

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We recast these results in a many-body context by connecting the error threshold to an anyon-condensation transition out of a topologically ordered quantum phase, which also gives an intuitive understanding of the remarkable robustness for certain Bell-measurement angles. We provide additional analytical insights via a mapping of the problem to a classical Ashkin-Teller model [18] with random (non-Hermitian) couplings, the statistical mechanics of which is reminiscent of Nishimori physics [19] in the random-bond Ising model (RBIM). On a conceptual level, our work goes beyond the widely studied phenomenology of Clifford decoherence and highlights the effect of non-*Clifford* decoherence on the quantum many-body physics of a long-range entangled state, thereby shedding light on the stability of topological order in mixed states [20–24], making a connection with the physics of coherent errors [25–28] and weak measurements [29–41], as well as generalized wave-function deformations [42–45]. With an eye toward experimental realization, we believe that configurable Rydberg-atom arrays [46-48] can readily implement our protocol, controllably inject the coherent errors, and establish its fault tolerance.

## **II. TELEPORTATION PROTOCOL**

The prototypical protocol for nonlocal quantum state teleportation [6] between two points in space requires three principal ingredients. First, it needs Bell pairs to establish entanglement at arbitrary distance. Second, quantum measurements enter with two consequences, steering the flow of quantum information from source to target and collapsing the initial wave function into a classical state. Finally, it requires a classical communication channel to transfer the measurement outcomes from the source to the target location, which is necessary to perform a round of corrections on the transferred quantum state (such as local-qubit or basis rotations).

A schematic of our protocol for teleporting the quantum many-body state underlying a logical qubit is illustrated in Fig. 1(a), where we employ N Bell pairs to steer the information between the spatially separated N-qubit systems A (Alice) and B (Bob). A more concise formulation is given in the quantum circuit of Fig. 1(b). In the initialization stage, we prepare (i) a surface-code state by an encoder for Alice's qubits A and (ii) a bunch of N Bell pairs, between the ancilla system A' and Bob's qubits B. Concerning the degeneracy of the surface-code states, we maximally entangle the logical state with a reference qubit R [49], such that when R is traced out, the surface code becomes maximally mixed in the logical space. To perform the teleportation, we (spatially) align Alice's qubits A with the ancilla qubits A' on a one-to-one basis and perform a rotated Bell-pair measurement for each pair, which is carried out by first applying transversal unitary  $R_{XX}$  =  $e^{-i(\pi/4-t)X_1X_2}$  gates (Ising interaction evolution) [48] and then projectively measuring both in the Pauli Z basis. It is in this entangling step that we introduce a tuning knob (indicated by the orange circle), which allows us to go to an imperfect entangling time (rotation),  $0 < t < \pi/4$ , that weakens the entanglement between A and A'. As a result, Alice effectively measures a weakly entangled pair,  $(1 - i \tan(\pi/4 - t)X_1X_2) |ss'\rangle$ , depending on the two measurement outcomes  $s, s' = \pm 1$ . Here, the weakly entangled Bell pair breaks the SU(2) symmetry down to U(1) with an axis dependence, where the axis can be rotated by a unitary  $U_{\theta,\phi}$  from the Pauli Z to an arbitrary direction on the Bloch sphere,  $\hat{\sigma}^{\theta,\phi} = \sin(\theta)\cos(\phi)X + \sin(\theta)\sin(\phi)Y +$  $\cos(\theta)Z$ , characterized by angles  $\theta$  and  $\phi$ . An example is  $| \nearrow \nearrow \rangle - i \tan(\pi/4 - t) | \swarrow \checkmark \rangle$  for  $\theta = \pi/4, \phi = 0$  rotating Z to Z + X, with  $\nearrow (\swarrow)$  denoting the positive (negative) eigenstate of Z + X. We find that the teleportation channel with such weakly entangled Bell pairs is given, up to a local unitary correction (see Appendix B 1 for additional information), by the following Kraus operator:

$$M_s = \exp\left(\frac{\beta}{2}s\hat{\sigma}^{\theta,\phi}\right)/\sqrt{2\cosh(\beta)},\tag{1}$$

where  $s = \pm 1$  indicates Alice's measurement outcome and  $\beta = \tanh^{-1} \sin(2t)$  characterizes the effective measurement strength. This channel can alternatively be interpreted as a weak-measurement channel, due to the imperfect Bell measurement. On a conceptual level, these measurement gates implement a nonunitary local wavefunction deformation of the underlying topological state [42–45,50–53]. Note also that  $M_s \propto (1 + \tanh(\beta/2)s\hat{\sigma}^{\theta,\phi})$ is a superposition of the identity and Pauli operators, which maps the density matrix onto  $M_s \rho M_s^{\dagger} \propto \rho + \tanh^2(\beta/2) \hat{\sigma}^{\theta,\phi} \rho \hat{\sigma}^{\theta,\phi} + s \tanh(\beta/2) \{\hat{\sigma}^{\theta,\phi}, \rho\}$ , where the off-diagonal terms render it distinct from the conventional Clifford Pauli errors.

Due to the quantum no-cloning theorem [54], the logical information either successfully flows to B or leaks to the measurement outcomes of A. When t = 0, a perfect Bell measurement is performed (which does not extract the logical information but, rather, propagates it across space) and the surface-code state is successfully teleported to B, visualized as an information flow through the wire of the circuit from A to B. When  $t = \pi/4$ , A is decoupled from B, and the measurement collapses every qubit in the surface code, such that the information gets pumped out to the measurement outcomes of A and cannot flow into B. When  $0 < t < \pi/4$ , the variable strength can turn the teleportation of the surface code on and off, and the teleportation will be shown to exhibit topological quantum phase transitions. However, the post-teleportation state depends on the measurement outcomes s (a bit string of A qubits):

$$|\Psi(\mathbf{s})\rangle := \frac{M_{\mathbf{s}} |\Psi\rangle}{\sqrt{P(\mathbf{s})}},$$
 (2)



FIG. 1. The teleportation of a logical-qubit or surface-code phase. (a) The schematics of our teleporting protocol for N-qubit systems from A to B. Starting from a surface-code state in A, it requires preparation of N Bell pairs between auxiliary system A' and B, followed by subsequent Bell measurements for A and A'. The shaded arrow indicates the flow of information. (b) The quantum circuit. An encoder prepares a surface-code state for A qubits, the logical qubit of which is maximally entangled with a reference qubit R (via the green wire), while preserving the logical information. A' and B are initialized in Bell pairs. The Bell measurement is performed by entangling A and A' via a unitary  $R_{XX}$  gate and subsequent measurements of A and A'. A coherent error is introduced via the parameter t, which further depends on the Bell-measurement angles  $\theta$  and  $\phi$  induced by single-qubit rotations on the Bloch sphere  $U_{\theta,\phi} =$  $\exp\{-iZ\phi/2\}\exp\{-iY\theta/2\}$  (orange circle). The shaded arrow corresponds to the information flow in (a) (if one remains below the threshold  $t < t_c(\theta, \phi)$ ). (c) The phase diagram in the XZ plane with a cascade of transitions for the ensemble of postmeasurement states. The duality is equivalent to a Hadamard transformation that swaps the Z and X axes, which yields a symmetric phase diagram. The origin is the fixed-point surface-code state for every possible measurement outcome. The blue-shaded regions represent the topological surface-code phases with the protected code space maintaining 1 bit of *n*th-order Rényi coherent information Eq. (7). The varying shades of blue correspond to different fractions of the ensemble, separated by the red critical lines of varying replicas  $n = 2, ..., \infty$ . The transition line (in blue) serves as the optimal phase boundary, beyond which the von Neumann coherent information in Eq. (4) decays exponentially with the code distance. The replica index serves as a "lens" that can focus on the more probable configurations of the measurement outcomes. Compared with the n = 1 case that averages all configurations according to the Born rule, the n = 2 (outer red line) phase boundary falls into the Ising universality class, except for the four-state Potts point at  $t = \pi/4 = \theta$ . The  $n = \infty$  (inner red line) phase boundary describes the postselected s = +1 pure state among the ensemble, which is a cleanly deformed surfacecode state. At this phase boundary of  $n = \infty$ , the two Ising critical lines merge into a Kosterlitz-Thouless critical point at  $t = \pi/8$ and  $\theta = \pi/4$ , and open up a gapless critical line for  $\theta = \pi/4$ ,  $t > \pi/8$ . The dots are analytical or numerical data points. (d) The phase diagram not only describes a teleportation protocol with coherent errors but also surface code under weak measurement and wave-function deformations resulting in topological transitions.

where  $|\Psi\rangle := \frac{1}{\sqrt{2}}(|\psi_+\rangle_B |+\rangle_R + |\psi_-\rangle_B |-\rangle_R)$ , with  $\psi_{+(-)}$  denoting the two degenerate surface-code states, as eigenstates for the logical  $\hat{X}_L$  operator. The normalization constant  $P(\mathbf{s}) = \langle \Psi | M_{\mathbf{s}}^{\dagger} M_{\mathbf{s}} | \Psi \rangle$  is the probability of a measurement outcome according to the Born rule [29]. When all possible measurement outcomes are collected, with A' traced out, the global state is a *block-diagonal* mixed state:

$$\rho_{RAB} = \sum_{\mathbf{s}} P(\mathbf{s}) |\Psi(\mathbf{s})\rangle \langle \Psi(\mathbf{s})| \otimes |\mathbf{s}\rangle \langle \mathbf{s}|_{A}.$$
(3)

Under such effective decoherence induced by coherent error, the state remains topologically ordered if and only if it maintains a protected two-dimensional (2D) code space, which means that there exist two locally indistinguishable but global orthogonal states (akin to the degenerate ground states of a topological Hamiltonian [1]) in the thermodynamic limit of large code distances,  $d \rightarrow \infty$ . The size of the protected code space can be detected by the coherent information [20,49,55–57], which for  $\rho_{RAB}$  is

$$I_c = S_{RA} - S_A = S_{AB} - S_{RAB} = \sum_{\mathbf{s}} P(\mathbf{s}) S_B(\mathbf{s}).$$
 (4)

One can give physical meaning to this formula in three different ways: (i)  $S_{RA} - S_A$  as a conditional entropy expresses the quantum information of *R* being subtracted by the leakage into *A*, where *A* plays a role analogous (but not identical) to the environment; (ii)  $S_{AB} - S_{RAB}$  expresses the quantum information that *B* can decode with the assistance of classical information (measurement outcomes) from *A*; or (iii)  $S_B(\mathbf{s}) = S_R(\mathbf{s})$  is the von Neumann entropy of the *logical* qubit for each measurement outcome, which quantifies the size of the uncorrupted quantum code space, the average of which yields the coherent information. To calculate this quantity, note that the reduced density matrix of *R* can be derived by projecting  $M_{\mathbf{s}}^{\dagger}M_{\mathbf{s}}$  onto the logical

space of *B* [58],

$$\rho_R(\mathbf{s}) = \frac{1}{2P(\mathbf{s})} \begin{pmatrix} P_{++}(\mathbf{s}) & P_{+-}(\mathbf{s}) \\ P_{+-}^*(\mathbf{s}) & P_{--}(\mathbf{s}) \end{pmatrix} \equiv \frac{1 + \vec{\kappa}(\mathbf{s}) \cdot \vec{\sigma}_R}{2},$$
(5)

where  $P_{\mu\nu}(\mathbf{s}) := \langle \psi_{\mu} | M_{\mathbf{s}}^{\dagger} M_{\mathbf{s}} | \psi_{\nu} \rangle$  is the overlap between two logical states being connected by the weakmeasurement operators and  $P(\mathbf{s}) = (P_{++}(\mathbf{s}) + P_{--}(\mathbf{s}))/2$ . Notably, Eq. (5) can be interpreted as a qubit subject to a *polarization-field* vector  $\vec{\kappa}$ . It expresses the precise logical error based on a fixed "syndrome"  $\mathbf{s}$ ,

$$\mathcal{E}(\rho_L) = \sqrt{\rho_R(\mathbf{s})} \rho_L \sqrt{\rho_R(\mathbf{s})},\tag{6}$$

for any state  $\rho_L$  in the logical space. A finite  $\kappa_{x(z)}$  compresses the logical Bloch sphere along the X(Z) axis (Fig. 1), with density eigenvalues  $(1 \pm \kappa(\mathbf{s}))/2$ . For sufficiently large field strength, this shrinks the Bloch sphere to a *classical* bit, which is read out by Alice—indicating the teleportation phase transition mapped out in Fig. 1(c).

### III. TOPOLOGICAL DEGENERACY AND ANYON CONDENSATION

To understand the general shape of the phase diagram in Fig. 1(c), it is helpful to relate the breakdown of teleportation to the field-induced transition of the  $Z_2$  gauge-theory description underlying the surface code [45,59-67]. In this language, the surface code allows two types of elementary excitations: electric charge e and magnetic flux m particles, which due to their mutual semion statistics are referred to as anyons. For the surface-code open boundary condition [Fig. 1(a)], one can create two *e* particles and separate them away from each other, such that they disappear into the left and right e boundaries [68], which transforms the surfacecode state  $\psi_+$  into  $\psi_-$ , that are locally indistinguishable but globally orthogonal, yielding a twofold topological degeneracy-this is the logical-qubit space. Under small deformations as given in Eq. (2), the states  $M_{\rm s} |\psi_{+}\rangle$  and  $M_{\rm s} |\psi_{-}\rangle$  remain asymptotically orthogonal despite their anyon excitations starting to fluctuate. For large deformations, however, they become indistinguishable and the topological order breaks down. This phase transition is driven by the condensation of the anyons [69]. When an e particle is condensed [67],  $M_s^{\dagger}M_s$  can map  $\psi_+$  to  $\psi_-$ , leading to nonzero  $\kappa_z$ , which quantifies the *e*-condensation fraction. When an *m* particle is condensed, the *e* particles must be confined due to destructive interference with m. As a result, either  $M_{\rm s} |\psi_+\rangle$  or  $M_{\rm s} |\psi_-\rangle$  has an exponentially decaying norm and is ill defined, which is signaled by nonzero  $\kappa_x$ . In this aspect, the coherent information in Eq. (4) serves as a single order parameter that collects the deconfinement and the uncondensation contribution.

## IV. MIXED-STATE PHASE DIAGRAM OF BORN AVERAGE

In Fig. 1(c), the thresholds or critical points are shown as the blue dots forming the blue line, inside which the entire inner blue-shaded region represents the topological phase, where the average postmeasurement state maintains the protected code-space or topological degeneracy and constitutes a coherent superposition of loops [1]. In contrast, the states above the threshold decohere into a classical loop gas [22], the loops of which are indicated by the negative measurement outcomes  $\{s = -1\}$ . The two phases are separated by a phase transition, the precise location of which is obtained by a finite-size scaling collapse [70] of the coherent information in Eq. (4), which we have computed using a hybrid Monte Carlo and tensor network technique for shallow circuit sampling [29,71] (see Fig. 2). Changing the Bloch angle  $\theta$ , the threshold  $t_c$  is found to vary from a (Nishimori) transition with threshold  $t_c \approx 0.143\pi$  [19,29] for  $\theta = 0$  off to  $t_c = \pi/4$ , ( $\beta = \infty$ , an " $\infty$  threshold") for  $\theta = \pi/4$ , i.e., along the X + Z Bloch projection. To understand this, note that for finite  $\theta$ , both e and m particles are fluctuating and compete with each other. As a result,



FIG. 2. Coherent information and teleportation transition. Shown are two sets of traces for  $\theta = 0$  (Z direction) in blue and  $\theta = \pi/4$  (X + Z self-dual direction) in red. The vertical gray lines indicate the thresholds, obtained from finite-size scaling analysis, with data collapses shown in the insets. (i)  $\theta = 0$ :  $t_c/\pi = 0.143(1)$  and  $\nu = 1.6(1)$ , consistent with Nishimori criticality. The gauge symmetry of the Nishimori line allows us to uncorrelate the disorder and perform random sampling, where for each sample we perform tensor-network contraction for the coherent information, simulating code distances up to d = 32(1985 qubits), averaged over 1000-10000 random samples. (ii)  $\theta = \pi/4$ :  $t_c/\pi = 0.25(1)$  and  $\nu = 1.8(1)$ . Without explicit gauge symmetry for the disorder ensemble, we Monte Carlo sample the disorder and subsequently contract out the tensor network [29]. This hybrid approach allows us to simulate codes up to d = 16(481 qubits), averaged over 200–1000 random samples.

it takes stronger deformation to achieve anyon condensation, which explains the enhancement of the threshold by deviating  $\theta$  from 0 or  $\pi/2$ . When  $\theta = \pi/4$ , a higher symmetry emerges as the state remains invariant under Hadamard transformations that swap  $Z \leftrightarrow X$  for every physical qubit-this is the electric-magnetic self-duality [45,60,63,72], which along this line is respected not only by the pristine surface code (t = 0) but also all deformations (t > 0). As a consequence, the frustration from competing anyon condensation is strongest along this line and, as revealed in our calculations, pushes the threshold all the way to infinity. This implies a remarkable robustness of the teleportation protocol along this line. In reverse, it means that teleportation of the topologically ordered many-qubit state between Alice and Bob is successful with only infinitesimally weakly entangled pair resources-the critically entangled pair that we need to measure for a finite-size system is  $|\mathcal{I}\mathcal{I}\rangle + \mathcal{O}(d^{-1/\nu}) |\mathcal{I}\mathcal{I}\rangle$  (taking, e.g., s = s' = +1). For any experimental realization of surface code teleportation, this is thus the optimal angle. When recast in terms of weak measurement, this result tells us that a self-dual surface code is most robust against decoherence.

### V. REPLICAS AND CASCADE OF TRANSITIONS

To shed light on the ensemble of post-teleportation states, consider a Rényi variant of the coherent information,

$$I_{c}^{(n)} = \frac{1}{1-n} \ln \frac{\operatorname{tr}(\rho_{RA}^{n})}{\operatorname{tr}(\rho_{A}^{n})} = \frac{1}{1-n} \ln[\operatorname{tr}\rho_{R}(\mathbf{s})^{n}]_{n}, \quad (7)$$

which on the right-hand side is given as the logarithm of the average nth-order purity of the reference qubit. The nreplica average  $[\cdots]_n := \sum_{\mathbf{s}} P(\mathbf{s})^n \langle \cdots \rangle / \left( \sum_{\mathbf{s}} P(\mathbf{s})^n \right)$  can be viewed as the linear average over n replicas of the system carrying the same disorder. Compared with the Born (one-replica) average discussed above, the *n*-replica average enhances the contribution of states with higher probability. The numerically computed two-replica threshold is shown as the outer red line and dots in Fig. 1(c) and is generally smaller than the one-replica threshold for varying angles—with the exception of the  $\infty$  threshold at the self-dual angle  $\theta = \pi/4$ , which is preserved in the two-replica system. Unlike Eq. (4), the Rényi coherent information of the mixed state is distinct from the averaged *pure-state* Rényi entropy [73]. We conjecture that the tworeplica phase boundary will lower bound the one-replica and upper bound the higher-replica phase boundaries, for which a rigorous proof along the Z and X axes can be adapted from Ref. [74].

In the  $\infty$ -replica limit,  $P(\mathbf{s})^n$  distills out only those configurations **s** that have the highest probability [20,74], which usually postselects the clean frustration-free configuration  $\mathbf{s} = +1$  so as to minimize the energy, in the

language of statistical mechanics. This reduces Eq. (1) to a clean deformation operator, which can be treated analytically [45], and yields the inner phase boundaries marked by the red and purple lines in Fig. 1(c). The  $\infty$  replica generally exhibits the smallest threshold compared with n = 2 and n = 1, pointing to a *cascade of phase transitions* where higher-probability states generally have smaller thresholds. Note that along the self-dual line  $\theta = \pi/4$ , the  $\infty$  replica stands out. It does not exhibit the same level of robustness as found for n = 1, 2 but it exhibits a finite threshold at  $t_c = \pi/8$ , beyond which the system exhibits a critical line (described by a c = 1 conformal field theory with varying critical exponents) [45].

### VI. STATISTICAL MODEL

To gain insight into the nature of the phase transition for generic *n*, we now proceed to map the quantummechanical problem to a classical statistical model, akin to the approach of Ref. [29] (for Nishimori cat states). Following the Born rule, the probability function of the measurement outcome  $P(\mathbf{s})$  is identical to the wave-function amplitude, which here can be cast into a classical statistical model for two layers of spins (residing at the vertices)—dual to the surface-code wave-function ket and bra, respectively. As  $P \equiv \sum_{\sigma,\tau} \exp\left(-\sum_{\langle ij \rangle} E_{ij}\right)$ , the pairwise spin interactions are

$$-E_{ij} = Js_{ij}\frac{\sigma_i\sigma_j + \tau_i\tau_j}{2} + i\phi\frac{\sigma_i\sigma_j - \tau_i\tau_j}{2} + \left(2K + i\pi\frac{1 - s_{ij}}{2}\right)\frac{\sigma_i\sigma_j\tau_i\tau_j - 1}{2}, \quad (8)$$

with coupling strengths  $tanh(J) = sin(2t) cos(\theta)$  and  $e^{-2K} = sinh(J) tan(\theta)$  (see Appendix B 2 for additional information). This is an Ashkin-Teller model [18] with generalized intra- and interlayer couplings. First, there is a non-Hermitian term for finite Bloch angle  $\phi$ , i.e., when we consider a general deformation with Pauli *Y* operators. We defer a discussion of this case to future work. Second, the interlayer coupling exhibits random bond disorder introduced by the random measurement outcomes  $s_{ij}$ . For a general dictionary of the underlying quantum-classical correspondence, we refer to Table I. The coherent information is then effectively determined by the boundary correlation (see Appendix B 3 for additional information).

In the classical model, the electric-magnetic duality of the quantum model turns into a Kramers-Wannier duality. This is most transparent in the eight-vertex representation of the Ashkin-Teller model [see Fig. 3(a)], where it is equivalent to swapping the horizontal and vertical gates. For the Hermitian case ( $\phi = 0$ ), only six-vertex configurations appear [see Fig. 3(b)] and the transfer matrix of each slice describes a quantum *XXZ* chain of 2d + 2 spins

TABLE I. The quantum-classical correspondence between the wave-function and the statistical model [45]. The ordering of classical spins  $\sigma$  or their dual spins  $\mu$  corresponds to the Higgs or confinement phase transition for the quantum wave function, respectively.

$(2+0)\mathrm{D}  \psi(\mathbf{s})\rangle$	2D Ashkin-Teller model	(1 + 1)D XXZ chain
e	σ	
т	$\mu$	
$P_{}/P_{++}$	$\langle \sigma_0  au_0 \sigma_d  au_d \rangle$	$-\langle Z_0 Z_{2d+1} \rangle$
$P_{+-}/P_{++}$	$\langle \sigma_0 \sigma_d \rangle$	$\langle X_0 X_{2d+1} \rangle$

with randomness. In such an *XXZ* representation, the logical operator becomes simply the correlation between the boundary spins (Table I).

These general quantum-classical mappings offer several merits. First, the numerical exploration of the phase diagram is considerably more affordable in the Ashkin-Teller, and particularly the XXZ, representation. Second, by recasting the various thresholds or phase boundaries in terms of classical transitions, we can infer their universality classes. For the single-component Z (X) transitions along the  $\theta = 0, \pi/2$  directions in our phase diagram, we can rigorously identify the one-replica transition to be the Nishimori transition [19,29,30,33] of the 2D RBIM,



FIG. 3. The effective (1 + 1)D nonunitary circuit and the 2D statistical model. (a) The surface-code state under decoherence can be mapped to a nonunitary (1 + 1)D circuit, by viewing one spatial dimension as fictitious "time." Each physical qubit in the surface code is mapped to a gate, where the gate elements depend on the measurement outcome at the same location. Their rectangular shape allows us to distinguish the gates on the horizontal versus vertical bonds. When  $tan(\theta) cos(\phi) = 1$ , the gate is self-dual: rotating the gate by  $90^{\circ}$  leaves it invariant and the network becomes invariant under vertex-plaquette duality, consistent with the electric-magnetic self-duality of the surface code. (b) The eight nonzero gate elements for the corresponding input and output spin configurations, which define a random eightvertex model. The duality swaps the second and third rows of vertices. When  $\phi = 0$ , the bottom two elements drop out and the system reduces to a random six-vertex model.

while the two-replica and  $\infty$ -replica transitions are nonrandom 2D Ising transitions [20,45]. In the asymptotic limit  $t \rightarrow \pi/4$ , the *n*-replica model is, for all Bloch angles  $\theta$ , an  $S_{2n}$  permutation symmetric model (see Appendix B 4 for additional information) with Kramers-Wannier duality, driven by the one-parameter coupling constant J = $\tanh^{-1}\cos(\theta)$ . At  $\theta \ll \pi/4$ , every layer is ordered independently and  $S_{2n}$  is spontaneously broken. For the tworeplica case, this leads us to conjecture that the Ising lines emanating from the  $\theta = 0$  transitions meet in a fourstate Potts point (at  $\theta = \pi/4, t_c = \pi/4$ ), akin to what happens in the  $\infty$ -replica case at finite threshold ( $\theta =$  $\pi/4, t_c = \pi/8$ ), where J = 2K gives rise to  $S_4$  symmetric energetics  $-K(\sigma_i\sigma_i + \tau_i\tau_i + \sigma_i\sigma_i\tau_i\tau_i)$ . This conjecture is corroborated by numerical simulations yielding a central charge estimate  $c \approx 1$  from entanglement scaling (see Appendix A 1 for additional information).

## VII. DISCUSSION AND OUTLOOK

Expanding our view, when the error preserves the selfduality of the surface code, the mutual frustration of anyon condensation points to a general guiding principle to dramatically enhance the code threshold. Beyond our work here, this is corroborated by the "ultrahigh threshold" of the surface code under incoherent Y noise [75–77] and that under random projective Pauli measurements [78–85] where self-duality is fulfilled on average [86,87]. The connection between percolation criticality for the latter case and the Nishimori transition reported here is left for future study [88]. In comparison with the self-dual Hamiltonian phase diagram of the 2D toric code or the  $Z_2$  lattice gauge theory [60,62–67], our surface code under imperfect teleportation or weak measurement exhibits a wave-function phase diagram with distinct topology: while the (e-condensed) Higgs phase and the (*m*-condensed) confinement phase can be adiabatically connected beyond the finite extent of a first-order transition line in the Hamiltonian phase diagram, in our wave-function phase diagram they are always separated by critical points [Fig. 1(c)]. Besides, the Hamiltonian criticality is usually three-dimensionally (3D) conformal symmetric [43,62,66], whereas the wave-function criticality typically exhibits a dimensional reduction and belongs instead to 2D conformal criticality [42-45]. This dimensional reduction is related to the fact that the quantum circuit of the whole protocol is *finite depth* in its time dimension.

To compare our protocol to other teleportation schemes, it is useful to consider their required resources. Our protocol needs a set of N independent Bell pairs, reflecting that it is a straightforward generalization of the standard teleportation protocol [6] from few-body to many-body, where Bob's qubits are just a maximally mixed *product* state when others are traced out. This Bell-state approach can be compared with (i) the measurement-based quantum computation (MBQC) scheme [10–12,89], which prepares a joint *cluster state* for Alice and Bob, or (ii) the transversal approach [48] or the lattice-surgery approach [13,14], where Bob prepares a *long-range entangled* surface code beforehand. Because of its minimal request for Bob, our protocol is also highly suitable for a *highly nonlocal* transfer of a surface code and thus amenable for quantum communication or distributed topological quantum computing architectures.

Despite the microscopic difference of the various teleportation protocols, they share some commonalities on the level of their collective many-body physics. This includes, e.g., that-independent of the type of teleportation protocol-the resource state prior to measurement exhibits symmetry-protected topological order [90]. Going further, one can even recast the existence of an error threshold and the nature of the ensuing phase transition between different protocols. Let us do this by restating our results for the Bell-state teleportation protocol in terms of the cluster-state-based MBQC approach. For our protocol, we can show that if the resource Bell pairs are imperfectly prepared, e.g., by weakly entangling gates, this will not immediately impede teleportation (see Appendix B 1 for additional information), as Bell measurement and Bell preparation play dual roles [91] in fostering a quantum channel across space [6]. This is what ultimately gives rise to the phase diagram of Fig. 1(c). Notably, one can also cast this phase diagram for the imperfectly prepared Bell-state teleportation protocol as a statement about the fault tolerance of the imperfectly prepared cluster-statebased MBQC approach [10–12] in the presence of coherent errors. To see this, consider two layers of the 3D Raussendorf cluster state [11], as illustrated in Fig. 4,



FIG. 4. Cluster-state-based teleportation with coherent errors. Shown is a 3D Raussendorf lattice [10], where the blue dots in the top layer are interpreted as Alice's qubits, while the hollow circles in the second layer are Bob's qubits. For convenience, we modify the conventional cluster state by a Hadamard transformation into  $\prod_{\langle ij \rangle} e^{-i(\pi/4)X_iX_j} |0\rangle^{\otimes N}$ . The coherent errors *t* weaken the otherwise maximally entangling gate  $R_{XX}$  into  $\exp\{-i(\pi/4 - t)XX\}$ . An additional unitary  $U_{\theta,\phi}$  rotation is introduced before the entangling. The transferred wave function suffers from the same nonunitary error  $M_s$  as in Eq. (1).

where a measurement by Alice (on the top layer) allows the surface code to be transferred to Bob (on the second layer). When one now considers a scenario in which the cluster state is prepared in an imperfect manner by nonmaximally entangling gates, this not only reduces the entanglement of the resource state (akin to what we have considered for the Bell state approach) but one finds that the effective error takes precisely the form of the Kraus operator in Eq. (1) (for a detailed derivation, see Appendix B). Consequently, the ensuing thresholds for successful transfer of the logical state in the cluster-state-based approach are described in one-to-one correspondence by our phase diagram in Fig. 1(c).

Finally, let us mention that an experimental realization of many-qubit teleportation [92] can benefit from implementing an error-correction scheme. For this, one can follow two different paths, which we dub "active" versus "passive" teleportation. To decode Bob's deformed surface code, one can use Alice's information s as the syndrome-this is what we call "active teleportation." Here, one can then implement a tensor-network decoder [93]. i.e., one can run our tensor-network calculation for the random circuit in Fig. 3 to compute the precise logical error in Eq. (6), which can then be inverted by feedback operation or postprocessing [Fig. 5(a)]. The teleportation transition can alternatively be diagnosed by investigating whether Alice learns half of the logical information from s via a decoder (see Appendix C for additional information), which causes the logical qubit to collapse to a logical classical bit, similarly to the scalable decoder for measurement-induced entanglement phase transitions [56] or learnability transition [94].



FIG. 5. Active versus passive teleportation. (a) Active teleportation: Alice sends her measurement outcome **s** to a classical computer that runs the tensor-network calculation to deduce the error on the logical qubit. The result is fed to Bob such that Bob can correct the code accordingly. (b) The threshold comparison between the active- and passive-teleportation protocols for  $\theta = 0$ ,  $\phi = 0$ .

Alternatively, if we do not use Alice's information s as the syndrome-a scenario that we dub "passive teleportation," different pure states  $\Psi(\mathbf{s})$  mix together, equivalent to an incoherent Pauli-noise channel (where the off-diagonal terms are erased, akin to the Pauli twirling for coherent errors [25,26]):  $\mathcal{N}(\rho) = (1-p)\rho + p\hat{\sigma}^{\theta,\phi}\rho\hat{\sigma}^{\theta,\phi}$  with p = $\sin^2(t)$ . Such "ignorance" of the measurement outcomes is generally expected to lead to smaller thresholds. Take, e.g.,  $\theta = 0, \phi = 0$  [Fig. 5(b)]: with Alice's knowledge, the RBIM with disorder probability  $p = \sin^2(\pi/4 - t)$  gives a threshold at  $t_c \approx 0.143\pi$ ; in contrast, without Alice's knowledge, the mixture is subjected to dephasing noise [5] described by an RBIM with disorder probability  $\sin^2(t)$ , which yields a threshold of  $t_c \approx 0.107\pi$  instead. The cascade of transitions for such passive teleportation is thus an example of the recently explored notion of *mixed*state topological order transitions [20,21,23,24] driven by anyon condensation in the double Hilbert space—as opposed to anyon condensation in the single Hilbert space for active teleportation. By tuning the "degree of ignorance," one can join the active- and passive-teleportation transitions unifying the two anyon-condensation mechanisms, which we leave to future investigations.

The numerical data shown in the figures and the data for sweeping the phase diagram are available on Zenodo [95].

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*Note added.*—Recently, Ref. [96] has reported the toric code under self-dual Pauli noise, which is equivalent to our passive-teleportation protocol, where the authors show

a phase diagram qualitatively similar to ours for the tworeplica model; Ref. [97] has reported the experimental realization of a deformed surface code used in a parity game to demonstrate quantum advantage, the phase transition of which is identical to the Z axis in our phase diagram; and Ref. [98] has reported logical teleportation using trapped ions, where Alice and Bob have both prepared long-range entangled states before the teleportation protocol.

## APPENDIX A: SUPPLEMENTARY DATA FOR SELF-DUAL LINE X + Z

### 1. Two-replica model

Here, we investigate the two-replica XXZ quantum spin chains. Along the self-dual X + Z line,  $\sinh(J) = e^{-2K}$ ,



FIG. 6. The two-replica model along the self-dual line  $\mathbf{X} + \mathbf{Z}$ . (a) The second-order Rényi coherent information sweeps of the phase diagram. The matrix-product-state (MPS) virtual bonds are truncated by dropping the density eigenvalues  $\leq 10^{-10}$  and are bounded by the maximal bond dimension  $\chi = 256$ . (b) The criticality at  $t \rightarrow \pi/4$  by von Neumann entanglement entropy of the (1 + 1)D boundary MPS ( $\chi = 1024$ ). The data points are shown as dots, while the lines indicate a fit that reveals a central charge  $c \approx 0.94$ .

which simplifies each local transfer matrix into

$$T^{(2)} \propto \left(\frac{1 - Z_j Z_{j+1}}{2}\right)^{\otimes 2} + \frac{\sin^2(2t)}{2} \left(\frac{1 + \vec{\sigma}_j \cdot \vec{\sigma}_{j+1}}{2}\right)^{\otimes 2},$$
(A1)

for both horizontal and vertical gates (see Fig. 3). Note that  $(1 + \vec{\sigma}_i \cdot \vec{\sigma}_j)/2$  is a swap operator, the eigenstates of which are the spin singlet and triplets with eigenvalues  $\mp 1$ , respectively. As shown in Fig. 6, the second-order Rényi coherent information is computed by the two-replica average of the purity  $I_c^{(2)} = -\ln[\operatorname{tr} \rho_R^2]_2 = \ln 2 - \ln(1 + [\kappa^2]_2)$ , as a function of the two-point correlation of the two-replicated *XXZ* chain

$$[\kappa^{2}]_{2} = \frac{\sum_{s} (P_{++} - P_{--})^{2} + 4|P_{+-}|^{2}}{\sum_{s} (P_{++} + P_{--})^{2}}$$
$$= \frac{\langle (1 - \sigma_{0}\tau_{0}\sigma_{d}\tau_{d})^{\otimes 2} \rangle + 4\langle (\sigma_{0}\sigma_{d})^{\otimes 2} \rangle}{\langle (1 + \sigma_{0}\tau_{0}\sigma_{d}\tau_{d})^{\otimes 2} \rangle}$$
$$= \frac{\langle (1 + Z_{0}Z_{2d+1})^{\otimes 2} \rangle + 4\langle (X_{0}X_{2d+1})^{\otimes 2} \rangle}{\langle (1 - Z_{0}Z_{2d+1})^{\otimes 2} \rangle}, \quad (A2)$$

where from the second line to the third line we use the operator map in Eq. (B8) and the conserved quantity  $\prod X = +1$ ,  $\prod Z = (-1)^{d+1}$  of the spin chain for both copies. At the asymptotic limit  $t \to \pi/4$ , the entanglement



FIG. 7. The postselection ( $\infty$ -replica) model along the selfdual line **X** + **Z**. The gray lines indicate  $(t = \pi/8, I_c^{\infty} = 0.6 \ln 2)$ . Note the clear level crossing for different code distances in the plotted coherent information occurring for the Kosterlitz-Thouless transition at the four-state Potts point  $t_c = \pi/8$ . While Eq. (7) at the  $n \to \infty$  limit reduces to  $I_c^{\infty} = -\ln(1+\kappa)/2$ , here we show the von Neumann version,  $\tilde{I}_c^{\infty} = -(1+\kappa)/2 \ln(1+\kappa)/2 - (1-\kappa)/2 \ln(1-\kappa)/2$ , where the two are in one-to-one correspondence and behave similarly. The MPS cutoff is  $10^{-10}$  and  $\chi = 1024$ .

entropy of the boundary state conforms to the Calabrese-Cardy formula  $S_{vN} = c/6 \ln \sin(\pi l/(2d)) + \cdots$ , consistent with the c = 1 conformal field theory (CFT), which, combined with the  $S_4$  permutation symmetry, points to the four-state Potts CFT. Numerical details for Fig. 6(b) are as follows: we elongate the depth of the circuit from drows to 4d rows to ensure a steady boundary state and drop the leftmost and rightmost dangling qubits, leaving a chain of 2d qubits under the conventional open boundary condition. The entanglement cut is performed on the even cuts through the vertical gates.

#### 2. $\infty$ -replica model (postselection)

The local transfer matrix describes a clean XXZ spin chain

$$T^{(\infty)} \propto \left(\frac{1-Z_j Z_{j+1}}{2}\right) + \frac{\sin(2t)}{\sqrt{2}} \left(\frac{1+\vec{\sigma}_j \cdot \vec{\sigma}_{j+1}}{2}\right).$$
(A3)

The critical point lies at  $t_c = \pi/8$ , which opens a continuously varying critical line for  $t \ge \pi/8$  [45]. While the Kosterlitz-Thouless transition usually does not show a clear level crossing for the spin-wave stiffness, we find that the von Neumann coherent information of the postselected case does show a perfect crossing here (Fig. 7), which means that  $-\langle Z_0 Z_{2d+1} \rangle = \langle \prod_{j=1}^{2d} Z_j \rangle$  has zero scaling dimension.  $I_c$  approaches 1 exponentially fast for  $t < t_c = \pi/8$  below the threshold but converges quickly to a continuously varying finite constant for  $t > t_c$ .

### **APPENDIX B: SUPPLEMENTARY DERIVATIONS**

#### 1. Teleportation-circuit block

To understand the teleportation circuit, first consider a two-qubit  $R_{ZZ}$  rotation gate followed by measurement in the X basis,



where the  $R_{ZZ}$  and the X-measurement outcomes all play the role of attaching a phase factor to the four possible input and output states, framed in a nonunitary two-by-two matrix. This matrix transports the input qubit wire from the bottom left to the output qubit wire at the top right. Next, we perform a Hadamard rotation that swaps  $X \leftrightarrow Z$ :

Finally, we perform the simple on-site correction conditioned on the measurement outcomes (without the need for a decoder),



the two-body coherent error *t* can result from the singlebody rotation. Last but not least, we make a comment about the more general situation in which the Bell pairs are created imperfectly by an imperfect entangling gate  $R_{XX}$  up to angle  $\alpha'$ :

$$S = e^{-i\frac{\pi}{4}} \left( \cos(\alpha) \frac{1+sZ}{2} + \sin(\alpha) \frac{1-sZ}{2} \right)$$

$$e^{-i\alpha X \otimes X} = e^{-i\frac{\pi}{4}} \left( \cos(\alpha) \frac{1+sZ}{2} + \sin(\alpha) \frac{1-sZ}{2} \right),$$
(B1)

this state being akin to a noninteracting thermofield double state at finite temperature  $-1/\ln \tan(\alpha')$ . Combining the joint actions of imperfect Bell preparation and imperfect Bell measurement, the post-teleportation state suffers from the following effective weak-measurement gate operator:

that leads to a real nonunitary matrix as purely imaginary time evolution. This is equivalent to the weak measurement in the Z basis [29],



which can be summarized as normalized Kraus operator  $\exp[(\beta/2)sZ]/\sqrt{2\cosh\beta}$ , where  $\tanh(\beta) = \cos(2\alpha) = \sin(2t)$  denotes the effective measurement strength. Here, we define  $t = \pi/4 - \alpha$  to characterize the deviation from the perfect teleportation protocol, because  $\alpha = \pi/4$  corresponds to a perfect Bell-pair measurement that teleports the input qubit state into the output qubit without decoherence, which is consistent with t = 0 zero-measurement strength that decoheres the state. Note that the randomness of s' is simply corrected and drops out from the formula but the randomness of s cannot be fully undone. As the Ising-evolution gate can be decomposed into a single-body rotation sandwiched by controlled-NOT (CNOT) gates,



which is akin to what we discuss in the main text with minor adaptations. One can verify that (i) when  $\alpha' = \pi/4$  for perfect Bell pairs, it recovers Eq. (B1); (ii) when  $\alpha = \pi/4$  for perfect Bell measurement, it also recovers Eq. (B1) by replacing  $s \to ss'$  and  $\alpha \to \alpha'$ ; and (iii) when both Bellpair preparation and Bell-pair measurement are imperfect, the random disorder now depends on both *s* and *s'* quaternary rather than binary disorder, details of which we leave to future study.

For completeness, here we also derive a building block for the quantum state transfer without using Bell pairs but simply transferring a state from Alice to Bob via directly entangling and measuring Alice, as follows:



Here, we initiate Bob in a zero state and rotate Alice by a unitary transformation  $U_{\theta,\phi}$  that rotates  $U_{\theta,\phi}^{\dagger}ZU_{\theta,\phi} = \hat{\sigma}^{\theta,\phi}$ . Then, we entangle them with an  $R_{XX}$  gate over an angle  $\pi/4 - t$ , where  $t \in [0, \pi/4]$  specifies the strength of the coherent error. Afterward, we projectively measure out Alice and perform an on-site correction for Bob, by means of an  $S = \text{diag}(1, i) \propto e^{-i(\pi/4)Z}$  gate and a conditional Xgate. In total, the state in Alice is transferred to Bob experiencing an effective nonunitary gate, which can be interpreted as a weak measurement (due to coherent error), with  $\beta$  being the measurement strength. As a result, this state-transfer protocol also shares exactly the same phase diagram as we chart out in this paper.

#### 2. Tensor-network representation

The surface code can be written as a projected entangled pair state (PEPS) as illustrated in Fig. 8. By stacking two layers of PEPSs together and tracing out the physical legs, we obtain a 2D classical tensor network composed of a delta tensor at each vertex, joined by bond matrices as follows. First, we write down and simplify the doubled Kraus-operator matrix:

$$M_{s}^{\dagger}M_{s} = e^{\beta s(Z\cos\theta + \sin\theta(\cos\phi X + \sin\phi Y))} / (2\cosh\beta)$$
  
=  $\frac{1}{2\cosh J} \begin{pmatrix} e^{sJ} & se^{-2K}e^{-i\phi} \\ se^{-2K}e^{+i\phi} & e^{-sJ} \end{pmatrix}$ , (B4)

where we introduce J and K as functions of t and  $\theta$ :

$$tanh(J) = sin(2t) cos(\theta), \quad e^{-2K} = sinh(J) tan(\theta).$$
 (B5)

Then, we introduce the wave-function ket and bra:

$$\begin{aligned} |\psi\rangle &= \sum_{\{\sigma=\pm 1\}} \bigotimes_{ij} |Z_{ij} = \sigma_i \sigma_j\rangle, \\ \langle\psi| &= \sum_{\{\tau=\pm 1\}} \bigotimes_{ij} \langle Z_{ij} = \tau_i \tau_j |, \end{aligned} \tag{B6}$$



FIG. 8. The surface-code PEPS. Each solid dot denotes the diagonal delta tensor:  $T_{ijkl} = 1$  if and only if i = j = k = l[which represents a virtual Greenberger-Horne-Zeilinger (GHZ) state]; and each cross node denotes the off-diagonal tensor:  $T_{iik} =$ 1 if and only if  $i + j + k \mod 2 = 0$  (which represents a virtual GHZ state in the X basis:  $|+++\rangle + |---\rangle$ ). The left and right rough boundaries are captured by removing the dangling physical leg, such that the boundary forms a ferromagnetic GHZ chain, consistent with e condensation. The Z string acting on the physical legs can be *pulled through* the network to the virtual leg, which is further reduced into a two-point  $\sigma - \sigma$  operator connecting the two boundaries. Thus the logical qubit of the surface code that is changed by the Z string corresponds to the total Ising symmetry charge of the left and right GHZ chains. The top and bottom boundaries are smooth boundaries, where the *m* particle is condensed.

expressed as a dual paramagnet of classical spins  $\sigma$  and  $\tau$ . As a result,

$$\langle \psi | M_s^{\dagger} M_s | \psi \rangle \propto \sum_{\sigma, \tau} \exp\left(-\sum_{\langle ij \rangle} E_{ij}\right),$$
 (B7)

up to an s-independent constant prefactor, where  $E_{ij}$  is the generalized Ashkin-Teller model shown in the main text.

There are two alternative ways to further reduce the bilayer model into a single-layer vertex model or *XXZ* chain. One way is to perform duality for one layer only, resulting in the eight-vertex model discussed in the main text. An alternative way is to take a slice of the network as a quantum transfer matrix for 2(d + 1) spins subjected to Ashkin-Teller interactions, which can be rewritten in terms of *XXZ* interactions under the mapping

$$\sigma_{j}^{z}\sigma_{j+1}^{z} = X_{2j}X_{2j+1}, \quad \sigma_{j}^{x} = Y_{2j-1}Y_{2j}, \tau_{j}^{z}\tau_{j+1}^{z} = Y_{2j}Y_{2j+1}, \quad \tau_{j}^{x} = X_{2j-1}X_{2j},$$
(B8)

which preserves all Pauli commutation relations. Consequently, the interaction for the horizontal gates (see Fig. 3) is

$$-E_{2j,2j+1} = \frac{Js}{2}(X_{2j}X_{2j+1} + Y_{2j}Y_{2j+1}) + i\frac{\phi}{2}(X_{2j}X_{2j+1} - Y_{2j}Y_{2j+1}) -\left(K + i\frac{\pi}{2}\frac{1-s}{2}\right)(Z_{2j}Z_{2j+1} + 1), = Js(c_{2j}^{\dagger}c_{2j+1} + \text{h.c.}) + i\phi(c_{2j}c_{2j+1} + \text{h.c.}) -2\left(K + i\frac{\pi}{2}\frac{1-s}{2}\right)(2n_{2j}n_{2j+1} - n_{2j} - n_{2j+1} + 1),$$
(B9)

which is an antiferromagnetic anisotropic Heisenberg chain with  $\mathcal{PT}$ -symmetric non-Hermitian interactions, which is mapped to a complex fermion chain  $c_j$  with a non-Hermitian pairing term. Here,  $n_j = c_j^{\dagger} c_j$  denotes the fermion density. The matrix elements of  $\exp(-E)$  in the Z basis yield the Boltzmann weight for the corresponding vertex configuration, as shown in Fig. 3(b). The vertical gate can be obtained by rotating the horizontal gate, employing the duality of the model.

### 3. Coherent information cast in the classical model

For simplicity of explanation, let us first consider  $\theta = 0$ and  $\phi = 0$ , i.e., the *Z* direction, where  $P_{--} = P_{++}$  and the classical statistical model is a random-bond Ising model. The density matrix  $\rho_R$  describing the reference qubit is purely diagonal in the *Z* basis and classic: by adapting Eq. (5) from the Pauli *X* basis to the *Z* basis, we have  $\rho_R = \text{diag}(P_{+-}/P_{++}, P_{+-}/P_{++}),$  where, according to our dictionary (Table I in main text),  $P_{+-}/P_{++} = \langle \sigma_0 \sigma_d \rangle$ , which is the two-point correlation between the left and right boundaries. The classical model is placed on the following "fixed-boundary" geometry (inherited from the surface-code boundary condition):



where all spins on the left (right) boundary are fixed at identical values  $\sigma_0$  ( $\sigma_d$ ), respectively. In the original surface code, the "rough" boundary (illustrated above on the left-hand side) condenses the *e* particle on the left or right, thereby effectively gluing the whole boundary into one site (as shown on the right-hand side above). The parity of the left- and right-boundary sites can be used to define 1 bit,  $\kappa = \sigma_0 \sigma_d$ , the distribution function of which is  $(1 \pm \langle \kappa \rangle)/2$ . The classical entropy of this "boundary bit,"

$$S = -\frac{1+\langle\kappa\rangle}{2}\ln\frac{1+\langle\kappa\rangle}{2} - \frac{1-\langle\kappa\rangle}{2}\ln\frac{1-\langle\kappa\rangle}{2}, \quad (B11)$$

exactly equals the quantum coherent information of a given postmeasurement state with randomness. The coherent information of the mixed state is the random average of this entropy according to the Born probability. Importantly, this entropy should not be confused with the boundary entropy of the boundary CFT.

When deviating from the Z direction, the RBIM turns into a random Ashkin-Teller model in a similar geometry, where the coherent information is again determined by the correlation between the leftmost and rightmost sites in this special boundary condition. With two layers of classical spins, in general we can define a boundary quantum bit subject to the following "magnetic field"  $\vec{\kappa}$ :

$$\begin{aligned} \kappa_x &= \frac{P_{++} - P_{--}}{P_{++} + P_{--}} = \frac{1 - \langle \sigma_0 \tau_0 \sigma_d \tau_d \rangle}{1 + \langle \sigma_0 \tau_0 \sigma_d \tau_d \rangle}, \\ \kappa_z &+ i\kappa_y = \frac{2P_{+-}}{P_{++} + P_{--}} = \frac{2\langle \sigma_0 \sigma_d \rangle}{1 + \langle \sigma_0 \tau_0 \sigma_d \tau_d \rangle} \end{aligned}$$

Then, the density-matrix eigenvalues of this boundary quantum bit are  $(1 \pm \|\vec{\kappa}\|)/2$ , the corresponding entropy of which is again equivalent to the coherent information of the quantum system.

#### 4. Rényi coherent information and replica symmetry

The *n*th-order Rényi coherent information in the main text can be easily derived by tracing out *B*, resulting in the

reduced density matrices for RA and A:

$$\rho_{RA} = \sum_{\mathbf{s}} P(\mathbf{s}) \cdot \rho_{R}(\mathbf{s}) \otimes |\mathbf{s}\rangle \langle \mathbf{s}|, \quad \rho_{A} = \sum_{\mathbf{s}} P(\mathbf{s}) |\mathbf{s}\rangle \langle \mathbf{s}|,$$
(B12)

whose conditional *n*th-order Rényi entropy is determined by the *n*-replica of the classical random Ashkin-Teller model composed of 2*n* layers of Ising spins, denoted by a flavor index  $\alpha = 1, ..., 2n$ . For the asymptotic limit  $\phi = 0, t \rightarrow \pi/4$  for any  $\theta$  [the circular edge of our phase diagram in Fig. 1(c)], the Ashkin-Teller coupling between every two layers vanishes,  $K \rightarrow 0$ , leaving only the phase factor  $s^{(1-\sigma_i\sigma_j\tau_i\tau_j)/2}$ . Consequently, there is an emergent  $S_{2n}$ permutation symmetry in the flavor space:

$$\sum_{\mathbf{s}} P(\mathbf{s})^n = \sum_{\sigma} \sum_{\mathbf{s}} \exp\left\{\sum_{\langle ij \rangle} \frac{J}{2} s_{ij} \sum_{\alpha=1}^{2n} \sigma_i^{\alpha} \sigma_j^{\alpha} + i\pi \frac{1 - s_{ij}}{2} \frac{1 - \prod_{\alpha=1}^{2n} \sigma_i^{\alpha} \sigma_j^{\alpha}}{2}\right\},$$
(B13)

where the 2*n* Ising layers are only coupled by an  $S_{2n}$ -symmetric interaction. The  $S_{2n}$ -symmetric model is self dual at  $\theta = \pi/4$ . For example, the two-replica model has emergent  $S_4$  permutation symmetry and self-duality at ( $t = \pi/4, \theta = \pi/4, \phi = 0$ ), which is expected to be described by the four-state Potts CFT, the same as the postselection  $\infty$ -replica at ( $t = \pi/8, \theta = \pi/4, \phi = 0$ ).

### APPENDIX C: EXPERIMENT AND QUANTUM ERROR CORRECTION

We believe that our protocol is amenable to a direct implementation in reconfigurable Rydberg-atom arrays [46-48]. In this appendix, we address how to decode the teleportation transition in such experimental implementations.

As the logical qubit cannot be cloned, the basic idea is to investigate whether it falls into Alice's hand or Bob's hand. The teleportation succeeds if Bob has the key and fails if Alice has it. This teleportation of a single logical qubit is in one-to-one correspondence with teleporting the many-body  $Z_2$  topological order, quantified by the topological degeneracy. When there is no other eavesdropper, detecting Alice and detecting Bob will show the same transition. The difference between Alice and Bob, however, is that Alice has a classical ensemble of measurement bit strings {s}, while Bob has an ensemble of quantum wave functions { $\psi(s)$ } that come along with the message s shared from Alice.

#### 1. Decoding Alice

Here, we investigate whether Alice's measurement outcomes are sufficient to infer the information stored in the reference qubit [56]. To make the teleportation transition observable, we will follow a three-step protocol.

First, we projectively measure the reference qubit R in the Z basis, together with projective measurements of Alice's qubits A in every run of the experiment. As a result, we obtain an ensemble of bit strings capturing **s** for Alice qubits and k = 0(1) for the reference qubit.

Second, to see the statistical correlation between R and A, we use a standard diagnostic, the classical Shannon relative entropy. This quantity is equivalent to the entropy of R averaged over the A measurement outcomes:

$$I_c^z = [S_R^z(\mathbf{s})] = \left[ \langle \ln \frac{P(\mathbf{s})}{P_k(\mathbf{s})} \rangle \right], \tag{C1}$$

which is to take  $\ln P(\mathbf{s})/P_k(\mathbf{s})$  (calculated by a classical computer) being averaged over all the experimentally obtained bit strings. Here, for the Shannon entropy, we need a classical decoder (running the same tensor-network calculation as we do here) that computes the nonlinear part  $\ln P_k(\mathbf{s})$ , corresponding to each important sample bit string obtained from the experiment.

Finally, we make use of the symmetry for the protocol to reconstruct the full coherent quantum information:

- (1)  $\theta = 0$ : the logical space is compressed only in the *X* direction and thus the *Z*-basis measurement leads directly to the full coherent information:  $I_c = I_c^z$ .
- (2)  $\theta = \pi/4$ : the model maintains self-duality, such that the *X* contribution is equal to the *Z* contribution. And thus when we consider only the diagonal entry, the density eigenvalues of the reference qubit (for each **s**) reduce from  $(1 \pm \kappa)/2 \mapsto (1 \pm \kappa/\sqrt{2})/2$ , which would rescale the coherent information:  $I_c \mapsto I_c^z$ . We can revert this map to obtain the full coherent information from each bit string (for only  $\theta = \pi/4$ ).

### 2. Decoding Bob

Besides guiding the simple local correction as in the standard *few-body* teleportation experiment, Alice's measurement outcome **s** can additionally serve as the *many-body* syndrome. In the active-teleportation protocol, where Bob has access to these syndromes, he can deduce the underlying errors by running our classical calculation to contract out the tensor network of Fig. 3, which works for general angles.

Let us comment on the active-teleportation versus passive-teleportation protocol in the following. For active teleportation along  $\theta = 0$  ( $\phi = 0$ ), the quantum wave function is mapped to the RBIM from high to low temperature, as visualized in the Fig. 4(b) "hot" to "cool" regimes. Here,

"temperature" describes the uncertainty of the measurement outcomes. Consequently, it exhibits a finite threshold at the Nishimori critical point at  $t_c \approx 0.143\pi$ . The passiveteleportation protocol, in contrast, does not use a classical decoder on the way to processing the information but instead mixes the pure states into a noisy surface-code mixed state:  $\mathcal{N}(\rho) = \cos^2(t)\rho + \sin^2(t)Z\rho Z$ , which can be mapped to the RBIM from low to high temperatures. In this case, "temperature" describes the fluctuations induced by noise. Consequently, its optimal threshold lies at the opposite location,  $t_c \approx 0.107\pi = \pi/4 - 0.143\pi$ .

- [1] A. Y. Kitaev, Fault-tolerant quantum computation by anyons, Ann. Phys. (NY) **303**, 2 (2003).
- [2] X. Chen, Z.-C. Gu, and X.-G. Wen, Local unitary transformation, long-range quantum entanglement, wave function renormalization, and topological order, Phys. Rev. B 82, 155138 (2010).
- [3] S. B. Bravyi and A. Y. Kitaev, Quantum codes on a lattice with boundary, ArXiv:quant-ph/9811052.
- [4] Strictly speaking, when the code is defined on a planar (torus) geometry, it is called the surface (toric) code. Both surface and toric codes realize a  $Z_2$  topologically ordered phase with the same anyon excitations and statistics.
- [5] E. Dennis, A. Kitaev, A. Landahl, and J. Preskill, Topological quantum memory, J. Math. Phys. 43, 4452 (2002).
- [6] C. H. Bennett, G. Brassard, C. Crépeau, R. Jozsa, A. Peres, and W. K. Wootters, Teleporting an unknown quantum state via dual classical and Einstein-Podolsky-Rosen channels, Phys. Rev. Lett. **70**, 1895 (1993).
- [7] D. Bouwmeester, J.-W. Pan, K. Mattle, M. Eibl, H. Weinfurter, and A. Zeilinger, Experimental quantum teleportation, Nature **390**, 575 (1997).
- [8] W. Pfaff, B. J. Hensen, H. Bernien, S. B. van Dam, M. S. Blok, T. H. Taminiau, M. J. Tiggelman, R. N. Schouten, M. Markham, D. J. Twitchen, and R. Hanson, Unconditional quantum teleportation between distant solid-state quantum bits, Science 345, 532 (2014).
- [9] J.-G. Ren, *et al.*, Ground-to-satellite quantum teleportation, Nature 549, 70 (2017).
- [10] R. Raussendorf, J. Harrington, and K. Goyal, A faulttolerant one-way quantum computer, Ann. Phys. (NY) 321, 2242 (2006).
- [11] R. Raussendorf, J. Harrington, and K. Goyal, Topological fault-tolerance in cluster state quantum computation, New J. Phys. 9, 199 (2007).
- [12] H. J. Briegel, D. E. Browne, W. Dür, R. Raussendorf, and M. Van den Nest, Measurement-based quantum computation, Nat. Phys. 5, 19 (2009).
- [13] D. Horsman, A. G. Fowler, S. Devitt, and R. V. Meter, Surface code quantum computing by lattice surgery, New J. Phys. 14, 123011 (2012).
- [14] A. Erhard, H. Poulsen Nautrup, M. Meth, L. Postler, R. Stricker, M. Stadler, V. Negnevitsky, M. Ringbauer, P. Schindler, H. J. Briegel, R. Blatt, N. Friis, and T. Monz, Entangling logical qubits with lattice surgery, Nature 589, 220 (2021).

- [15] K. S. Chou, J. Z. Blumoff, C. S. Wang, P. C. Reinhold, C. J. Axline, Y. Y. Gao, L. Frunzio, M. H. Devoret, L. Jiang, and R. J. Schoelkopf, Deterministic teleportation of a quantum gate between two logical qubits, Nature 561, 368 (2018).
- [16] X.-G. Wen, Choreographed entanglement dances: Topological states of quantum matter, Science 363, eaal3099 (2019).
- [17] In our context, the coherent error is characterized by a tunable parameter that quantifies the imperfection of the unitary entangling gates in the quantum circuit away from the maximally entangled Clifford protocol.
- [18] J. Ashkin and E. Teller, Statistics of two-dimensional lattices with four components, Phys. Rev. 64, 178 (1943).
- [19] H. Nishimori, Internal energy, specific heat and correlation function of the bond-random Ising model, Prog. Theor. Phys. 66, 1169 (1981).
- [20] R. Fan, Y. Bao, E. Altman, and A. Vishwanath, Diagnostics of mixed-state topological order and breakdown of quantum memory, PRX Quantum, 5, 020343 (2024).
- [21] Y. Bao, R. Fan, A. Vishwanath, and E. Altman, Mixed-state topological order and the errorfield double formulation of decoherence-induced transitions, ArXiv:2301.05687.
- [22] J. Y. Lee, C.-M. Jian, and C. Xu, Quantum criticality under decoherence or weak measurement, PRX Quantum 4, 030317 (2023).
- [23] Y.-H. Chen and T. Grover, Separability transitions in topological states induced by local decoherence, Phys. Rev. Lett. 132, 170602 (2024).
- [24] Z. Li and R. S. K. Mong, Replica topological order in quantum mixed states and quantum error correction, ArXiv:2402.09516.
- [25] S. Bravyi, M. Englbrecht, R. König, and N. Peard, Correcting coherent errors with surface codes, npj Quantum Inf. 4, 55 (2018).
- [26] F. Venn, J. Behrends, and B. Béri, Coherent-error threshold for surface codes from Majorana delocalization, Phys. Rev. Lett. 131, 060603 (2023).
- [27] J. Behrends, F. Venn, and B. Béri, Surface codes, quantum circuits, and entanglement phases, Phys. Rev. Res. 6, 013137 (2024).
- [28] J. K. Iverson and J. Preskill, Coherence in logical quantum channels, New J. Phys. 22, 073066 (2020).
- [29] G.-Y. Zhu, N. Tantivasadakarn, A. Vishwanath, S. Trebst, and R. Verresen, Nishimori's cat: Stable long-range entanglement from finite-depth unitaries and weak measurements, Phys. Rev. Lett. 131, 200201 (2023).
- [30] E. H. Chen, G.-Y. Zhu, R. Verresen, A. Seif, E. Bäumer, D. Layden, N. Tantivasadakarn, G. Zhu, S. Sheldon, A. Vishwanath, S. Trebst, and A. Kandala, Realizing the Nishimori transition across the error threshold for constant-depth quantum circuits, ArXiv:2309.02863.
- [31] K. Su, N. Myerson-Jain, C. Wang, C.-M. Jian, and C. Xu, Higher-form symmetries under weak measurement, Phys. Rev. Lett. 132, 200402 (2024).
- [32] S. J. Garratt, Z. Weinstein, and E. Altman, Measurements conspire nonlocally to restructure critical quantum states, Phys. Rev. X 13, 021026 (2023).
- [33] J. Y. Lee, W. Ji, Z. Bi, and M. P. A. Fisher, Decoding measurement-prepared quantum phases and

transitions: From Ising model to gauge theory, and beyond, ArXiv:2208.11699.

- [34] S. Murciano, P. Sala, Y. Liu, R. S. K. Mong, and J. Alicea, Measurement-altered Ising quantum criticality, Phys. Rev. X 13, 041042 (2023).
- [35] G.-Y. Zhu and S. Trebst, Qubit fractionalization and emergent Majorana liquid in the honeycomb Floquet code induced by coherent errors and weak measurements, ArXiv:2311.08450.
- [36] M. Szyniszewski, A. Romito, and H. Schomerus, Entanglement transition from variable-strength weak measurements, Phys. Rev. B 100, 064204 (2019).
- [37] C.-M. Jian, Y.-Z. You, R. Vasseur, and A. W. W. Ludwig, Measurement-induced criticality in random quantum circuits, Phys. Rev. B 101, 104302 (2020).
- [38] Y. Bao, S. Choi, and E. Altman, Theory of the phase transition in random unitary circuits with measurements, Phys. Rev. B 101, 104301 (2020).
- [39] X. Turkeshi, A. Biella, R. Fazio, M. Dalmonte, and M. Schiró, Measurement-induced entanglement transitions in the quantum Ising chain: From infinite to zero clicks, Phys. Rev. B 103, 224210 (2021).
- [40] A. C. Potter and R. Vasseur, Entanglement dynamics in hybrid quantum circuits, ArXiv:2111.08018.
- [41] M. P. Fisher, V. Khemani, A. Nahum, and S. Vijay, Random quantum circuits, Annu. Rev. Condens. Matter Phys. 14, 335 (2023).
- [42] E. Ardonne, P. Fendley, and E. Fradkin, Topological order and conformal quantum critical points, Ann. Phys. (NY) 310, 493 (2004).
- [43] C. Castelnovo, S. Trebst, and M. Troyer, in Understanding Quantum Phase Transitions (CRC press, Boca Raton, 2010), p. 169.
- [44] S. V. Isakov, P. Fendley, A. W. W. Ludwig, S. Trebst, and M. Troyer, Dynamics at and near conformal quantum critical points, Phys. Rev. B 83, 125114 (2011).
- [45] G.-Y. Zhu and G.-M. Zhang, Gapless Coulomb state emerging from a self-dual topological tensor-network state, Phys. Rev. Lett. **122**, 176401 (2019).
- [46] A. Browaeys and T. Lahaye, Many-body physics with individually controlled Rydberg atoms, Nat. Phys. 16, 132 (2020).
- [47] S. Ebadi, T. T. Wang, H. Levine, A. Keesling, G. Semeghini, A. Omran, D. Bluvstein, R. Samajdar, H. Pichler, W. W. Ho, S. Choi, S. Sachdev, M. Greiner, V. Vuletić, and M. D. Lukin, Quantum phases of matter on a 256atom programmable quantum simulator, Nature 595, 227 (2021).
- [48] D. Bluvstein, *et al.*, Logical quantum processor based on reconfigurable atom arrays, Nature **626**, 58 (2024).
- [49] B. Schumacher and M. A. Nielsen, Quantum data processing and error correction, Phys. Rev. A 54, 2629 (1996).
- [50] N. Schuch, D. Poilblanc, J. I. Cirac, and D. Pérez-García, Topological order in the projected entangled-pair states formalism: Transfer operator and boundary Hamiltonians, Phys. Rev. Lett. 111, 090501 (2013).
- [51] W.-T. Xu, Q. Zhang, and G.-M. Zhang, Tensor network approach to phase transitions of a non-Abelian topological phase, Phys. Rev. Lett. **124**, 130603 (2020).
- [52] Q. Zhang, W.-T. Xu, Z.-Q. Wang, and G.-M. Zhang, Non-Hermitian effects of the intrinsic signs in topologically

ordered wavefunctions, Commun. Phys. **3**, 209 (2020).

- [53] G.-Y. Zhu, J.-Y. Chen, P. Ye, and S. Trebst, Topological fracton quantum phase transitions by tuning exact tensor network states, Phys. Rev. Lett. 130, 216704 (2023).
- [54] W. K. Wootters and W. H. Zurek, A single quantum cannot be cloned, Nature 299, 802 (1982).
- [55] S. Lloyd, Capacity of the noisy quantum channel, Phys. Rev. A 55, 1613 (1997).
- [56] M. J. Gullans and D. A. Huse, Scalable probes of measurement-induced criticality, Phys. Rev. Lett. 125, 070606 (2020).
- [57] L. Colmenarez, Z.-M. Huang, S. Diehl, and M. Müller, Accurate optimal quantum error correction thresholds from coherent information, ArXiv:2312.06664.
- [58] E. Knill and R. Laflamme, Theory of quantum errorcorrecting codes, Phys. Rev. A 55, 900 (1997).
- [59] F. J. Wegner, Duality in generalized Ising models and phase transitions without local order parameters, J. Math. Phys. 12, 2259 (1971).
- [60] E. Fradkin and S. H. Shenker, Phase diagrams of lattice gauge theories with Higgs fields, Phys. Rev. D 19, 3682 (1979).
- [61] J. B. Kogut, An introduction to lattice gauge theory and spin systems, Rev. Mod. Phys. 51, 659 (1979).
- [62] S. Trebst, P. Werner, M. Troyer, K. Shtengel, and C. Nayak, Breakdown of a topological phase: Quantum phase transition in a loop gas model with tension, Phys. Rev. Lett. 98, 070602 (2007).
- [63] I. S. Tupitsyn, A. Kitaev, N. V. Prokof'ev, and P. C. E. Stamp, Topological multicritical point in the phase diagram of the toric code model and three-dimensional lattice gauge Higgs model, Phys. Rev. B 82, 085114 (2010).
- [64] S. Dusuel, M. Kamfor, R. Orús, K. P. Schmidt, and J. Vidal, Robustness of a perturbed topological phase, Phys. Rev. Lett. 106, 107203 (2011).
- [65] J. Haegeman, V. Zauner, N. Schuch, and F. Verstraete, Shadows of anyons and the entanglement structure of topological phases, Nat. Commun. 6, 8284 (2015).
- [66] A. M. Somoza, P. Serna, and A. Nahum, Self-dual criticality in three-dimensional  $F_2$  gauge theory with matter, Phys. Rev. X 11, 041008 (2021).
- [67] R. Verresen, U. Borla, A. Vishwanath, S. Moroz, and R. Thorngren, Higgs condensates are symmetryprotected topological phases: I. Discrete symmetries, ArXiv:2211.01376.
- [68] A. Kitaev and L. Kong, Models for gapped boundaries and domain walls, Commun. Math. Phys. 313, 351 (2012).
- [69] F. Burnell, Anyon condensation and its applications, Annu. Rev. Condens. Matter Phys. 9, 307 (2018).
- [70] M. Pütz, Scalingcollapse.jl (2024).
- [71] M. Fishman, S. R. White, and E. M. Stoudenmire, The ITensor software library for tensor network calculations, SciPost Phys. Codebases 4, 4 (202).
- [72] K. Su, Z. Yang, and C.-M. Jian, Tapestry of dualities in decohered quantum error correction codes, Phys. Rev. B 110, 085158 (2024).
- [73] The averaged pure-state Rényi entropy is  $S^{(n)} = \sum_{s} P(s)$  $(1/(1-n) \ln \operatorname{tr}(\rho^{n}))$ , in contrast to the Rényi coherent

- [74] J.-M. Maillard, K. Nemoto, and H. Nishimori, Symmetry, complexity and multicritical point of the two-dimensional spin glass, J. Phys. A: Math. Gen. 36, 9799 (2003).
- [75] D. K. Tuckett, S. D. Bartlett, and S. T. Flammia, Ultrahigh error threshold for surface codes with biased noise, Phys. Rev. Lett. **120**, 050505 (2018).
- [76] J. P. Bonilla Ataides, D. K. Tuckett, S. D. Bartlett, S. T. Flammia, and B. J. Brown, The XZZX surface code, Nat. Commun. 12, 2172 (2021).
- [77] J. Claes, J. E. Bourassa, and S. Puri, Tailored cluster states with high threshold under biased noise, npj Quantum Inf. 9, 9 (2023).
- [78] B. Skinner, J. Ruhman, and A. Nahum, Measurementinduced phase transitions in the dynamics of entanglement, Phys. Rev. X 9, 031009 (2019).
- [79] A. Lavasani, Y. Alavirad, and M. Barkeshli, Topological order and criticality in (2 + 1)D monitored random quantum circuits, Phys. Rev. Lett. 127, 235701 (2021).
- [80] A. Lavasani, Z.-X. Luo, and S. Vijay, Monitored quantum dynamics and the Kitaev spin liquid, Phys. Rev. B 108, 115135 (2023).
- [81] A. Sriram, T. Rakovszky, V. Khemani, and M. Ippoliti, Topology, criticality, and dynamically generated qubits in a stochastic measurement-only Kitaev model, Phys. Rev. B 108, 094304 (2023).
- [82] G.-Y. Zhu, N. Tantivasadakarn, and S. Trebst, Structured volume-law entanglement in an interacting, monitored Majorana spin liquid, ArXiv:2303.17627.
- [83] A.-R. Negari, S. Sahu, and T. H. Hsieh, Measurementinduced phase transitions in the toric code, Phys. Rev. B 109, 125148 (2024).
- [84] Y. Kuno, T. Orito, and I. Ichinose, Bulk-measurementinduced boundary phase transition in toric code and gauge Higgs model, Phys. Rev. B 109, 054432 (2024).
- [85] H. Sukeno, K. Ikeda, and T.-C. Wei, Bulk and boundary entanglement transitions in the projective gauge-Higgs model, ArXiv:2402.11738.

- [86] T. Botzung, M. Buchhold, S. Diehl, and M. Müller, Robustness and measurement-induced percolation of the surface code, ArXiv:2311.14338.
- [87] D. Lee and B. Yoshida, Randomly monitored quantum codes, ArXiv:2402.00145.
- [88] M. Pütz, S. Trebst, and G.-Y. Zhu, in preparation (2024).
- [89] H. J. Briegel and R. Raussendorf, Persistent entanglement in arrays of interacting particles, Phys. Rev. Lett. 86, 910 (2001).
- [90] Y. Hong, D. T. Stephen, and A. J. Friedman, Quantum teleportation implies symmetry-protected topological order, ArXiv:2310.12227.
- [91] For instance, along the self-dual direction, the critical entanglement entropy between A' and B scales as  $\mathcal{O}(d^{2-1/\nu})$ , contributed by  $\mathcal{O}(d^2)$  weakly entangled pairs, each of which has power-law scaling entanglement entropy  $\mathcal{O}(d^{-1/\nu})$ .
- [92] P. Sala, S. Murciano, Y. Liu, and J. Alicea, Quantum criticality under imperfect teleportation, PRX Quantum, 5, 030307 (2024).
- [93] S. Bravyi, M. Suchara, and A. Vargo, Efficient algorithms for maximum likelihood decoding in the surface code, Phys. Rev. A 90, 032326 (2014).
- [94] F. Barratt, U. Agrawal, A. C. Potter, S. Gopalakrishnan, and R. Vasseur, Transitions in the learnability of global charges from local measurements, Phys. Rev. Lett. **129**, 200602 (2022).
- [95] F. Eckstein, B. Han, S. Trebst, and G.-Y. Zhu, Data for "Robust teleportation of a topological surface code" (2024), https://doi.org/10.5281/zenodo.10717648.
- [96] Y.-H. Chen and T. Grover, Unconventional topological mixed-state transition beyond anyon condensation induced by self-dual coherent errors, ArXiv:2403.06553.
- [97] O. Hart, D. T. Stephen, D. J. Williamson, M. Foss-Feig, and R. Nandkishore, Playing nonlocal games across a topological phase transition on a quantum computer, ArXiv:2403.04829.
- [98] C. Ryan-Anderson, *et al.*, High-fidelity and fault-tolerant teleportation of a logical qubit using transversal gates and lattice surgery on a trapped-ion quantum computer, ArXiv:2404.16728.