Symmetry Classes of Disordered Fermions and Topological Insulators

M. Zirnbauer

Sommerfeld Theory Colloquium, LMU München (June 1, 2011)

The uncanny power of prediction by random matrix theory

universal fluctuations in energy spectra, scattering cross sections, …

Compound nucleus resonances

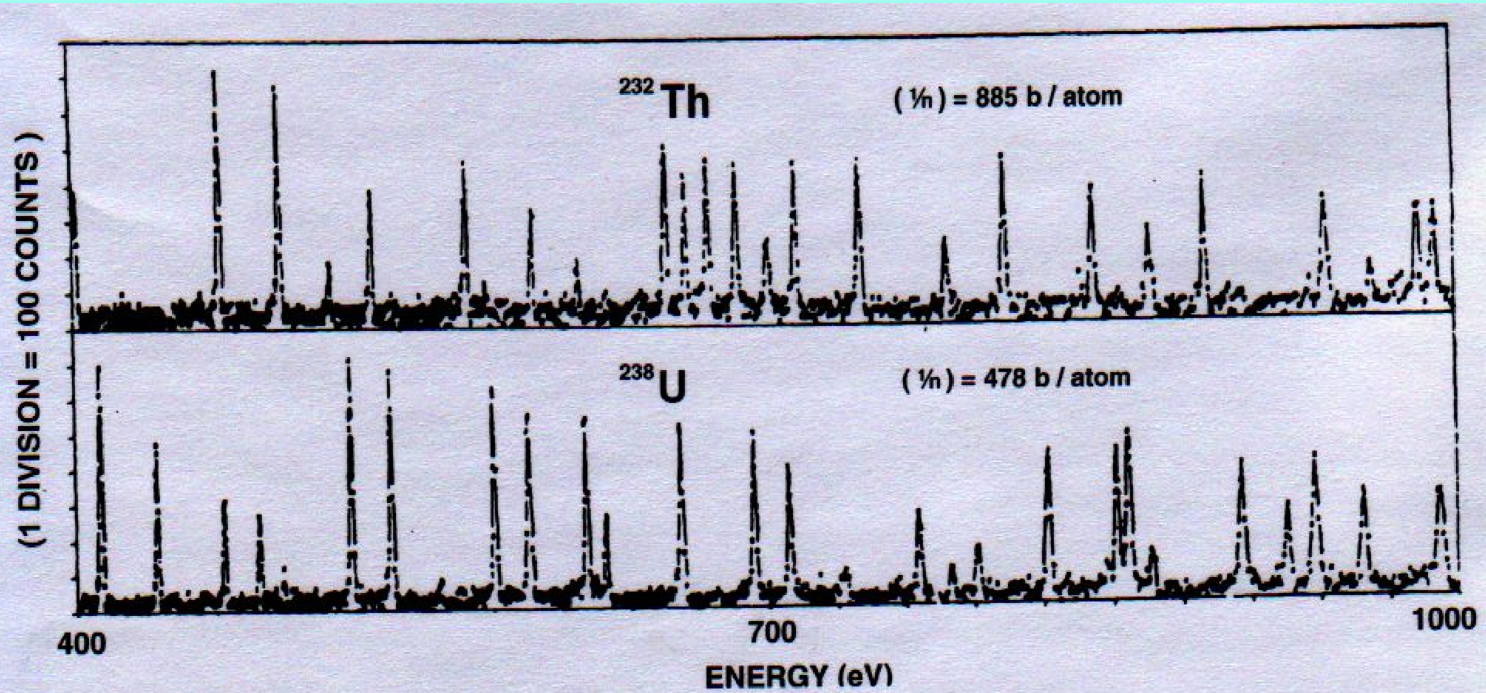


Figure 1.1. Slow neutron resonance cross-sections on thorium 232 and uranium 238 nuclei. Reprinted with permission from The American Physical Society, Rahn et al., Neutron resonance spectroscopy, X, *Phys. Rev. C* 6, 1854–1869 (1972).

Total cross section versus center of mass energy for scattering of slow neutrons on 232Th and 238U. The resonances all have the same spin 1/2 and positive parity.

Eugene Wigner

The Nobel Prize in Physics 1963

Biography

Eugene Paul Wigner, born in Budapest, Hungary, on November 17, 1902, naturalized a citizen of the United States on January 8, 1937, has been since 1938 Thomas D. Jones Professor of Mathematical Physics at Princeton University - he retired in 1971. His formal education was acquired in Europe; he obtained the Dr. Ing. degree at the Technische Hochschule Berlin. Married in 1941 to Mary Annette Wheeler, he is the father of two children, David and Martha. His son, David, is teaching mathematics at the University of California in Berkeley. His daughter, Martha, is with the Chicago area transportation system, an organization endeavoring to improve the internal transportation system of that city. Dr.Wigner worked on the Manhattan Project at the University of Chicago during

Niels Bohr’s picture of the compound nucleus

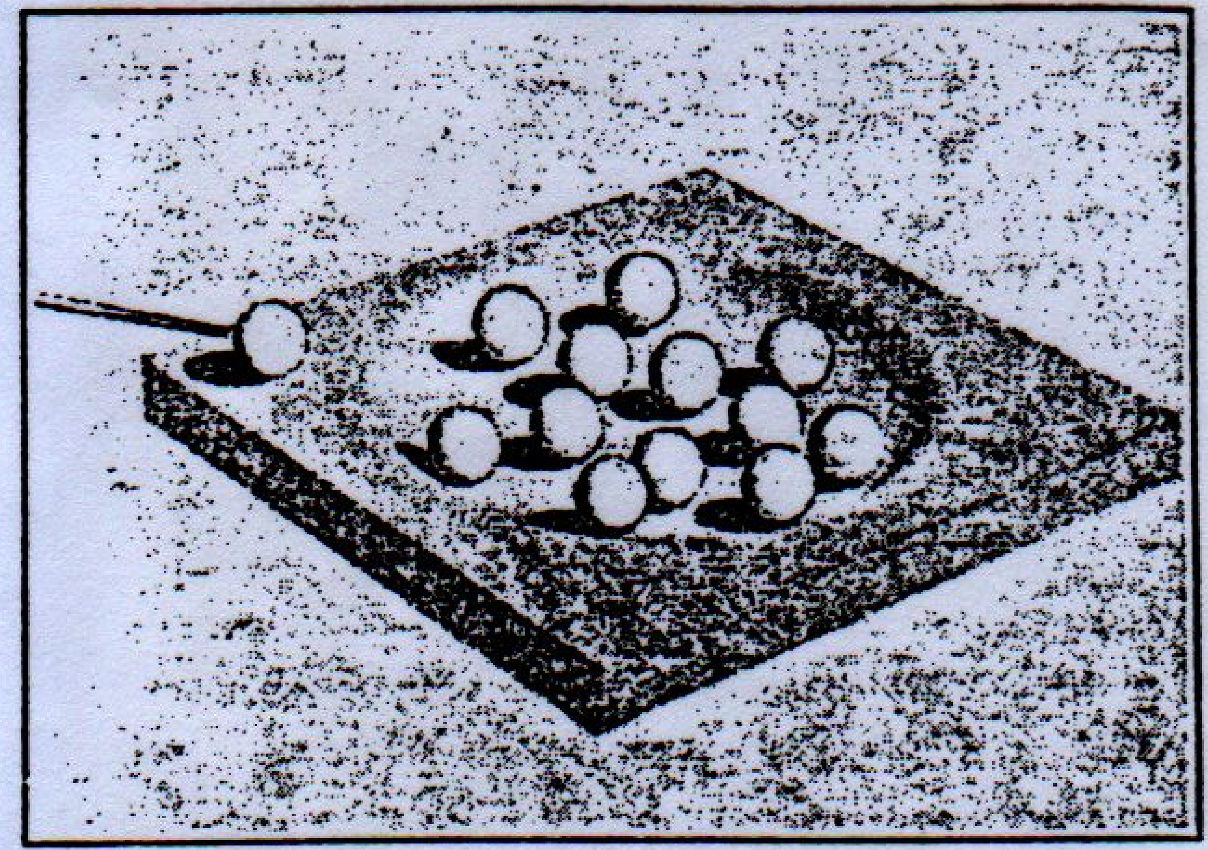


Fig. 35. Picture illustrating the compound nucleus idea, as presented by N. Bohr in 1936. In a neutron-nucleus collision the constituent nucleons are viewed as billiard balls and the nuclear binding as a shallow basin (taken from [112]).

Nuclear Data Ensemble (1726 levels)

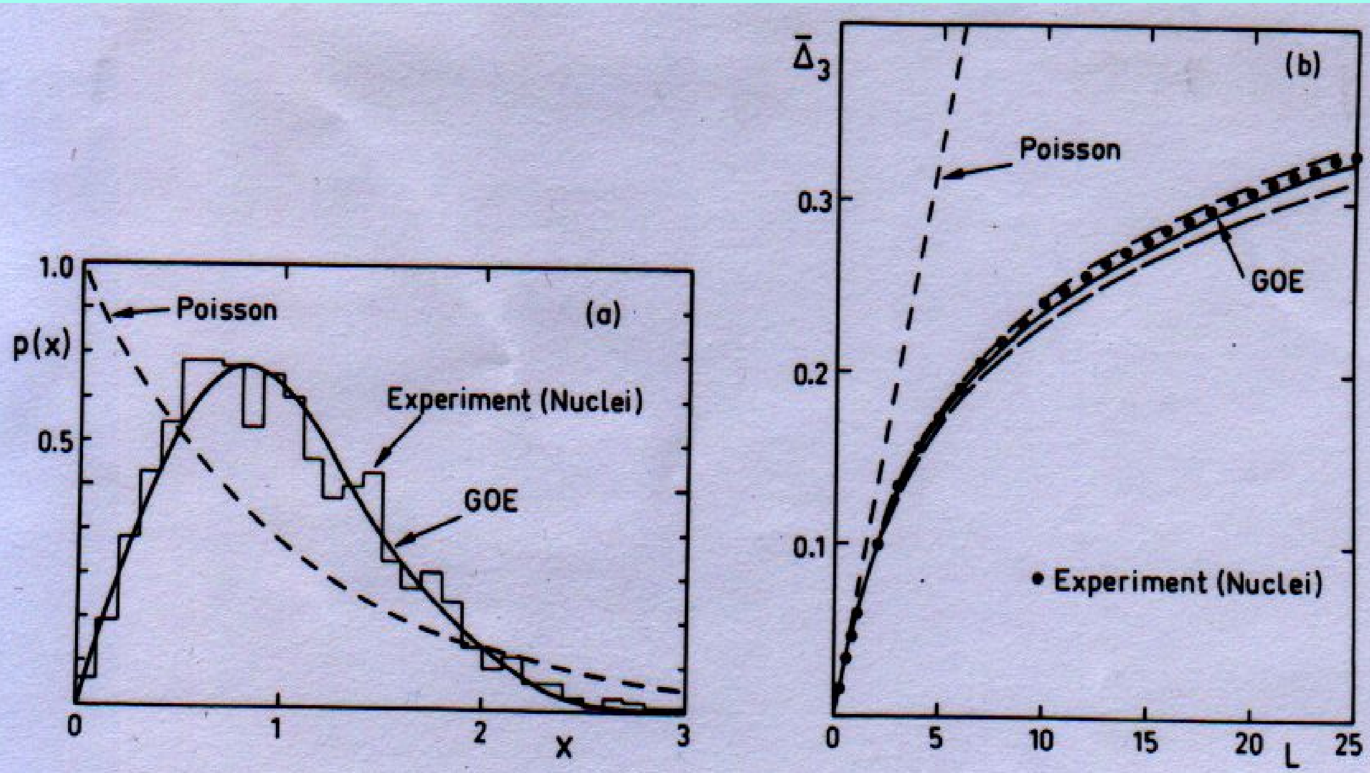


Fig. 8. Left: nearest neighbour spacing histogram for the nuclear data ensemble (NDE) (taken from [53]). Right: Dyson–Mehta statistic $\bar{\Delta}$ for NDE (taken from [54]). GOE and Poisson predictions are plotted for the sake of comparison.

Quantum chaotic billiard

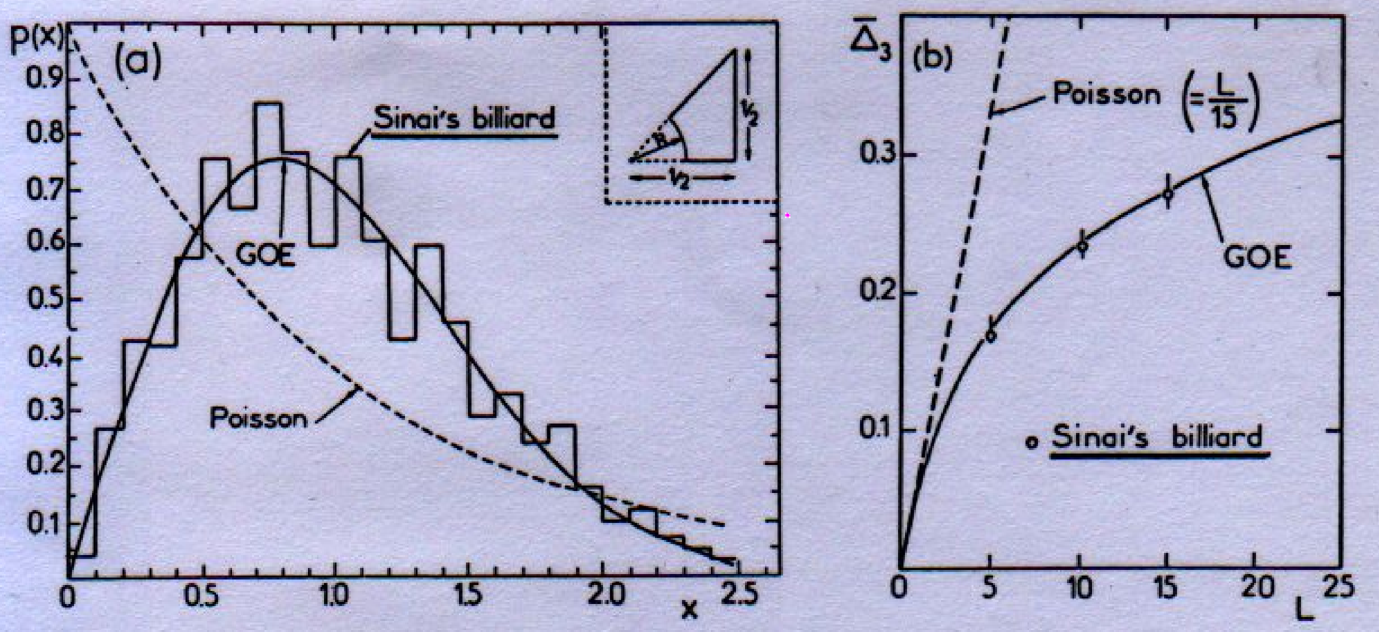


Fig. 12. Spectral fluctuations for Sinai's billiard (see fig. 11). (a) spacing distribution histogram; (b) Dyson–Mehta statistic. GOE and Poisson predictions are plotted for the sake of comparison (taken from [62]).

Random matrix conjecture by Bohigas, Giannoni, and Schmit (1984)

Chiral random matrices

Nonabelian gauge field A_μ (vacuum fluctuations)

Dirac operator : $D = \gamma^\mu(\partial_\mu - A_\mu) = -\gamma_5 D \gamma_5$

Verbaarschot, Zahed (1993):

$$\gamma_5 = \begin{pmatrix} 1_p & 0 \\ 0 & -1_q \end{pmatrix}, \quad D = \begin{pmatrix} 0 & Z \\ Z^* & 0 \end{pmatrix},$$

random matrix Z (rectangular : $p \times q$),

$p - q =$ topological charge of gauge field.

Chiral random matrix ensembles for the QCD Dirac operator

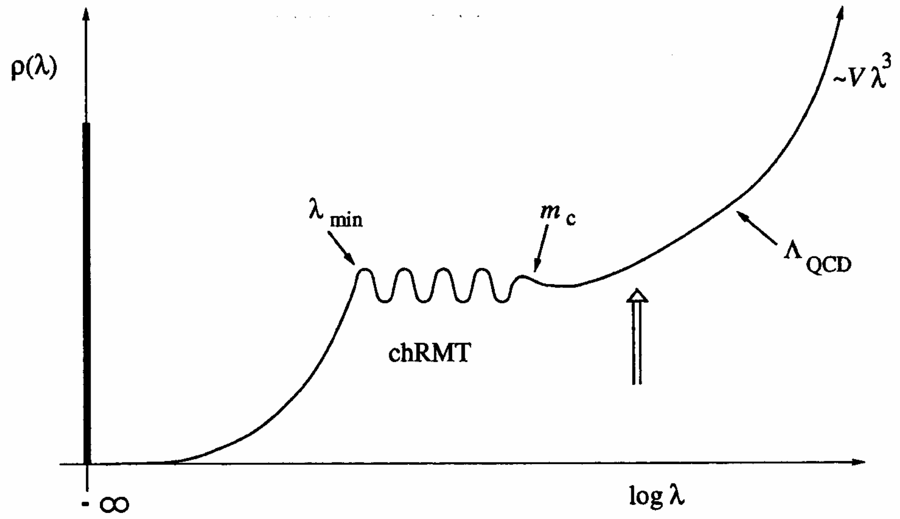


Figure 2.
Schematic picture of the average spectral density of QCD Dirac operator.

Verbaarschot, Zahed (1993)

QCD Dirac spectra

from Berbenni-Bitsch et al. (1997)

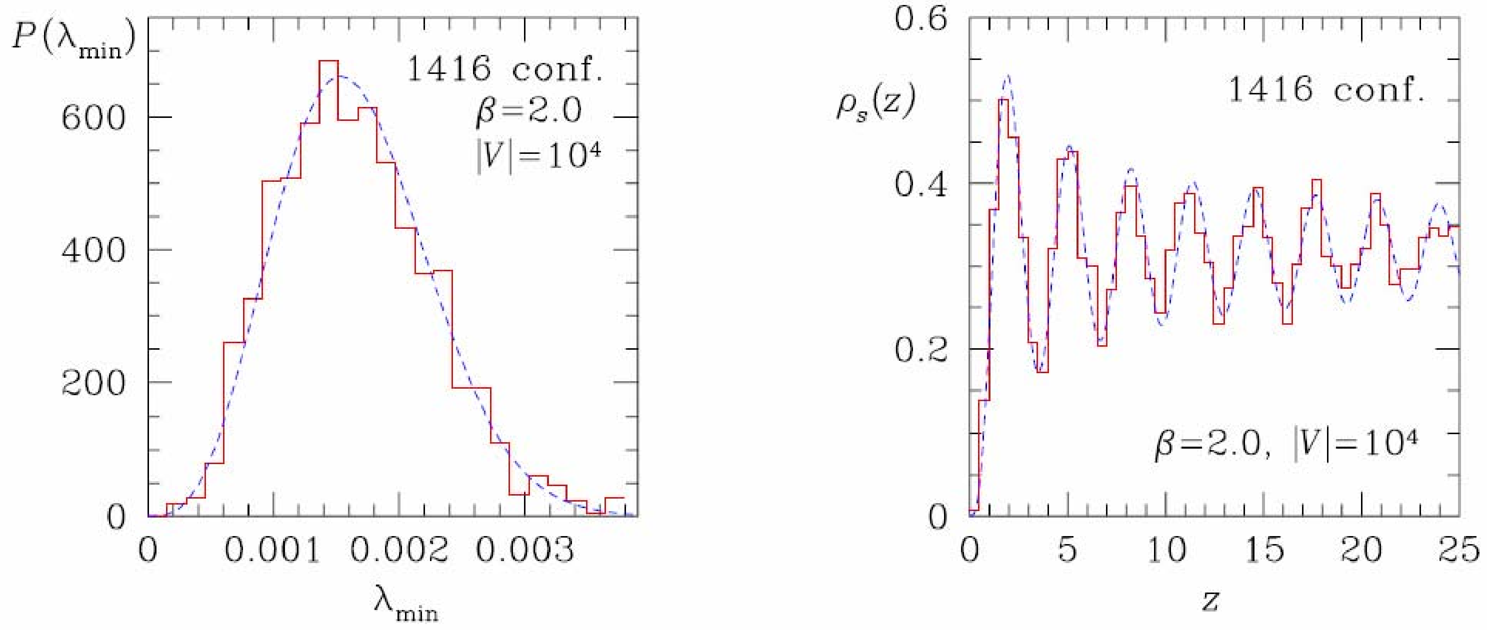


Figure 5: Distribution of the smallest eigenvalue *(left)* and microscopic spectral density *(right)* of the staggered Dirac operator in quenched SU(2). The dashed lines are the predictions of the chSE for $N_f = 0$ and $\nu = 0$.

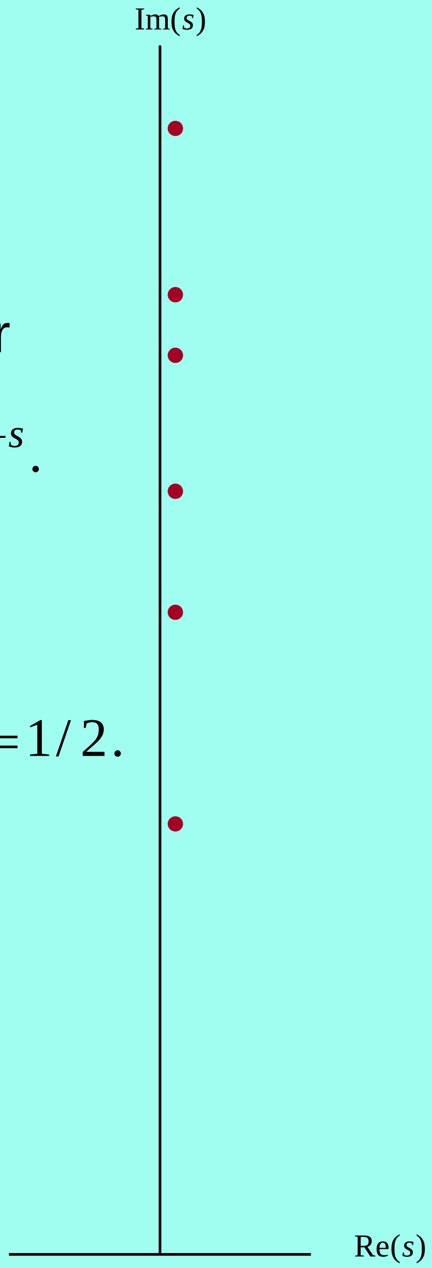
Riemann zeroes

The Riemann zeta function $\zeta(s)$ is defined for

$\mathrm{Re}(s) > 1$ by its Dirichlet series: $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}.$

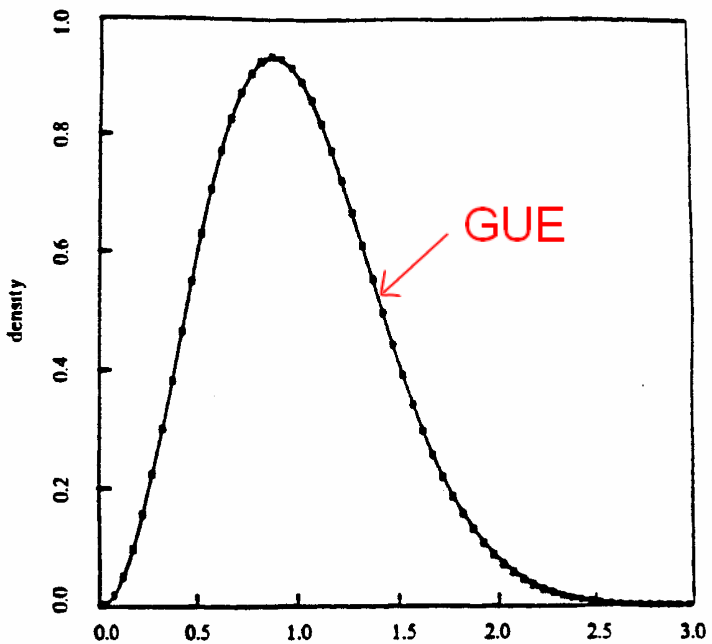
According to the Riemann hypothesis, all nontrivial zeroes of $\zeta(s)$ lie on the line $\mathrm{Re}(s) = 1/2$.

The six lowest zeroes have imaginary parts
14.13, 21.02, 25.01, 30.42, 32.93, 37.58



Spacing Distribution of the Riemann Zeroes

from A. Odlyzko (1987)



Normalized spacings between neighboring Riemann zeroes.
The data set consists of 70 x 10^6 consecutive zeroes,
starting at the zero of order 10^20.

Universality of Spectral Fluctuations

In the spectrum of the Schrödinger, wave, or Dirac operator
for a large variety of physical systems, such as

- atomic nuclei (neutron resonances),
- disordered metallic grains,
- chaotic billiards (Sinai, Bunimovich),
- microwaves in a cavity,
- acoustic modes of a vibrating solid,
- quarks in a nonabelian gauge field,
- zeroes of the Riemann zeta function,

one observes fluctuations that obey the laws of random matrix theory for the appropriate symmetry class and in the ergodic limit.

Spectral fluctuations are universal.

Why?

Supersymmetric non-linear sigma models …

Wilson’s renormalization group …

Universality at RG-fixed points …

The Threefold Way

Freeman Dyson

Born	December 15, 1923 Crowthorne, Berkshire, England
Residence	United States
Nationality	UK USA
Fields	Physicist,mathematics
Institutions	Royal Air Force Institute for Advanced Study Duke University Cornell University
Alma mater	University of Cambridge
Doctoral advisor	None
Known for	Dyson sphere Dyson operator Advocacy against nuclear weapons
Notable awards	Templeton Prize (2000)

JOURNAL OF MATHEMATICAL PHYSICS VOLUME 3, NUMBER 6 NOVEMBER-DECEMBER 1962

The Threefold Way. Algebraic Structure of Symmetry Groups and Ensembles in Quantum Mechanics

FREEMAN J. DYSON
Institute for Advanced Study, Princeton, New Jersey
(Received June 22, 1962)

Using mathematical tools developed by Hermann Weyl, the Wigner classification of group-representations and co-representations is clarified and extended. The three types of representation, and the three types of co-representation, are shown to be directly related to the three types of division algebra with real coefficients, namely, the real numbers, complex numbers, and quaternions. The author's theory of matrix ensembles, in which again three possible types were found, is shown to be in exact correspondence with the Wigner classification of co-representations. In particular, it is proved that the most general kind of matrix ensemble, defined with a symmetry group which may be completely arbitrary, reduces to a direct product of independent irreducible ensembles each of which belongs to one of the three known types.

I. INTRODUCTION

THE purpose of this paper is to bring together and unify three trends of thought which have

In each of the three theories which we aim to unify, there appears a triple alternative, a choice between three mutually exclusive possibilities. (i) The ir-

Unitary and anti-unitary symmetries

In quantum mechanics, one is given
a Hilbert space V with Hermitian scalar product $\langle \cdot \, , \cdot \rangle$
and the Hamiltonian H is a Hermitian operator on V.

Unitary symmetries (e.g., space rotations) :

$$\langle \psi_1 \, , \psi_2 \rangle = \langle U\psi_1 \, , U\psi_2 \rangle \qquad\qquad \mathrm{e}^{-\mathrm{i}tH} U = U \, \mathrm{e}^{-\mathrm{i}tH}$$

Anti-unitary symmetries (e.g., time reversal) :

$$\langle \psi_1 \, , \psi_2 \rangle = \overline{\langle T\psi_1 \, , T\psi_2 \rangle} \qquad\qquad \mathrm{e}^{-\mathrm{i}tH} T = T \, \mathrm{e}^{+\mathrm{i}tH}$$

Dyson's Setting

The basic data is (V,G), a Hilbert space V carrying the action of a group G.

G is the group of unitary and anti-unitary symmetries of an ensemble of quantum systems with Hilbert space V.
G may be (Dyson:) "a rotation group, or an isotopic-spin rotation group, or a time-inversion group, or all of these in combination".

The Hamiltonians to be used for random matrix modeling are the Hermitian linear operators on V which commute with all of the symmetries G.

Question (Dyson): What can one say about the set of random matrix Hamiltonians which occur in this setting?

Double Commutant Theorem

G_U : a group of unitary operators acting on V.
A = the group algebra of G_U. Let dimV < ∞.

Thm. Let the action of A on V be reductive. If B = Z(A) is the commutant of A in End(V), then

1. B acts reductively on V.
2. Z(B) = A (the double commutant property).
3. V is a direct sum of G_U-isotypic components:

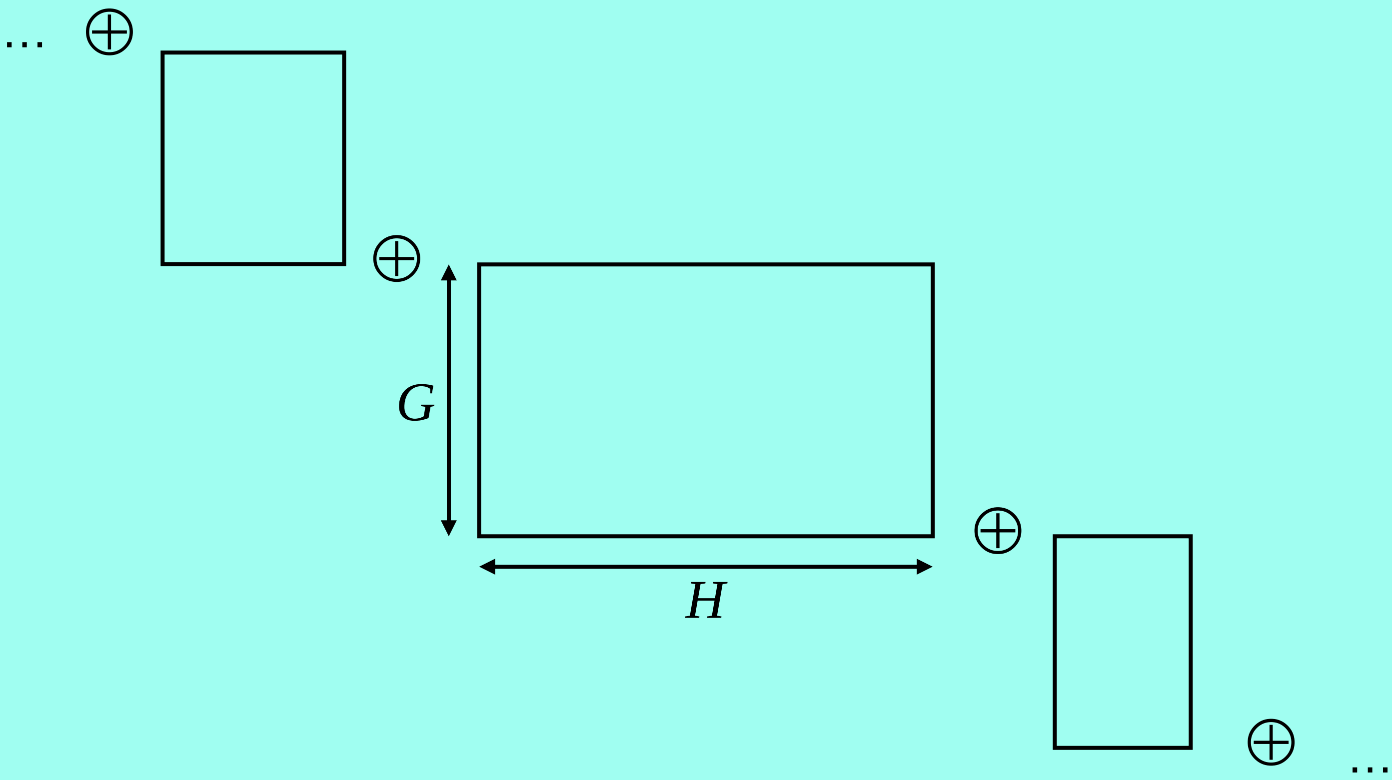
$$V = \oplus_\lambda V_\lambda \cong \oplus_\lambda R_\lambda \otimes \mathrm{Hom}(R_\lambda, V_\lambda)^{G_U}$$

here act the unitary symmetries (on R_λ)

there act the Hamiltonians (on $\mathrm{Hom}(R_\lambda, V_\lambda)^{G_U}$)

Reduction by unitary symmetries (H. Weyl)

$$V = \bigoplus V_\lambda = \bigoplus (R_\lambda \otimes S_\lambda) =$$



Example : $G = \mathrm{SO}(3)$. Rectangles are labeled by total angular momentum, L.
The rows of a rectangle are labeled by projection of angular momentum, M.

Enter the Anti-Unitaries

Symmetry group $G = G_U \cup TG_U$, $T^2 = \pm \mathrm{Id}$.

The decomposition $V = \oplus_\lambda V_\lambda \cong \oplus_\lambda R_\lambda \otimes \mathrm{Hom}(R_\lambda, V_\lambda)^{G_U}$ is preserved by T since $U \mapsto T^{-1}UT$ is an automorphism of G_U .

complex hermitian matrices

If $T(V_\lambda) = V_{\lambda'}$ and $\lambda \neq \lambda'$ then no condition on $H|_{V_\lambda}$ results but $T H|_{V_\lambda} = H|_{V_{\lambda'}} T$.

Hence let $T(V_\lambda) = V_\lambda$. By the G_U -irreducibility of R_λ the restriction must be a pure tensor: $T|_{V_\lambda} = \alpha \otimes \beta$.

real symmetric matrices

There exist but two possibilities: $\beta^2 = \pm \mathrm{Id}$.

quaternion self-dual matrices

Enter the anti-unitaries…

T_1, T_2 anti-unitary $\Rightarrow T_1 T_2$ unitary .

Let $T^2 = z \times \mathrm{Id}$, $z = \mathrm{e}^{\mathrm{i}\varphi}$.

Then associativity,

$zT = T^2 \cdot T = T \cdot T^2 = T z = \bar{z} T \;\Rightarrow\; z = \bar{z} \in \{\pm 1\}$,

leaves but two possibilities : $T^2 = \pm \mathrm{Id}$.

Consequences of anti-unitary symmetry

Recall $V = \bigoplus V_\lambda$.

Trichotomy :

1. No T, or $T : V_\lambda \leftrightarrow V_{\bar{\lambda}}$ $\Rightarrow$ complex hermitian matrices

2. $T : V_\lambda \to V_\lambda$ and $T^2 = +\mathrm{Id}$ $\Rightarrow$ real symmetric matrices
(use $e_i = Te_i$)

3. $T : V_\lambda \to V_\lambda$ and $T^2 = -\mathrm{Id}$ $\Rightarrow$ quaternion self-dual matrices
(use $Te_i = e_{\bar{i}}$, $Te_{\bar{i}} = -e_i$)

Example: Case 3 (class AII, symplectic ensemble)

Dyson (1962):

III. TIME-REVERSAL SYMMETRY. SYMPLECTIC ENSEMBLE

To find out whether the orthogonal ensemble is a reasonable one to use under all circumstances, a more careful analysis must be made of the consequences of time-reversal invariance. It will turn out that under some (perhaps not very realistic) circumstances a quite different ensemble should be used. The new ensemble will be called symplectic, because it bears the same relation to the symplectic group as E_1 bears to the orthogonal group.

Dyson (1970):

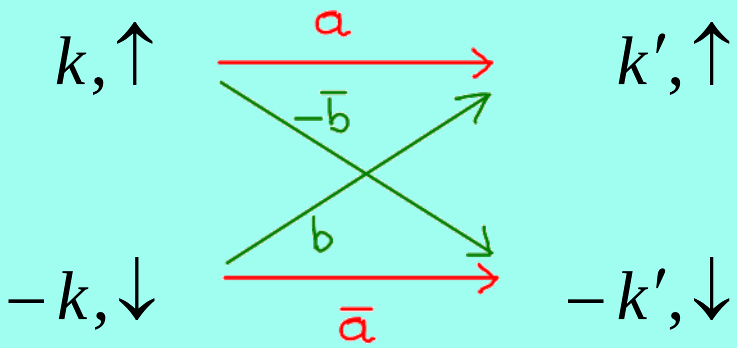
invariance. The case $\beta = 4$ would apply when H is invariant under time-reflection, without any rotation-invariance, for a system with half-integer spin. Until now no interesting physical examples have been found of the cases $\beta = 2$ and 4. The case $\beta = 1$ has been extensively studied in connection with the statistics of neutron capture levels in heavy nuclei

Example: Case 3 (class AII)

Time-reversal invariant disordered electrons with spin-orbit scattering:

$$H = \frac{p^2}{2m} + U(x) + V_{\mathrm{SO}}(x) \cdot (\sigma \times p)$$

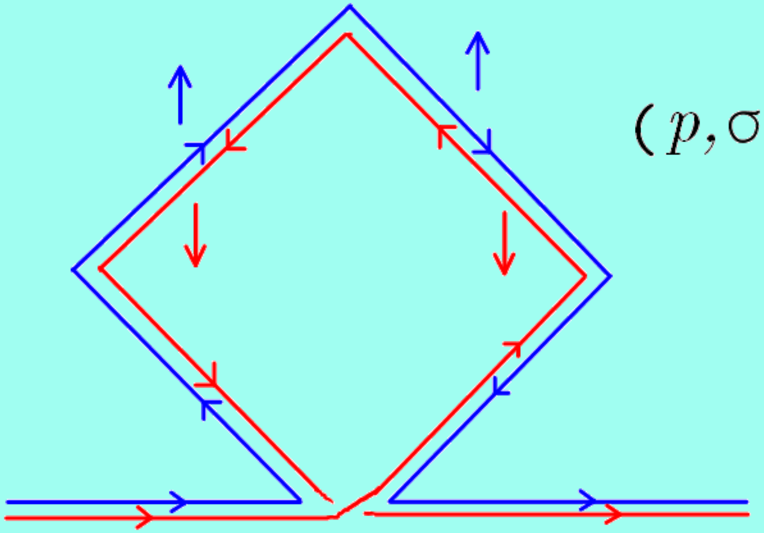
The matrix elements of the Hamiltonian



organize into quaternions: $\begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix}$.

Modes of quantum interference (AII)

Spin-singlet cooperon :



$$(p,\sigma) \longleftrightarrow (-p,-\sigma)$$

Weak anti-localization enhances conductivity.

Disordered Mg films with Au impurities (G. Bergmann, 1984)

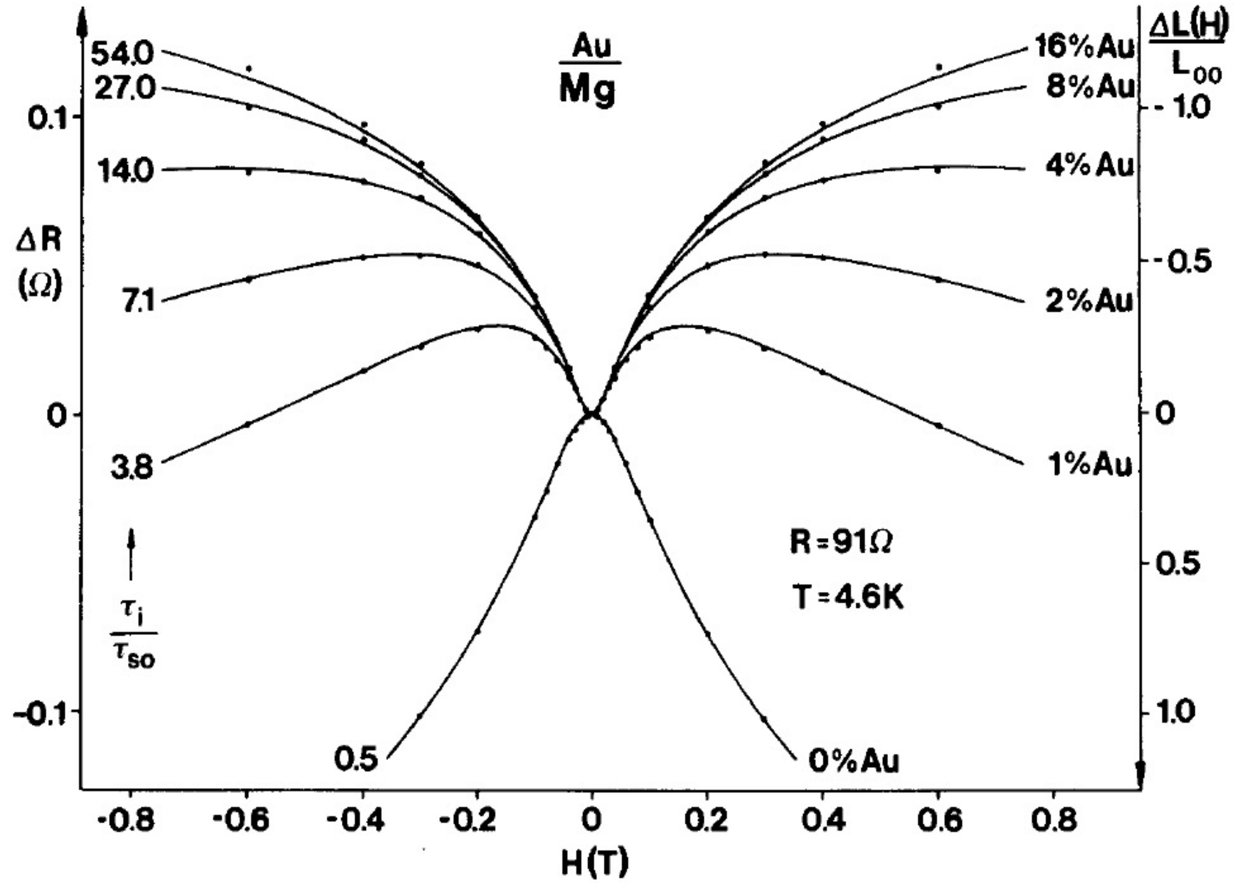


Fig. 2.10. The magneto-resistance of a thin Mg-film at 4.5 K for different coverages with Au. The Au thickness is given in % of an atomic layer on the right side of the curves. The superposition with Au increases the spin–orbit scattering. The points are measured. The full curves are obtained with the theory by Hikami et al. The ratio τ_i/τ_{so} on the left side gives the strength of the adjusted spin–orbit scattering. It is essentially proportional to the Au-thickness.

Wigner-Dyson symmetry classes:

- A : complex Hermitian matrices (‘unitary class’, GUE)
- AI : real symmetric matrices (‘orthogonal class’, GOE)
- AII : quaternion self-dual matrices (‘symplectic class’, GSE)

Dyson: ``The most general kind of matrix ensemble, defined with a symmetry group which may be completely arbitrary, reduces to a direct product of independent irreducible ensembles each of which belongs to one of three known types.’’

This classification has proved fundamental to various areas of theoretical physics, including the statistical theory of complex many-body systems, mesoscopic physics, disordered electron systems, and the field of quantum chaos.

The Tenfold Way

Beyond Dyson:

Random Matrix Theory and Chiral Symmetry in QCD

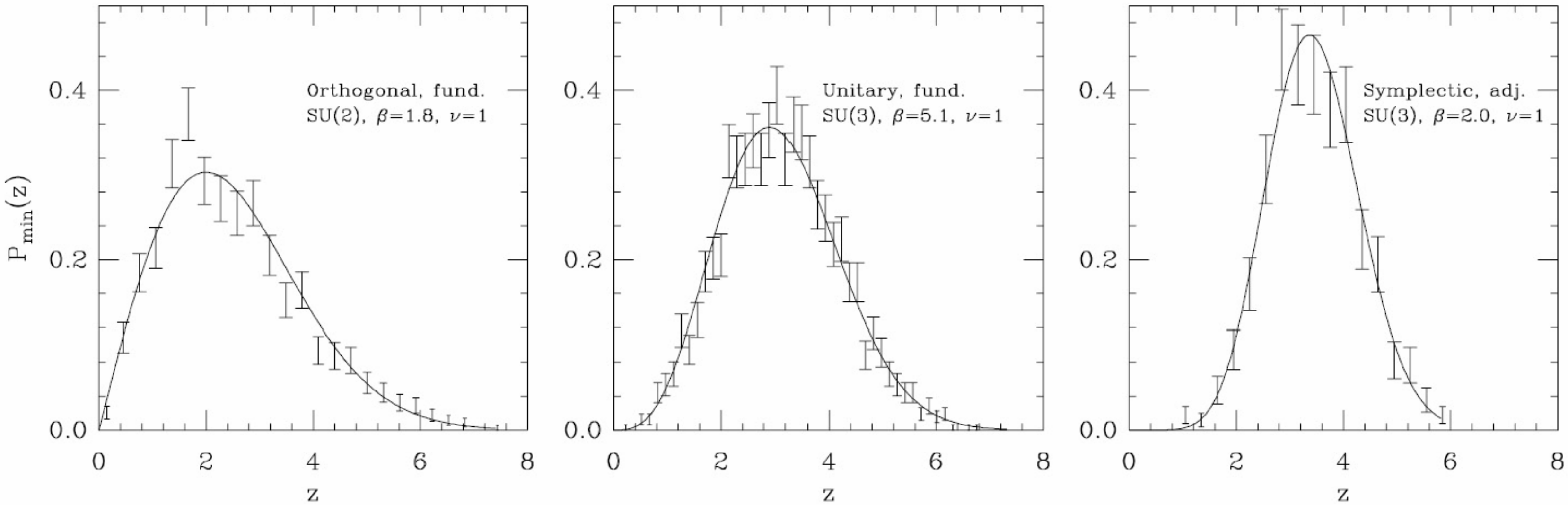


Figure 6: Distribution of the smallest Dirac eigenvalue in the $\nu = 1$ sector for all three symmetry classes. The data were obtained using the overlap Dirac operator on a 4^4 lattice. Solid lines represent the corresponding RMT results.

Metal / superconductor junctions

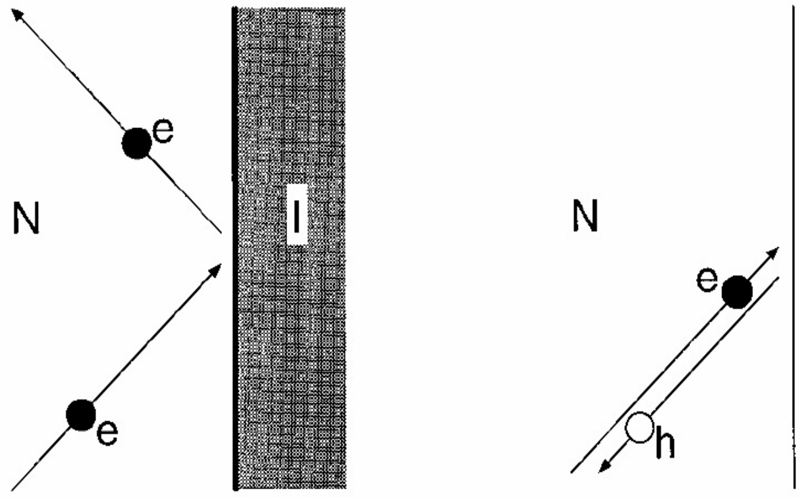
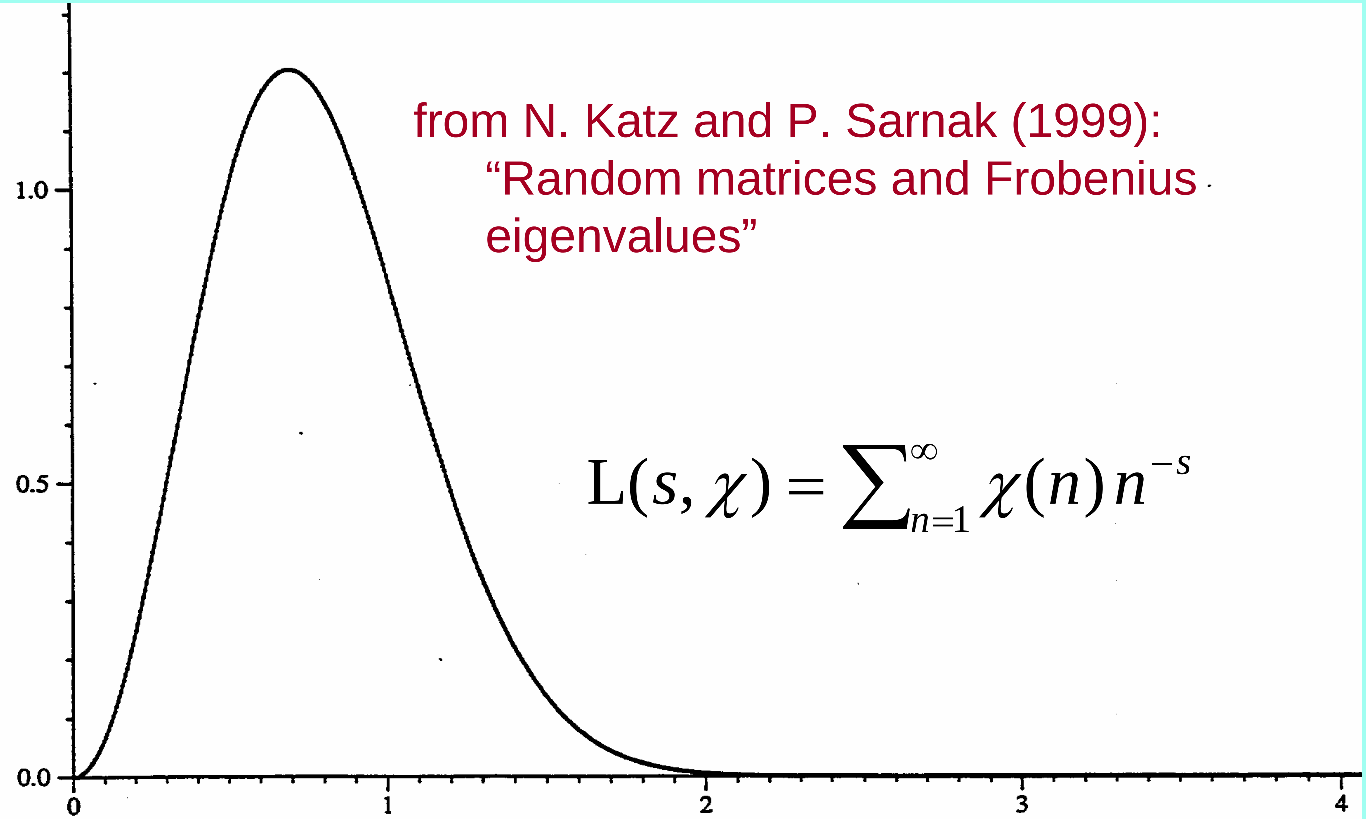


FIG. 27. Normal reflection by an insulator (I) versus Andreev reflection by a superconductor (S) of an electron excitation in a normal metal (N) near the Fermi level. Normal reflection (left) conserves charge but does not conserve momentum. Andreev reflection (right) conserves momentum but does not conserve charge: The electron (e) is reflected as a hole (h) with the same momentum and opposite velocity. The missing charge of $2e$ is absorbed as a Cooper pair by the superconducting condensate.

Beyond Dyson: Ensembles of L-functions

from N. Katz and P. Sarnak (1999): "Random matrices and Frobenius eigenvalues"

$$L(s,\chi) = \sum_{n=1}^{\infty} \chi(n)\, n^{-s}$$



Distribution of the first zero

Figure 2: density of $\nu(-,1)$

A. Altland, MZ: Non-standard symmetry classes in mesoscopic normal-/superconducting hybrid systems, Phys. Rev. B 55 (1997) 1142-1161

MZ: Riemannian symmetric superspaces and their origin in random matrix theory, J. Math. Phys. 37 (1996) 4986-5018

Our setting: Fock space

V = Hilbert space of a single particle; $\dim V = N$

F = Fock space for (identical) fermions :

$$= F_0 \oplus F_1 \oplus F_2 \oplus \ldots \oplus F_n \oplus \ldots \oplus F_N \; .$$

$F_n = \wedge^n (V)$ (Pauli principle).

$$c_\alpha c_\beta^{\,*} + c_\beta^{\,*} c_\alpha = \delta_{\alpha\beta}$$

Our setting: symmetries

Unitary symmetries :

any group of unitary operators defined on V

and extended to F in the natural way.

Anti-unitary symmetries :

1. Time reversal $T: V \to V$ extends to $T: F_n \to F_n$
2. Particle-hole conjugation $C: F_n \to F_{N-n}$

Statement of problem

$F :=$ fermionic Fock space with a $G-$action,

$G =$ arbitrary symmetry group made from generators as described above.

$\mathcal{H} :=$ (polynomials in) $G-$invariant one-body operators, i.e., operators which commute with all symmetry generators and are quadratic in particle creation and annihilation operators:

$$H = \sum W_{\alpha\beta}\, c_\alpha^* \, c_\beta + \tfrac{1}{2} \sum (Z_{\alpha\beta}\, c_\alpha^* \, c_\beta^* + \overline{Z}_{\alpha\beta}\, c_\beta \, c_\alpha)$$

Question: What types of irreducible block occur in this setting?

Communications in
Mathematical Physics

Symmetry Classes of Disordered Fermions

P. Heinzner[1], A. Huckleberry[1], M.R. Zirnbauer[2]

[1] Fakultät für Mathematik, Ruhr-Universität Bochum, Germany.
E-mail: heinzner@cplx.ruhr-uni-bochum.de; ahuck@cplx.ruhr-uni-bochum.de
[2] Institut für Theoretische Physik, Universität zu Köln, Germany.
E-mail: zirn@thp.uni-koeln.de

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Abstract: Building upon Dyson's fundamental 1962 article known in random-matrix theory as *the threefold way*, we classify disordered fermion systems with quadratic Hamiltonians by their unitary and antiunitary symmetries. Important physical examples are afforded by noninteracting quasiparticles in disordered metals and superconductors, and by relativistic fermions in random gauge field backgrounds.

The primary data of the classification are a Nambu space of fermionic field operators which carry a representation of some symmetry group. Our approach is to eliminate all

Theorem (H^2Z) :

Every irreducible block (of the Hamiltonians) occurring in this setting corresponds to a classical irreducible symmetric space,

and conversely,

every classical irreducible symmetric space occurs in this way.

What's a symmetric space?

Infinitesimal version :

Lie algebra $\mathfrak{g}$

with involution $\theta : \mathfrak{g} \rightarrow \mathfrak{g}$, $\theta^2 = 1$

$$\theta([X, Y]) = [\theta(X), \theta(Y)] .$$

The negative θ – eigenspace :

$$\mathfrak{p} = \{ X \in \mathfrak{g} : \theta(X) = -X \}$$

is an infinitesimal model of symmetric space.

Example : $\mathfrak{g} = \mathfrak{so}(3)$,

$$\theta(J_z) = J_z \ , \quad \theta(J_x) = -J_x \ , \quad \theta(J_y) = -J_y \ .$$

10-Way Table

Family name : CI

Symmetric space : $\mathrm{Sp}(2N)/\mathrm{U}(N)$

Standard form : $$H = \begin{pmatrix} 0 & Z \\ Z^* & 0 \end{pmatrix},$$

Z complex symmetric.

Realization : by quasiparticle excitations of disordered spin-singlet superconductors in the Meissner phase. (Important special case : d-wave superconductors)

10-Way Table

Family name : C

Symmetric space : $\mathrm{Sp}(2N)$

Standard form : $$H = \begin{pmatrix} W & Z \\ Z^* & -W^{\mathrm{t}} \end{pmatrix},$$

W hermitian, Z complex symmetric.

Realization : spin-singlet superconductor (same as CI), but in the mixed phase, with magnetic vortices.

10-Way Table

Family name: DIII

Symmetric space: $SO(2N)/U(N)$

Standard form: $H = \begin{pmatrix} 0 & Z \\ Z^* & 0 \end{pmatrix},$

Z complex skew.

Realization: spin-singlet superconductor with strong spin-orbit scattering (e.g., heavy-fermion sup. cond.); spin-triplet superconductor; B-phase of superfluid ^{3}He.

10-Way Table

Family name : D

Symmetric space : $\mathrm{SO}(2N)$

Standard form : $$H = \begin{pmatrix} W & Z \\ Z^* & -W^{\mathrm{t}} \end{pmatrix},$$

W hermitian, Z complex skew.

Realization : disordered superconductor with spin-triplet pairing and T-breaking p-wave symmetry (Sr_2RuO_4); A-phase of superfluid ^{3}He. ("Majorana fermions")

10-Way Table

Family name: AIII

Symmetric space: $\mathrm{U}(p+q)\,/\,\mathrm{U}(p)\times\mathrm{U}(q)$

Standard form: $$H = \begin{pmatrix} 0 & Z \\ Z^* & 0 \end{pmatrix},$$

Z complex $p\times q$ matrix.

Realization: massless Dirac fermions

in $\mathrm{SU}(N)$ gauge field background ($N>2$);

d-wave superconductor with soft impurity scattering.

10-Way Table

Family name : BDI

Symmetric space : $\mathrm{O}(p+q)\,/\,\mathrm{O}(p)\times\mathrm{O}(q)$

Standard form : $$H = \begin{pmatrix} 0 & Z \\ Z^* & 0 \end{pmatrix},$$

Z real $p \times q$ matrix.

Realization : massless Dirac fermions with gauge group $\mathrm{SU}(2)$ or $\mathrm{Sp}(N)$.

10-Way Table

Family name : CII

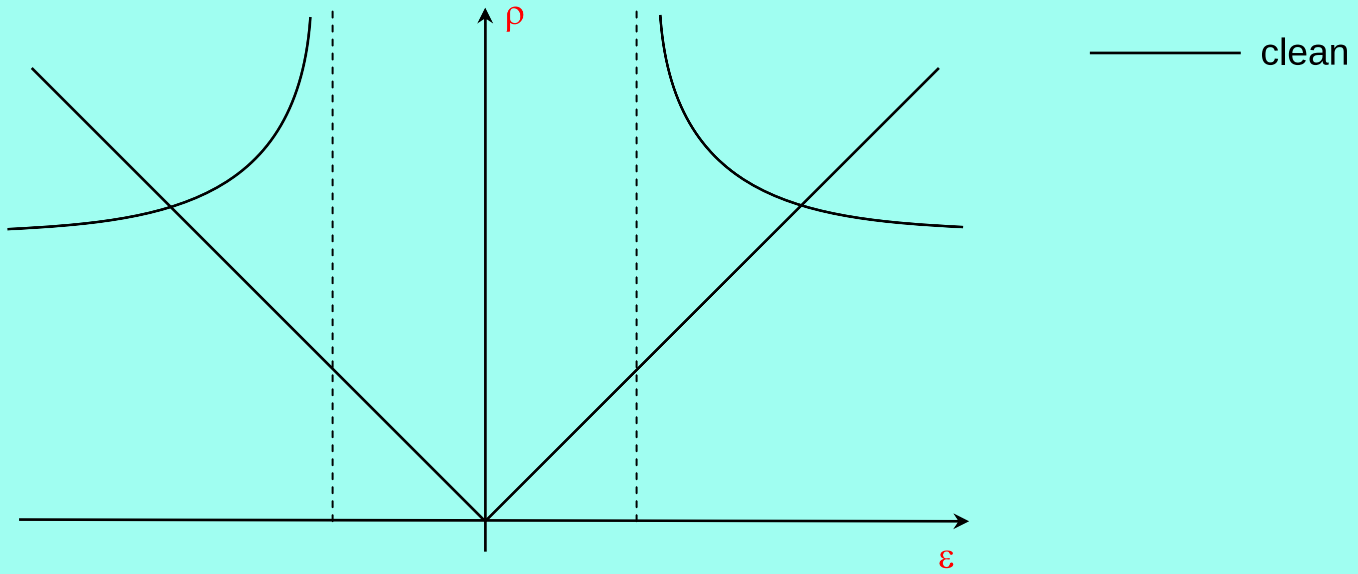
Symmetric space : $\mathrm{Sp}(p+q)\,/\,\mathrm{Sp}(p)\times\mathrm{Sp}(q)$

Standard form : $$H = \begin{pmatrix} 0 & Z \\ Z^* & 0 \end{pmatrix},$$

Z quaternion $p\times q$ matrix.

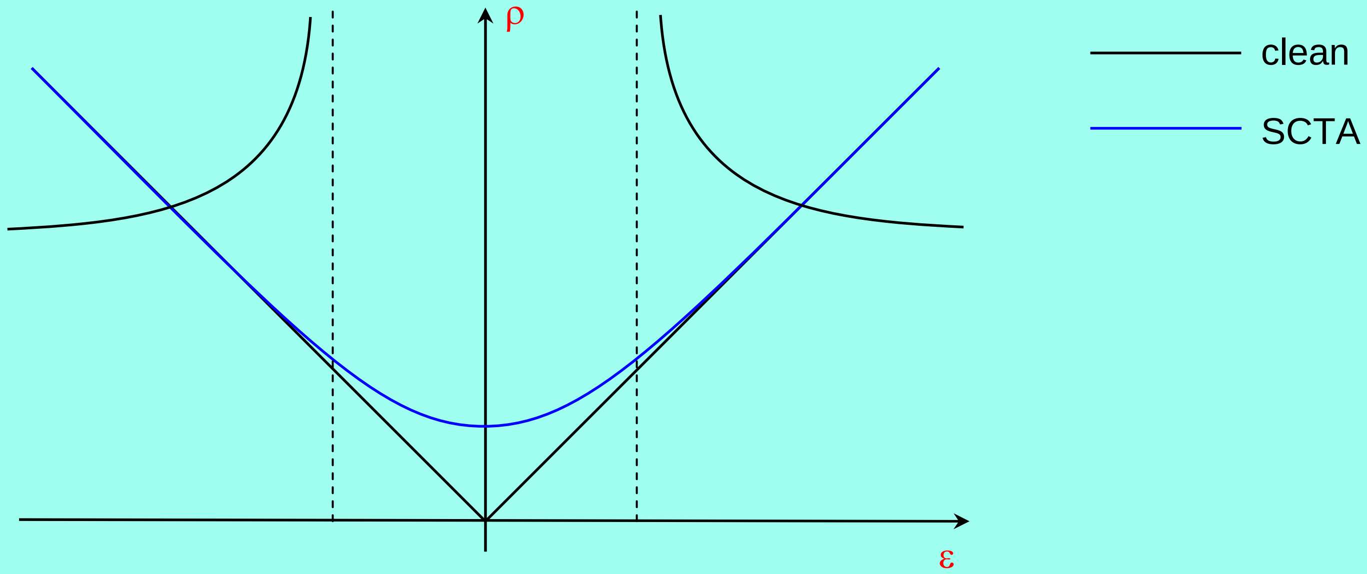
Realization : same as AIII and BDI, but with adjoint fermions or with gauge group $\mathrm{SO}(N)$.

Example



Effect of a random potential on the quasiparticle density of states of a d-wave superconductor

Example



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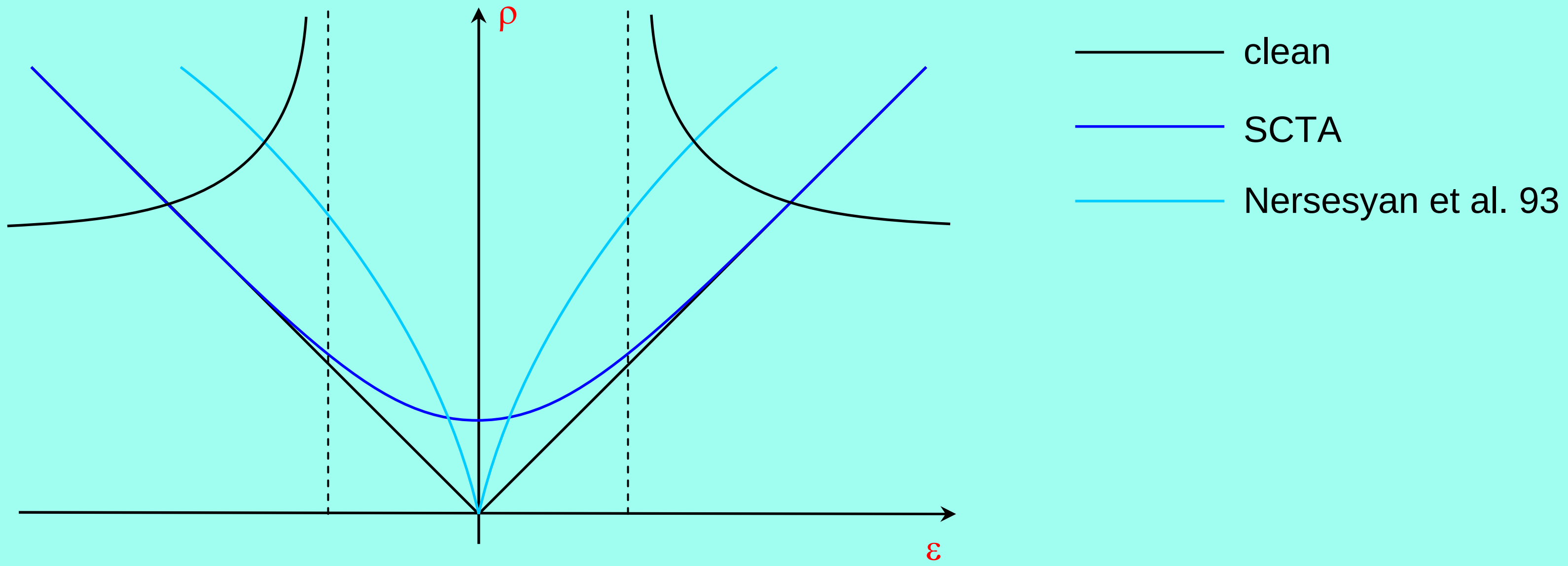

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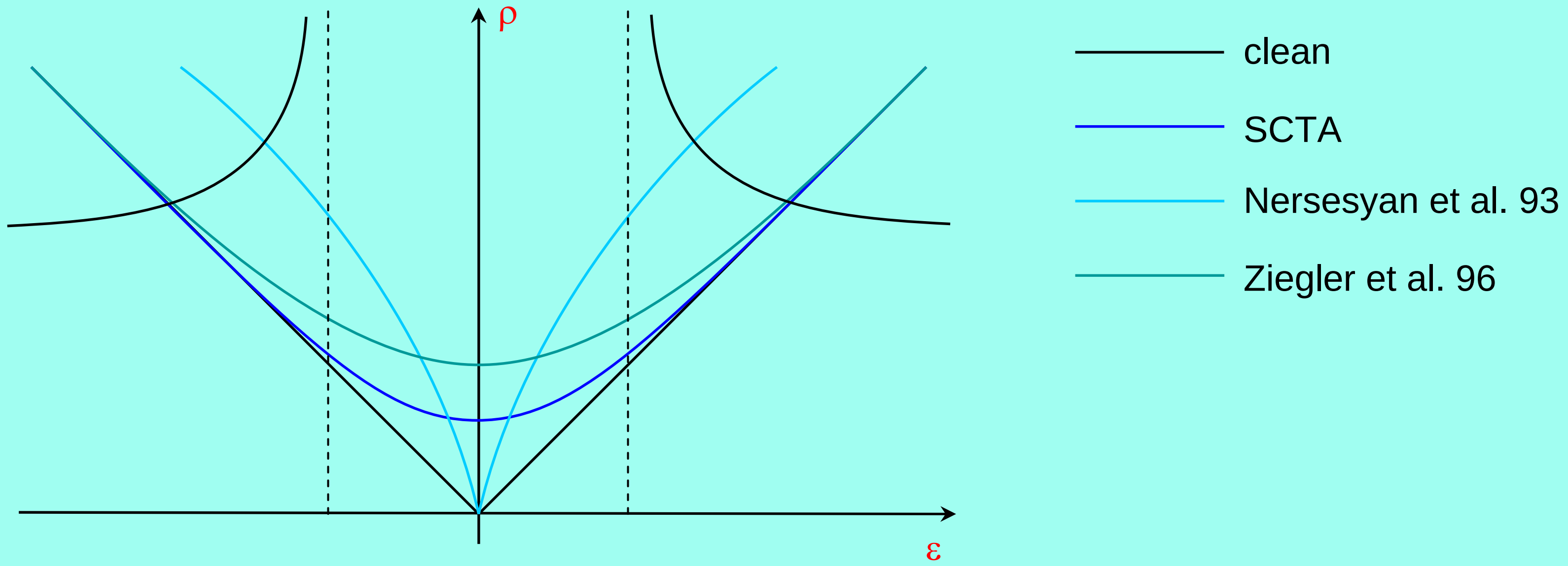

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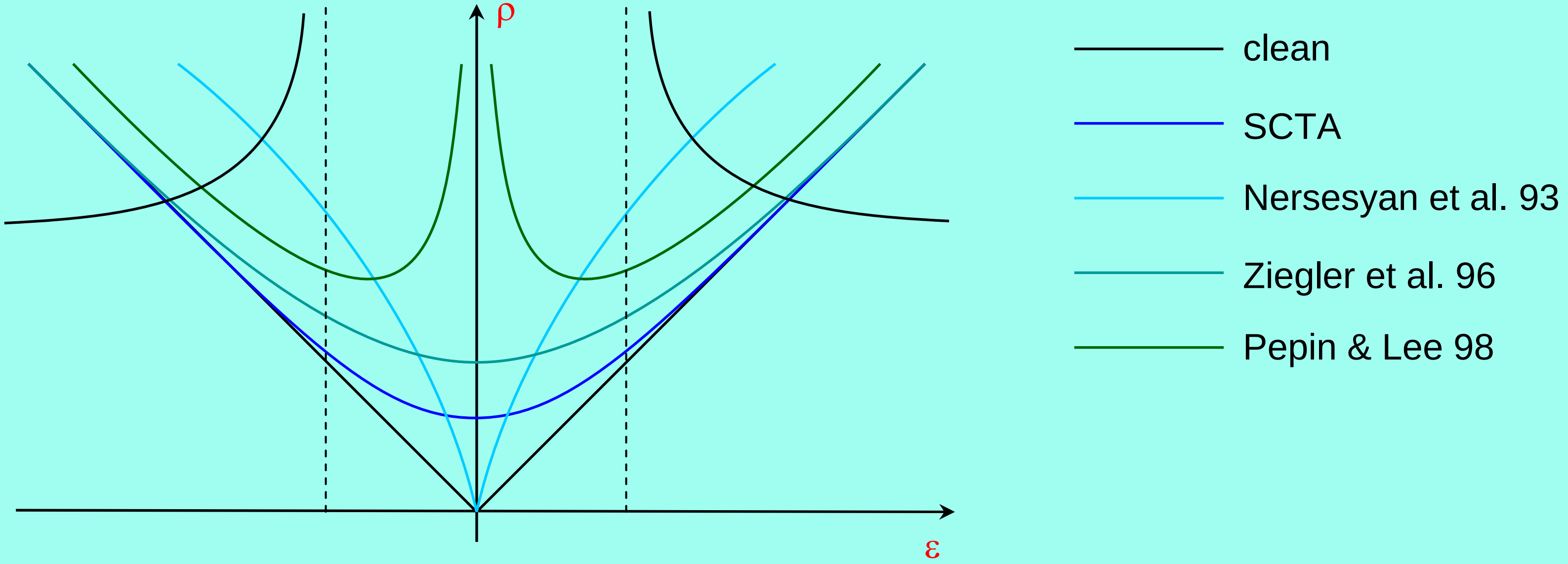

Effect of a random potential on the quasiparticle density of states of a d-wave superconductor

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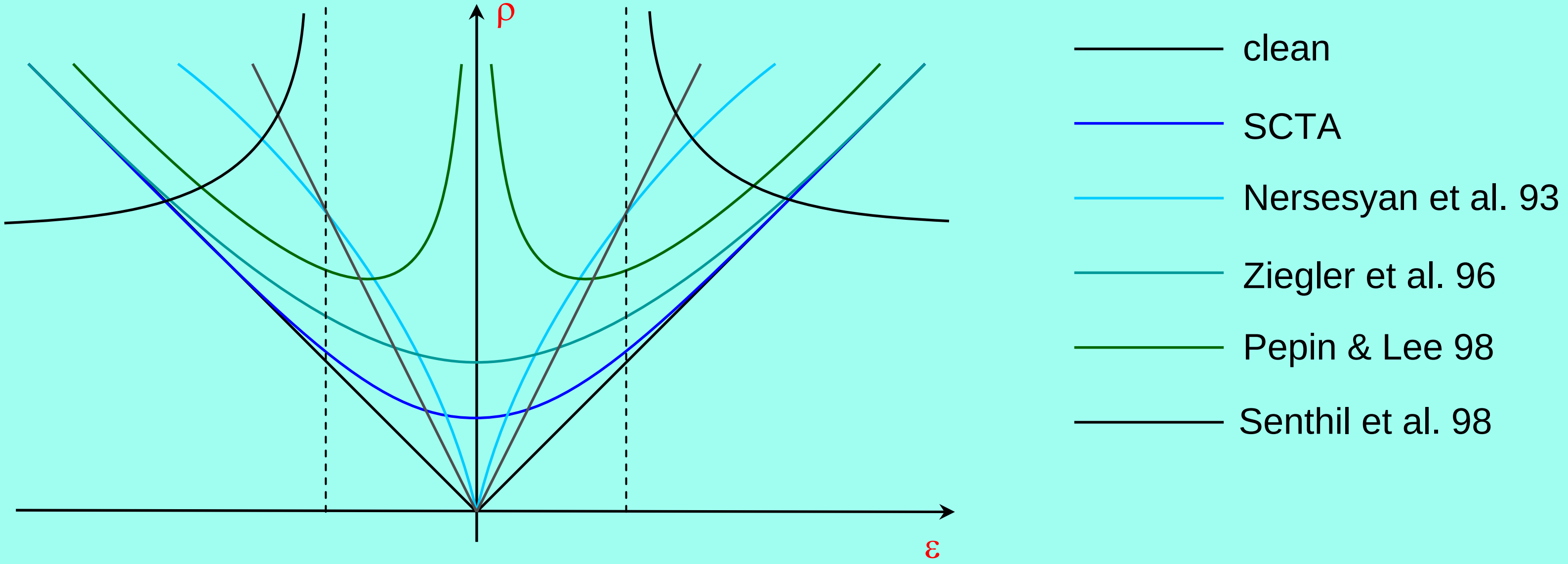


Effect of a random potential on the quasiparticle density of states of a d-wave superconductor

Example (continued)

Quasiparticle excitations in a spin-singlet superconductor with d-wave symmetry and potential disorder :

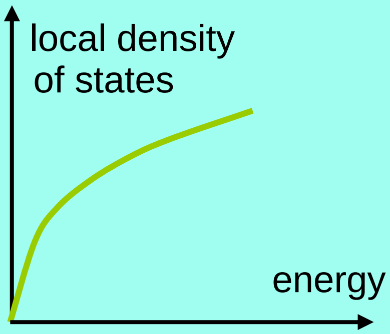
$$V = \mathbb{C}^N \otimes \mathbb{C}^2_{\text{spin}} \ ,$$

$G = G_{\text{U}} \cup T G_{\text{U}}$, $\quad G_{\text{U}} = \text{SU}(2)_{\text{spin}}$, $\quad T =$ time reversal.

The symmetry class is CI.

Altland, Simons & MRZ:

Phys. Rep. 359 (2002) 283



Directions of current research

- Analysis of supersymmetric nonlinear sigma models
- Symmetry classes of disordered bosons
- Classification of topological insulators and superconductors

Topological Insulators

topol. invariant for quantum Hall systems (Thouless et al., 1982)

$\mathbb{Z}_2$ topological insulator :

Kane & Mele (2005),

S.C. Zhang et al. (2006),

König et al. (2007)

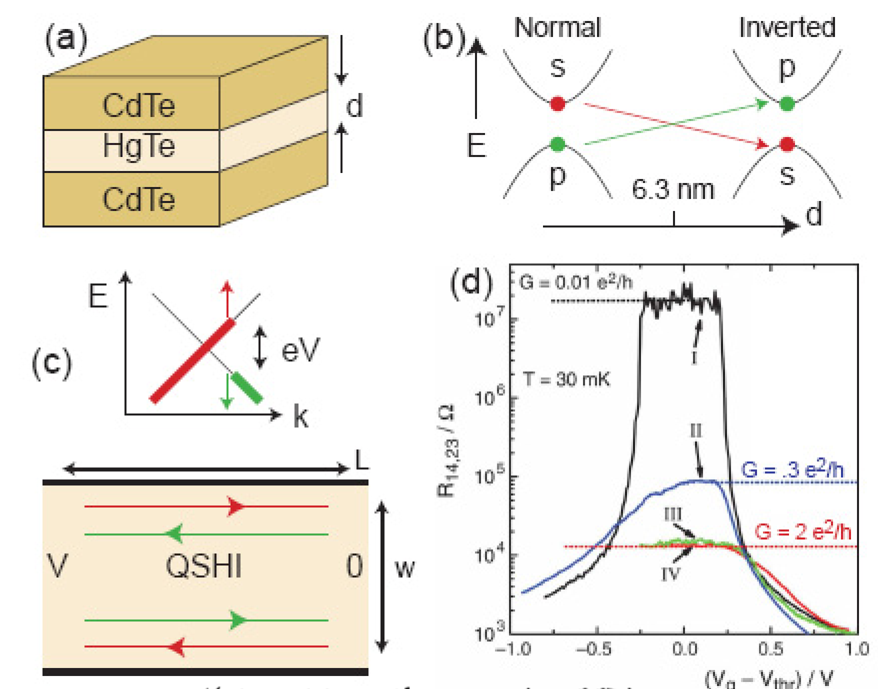
$d = 2$: HgCdTe,

$d = 3$: $Bi_{1-x}Sb_x$, Bi_2Se_3, Bi_2Te_3

insulating bulk,

but metallic surface !

(robust against disorder)



(Adapted from König et al., 2007)

FIG. 6 (a) A HgCdTe quantum well structure. (b) As a function of layer thickness d the 2D quantum well states cross at a band inversion transition. The inverted state is the QSHI

Crystal (solid body): regular graph $\Gamma \subset \mathbb{Z}^d \times \mathbb{C}^n$

single-electron Hilbert space $V = l^2(\Gamma)$

many-electron ground state $\Psi \in \wedge^N(V)$

Assume translation invariance $\Rightarrow V = \oplus V_k \;\; (V_k \cong \mathbb{C}^n)$

Then ground state in mean-field approximation

is given by a mapping

$\psi : k \mapsto \mathrm{Gr}_m(V_k) \to \wedge(V_k)$

homotopy classes

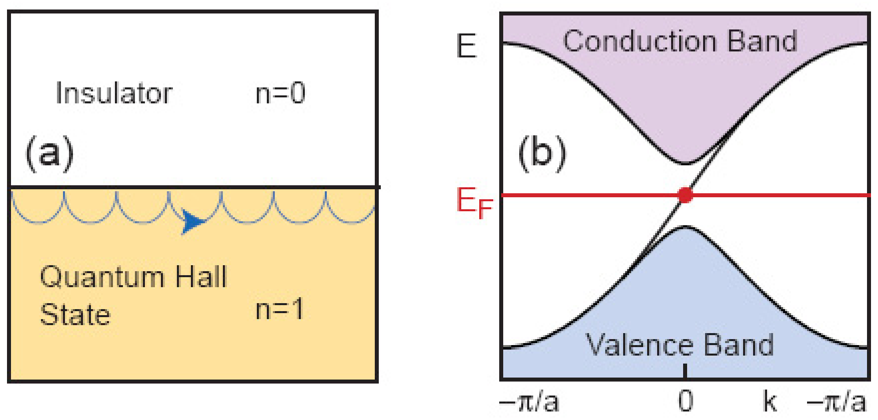


FIG. 2 The interface between a quantum Hall state and an insulator has chiral edge mode. (a) depicts the skipping cyclotron orbits, and (b) shows the electronic structure of a semi infinite quantum Hall state described by the Haldane model. A single edge state connects the valence band to the conduction band. (From Hasan & Kane, RMP 2010)

Classification of Topological Insulators

Retain setting of Tenfold Way !

G-invariant non-interacting ground state is mapping

Brillouin zone $\rightarrow$ symmetric space

Gapped single-particle spectrum (band insulator)

Classification of such ground states is problem in homotopy theory

A. Kitaev (2008) : Bott periodicity, K-theory

Periodic Table (Kitaev, 2008; Ludwig et al., 2009)

Symmetry				*d*							
AZ	Θ	Ξ	Π	1	2	3	4	5	6	7	8
A	0	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$
AIII	0	0	1	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0
AI	1	0	0	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$
BDI	1	1	1	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$
D	0	1	0	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$
DIII	-1	1	1	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0
AII	-1	0	0	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$
CII	-1	-1	1	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0
C	0	-1	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0
CI	1	-1	1	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0

TABLE I Periodic table of topological insulators and superconductors. The 10 symmetry classes are labeled using the notation of Altland and Zirnbauer (1997) (AZ) and are specified by presence or absence of $\mathcal{T}$ symmetry Θ, particle-hole symmetry Ξ and chiral symmetry $\Pi = \Xi\Theta$. ± 1 and 0 denotes

Quantum Hall Effect

He-3 (B phase)

HgTe

$Bi_{1-x}Sb_x$

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AI	1	0	0	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$
BDI	1	1	1	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$
D	0	1	0	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$
DIII	-1	1	1	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0
AII	-1	0	0	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$
CII	-1	-1	1	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0
C	0	-1	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0
CI	1	-1	1	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0

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D	0	1	0	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$
DIII	-1	1	1	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0
AII	-1	0	0	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$
CII	-1	-1	1	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0
C	0	-1	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0
CI	1	-1	1	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0

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