Superbosonization meets Free Probability

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- Introduction
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- Large-N characteristic function by free probability
- Superbosonization
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Introduction

'Free probability' introduced by D. Voiculescu (1986) in the study of von Neumann algebras.

Gives calculational scheme by which to handle invariant ensembles of $N \times N$ random matrices, $N = \infty$.

Large-N limit of density of states encoded in Voiculescu R-transform: $R(k) = \sum_{n=0}^{\infty} c_{n+1} k^n$.

Free cumulants c_n are additive under addition of independent random matrices (for $N \to \infty$).

Voiculescu's analytical approach : define R-transform by inverting $z \mapsto g(z)$ (average trace of resolvent).

Combinatorial description of free cumulants in terms of non-crossing partitions given by R. Speicher (1994).

Free probability theory has not yet produced results for spectral correlation functions in the microscopic limit.

Method of commuting and anti-commuting variables (Wegner, Efetov) : results for *correlation functions*

(e.g., level statistics of small metallic grains,

localization in thick disordered wires,

scaling exponents at the Anderson transition, etc.)

Traditional variant (Hubbard-Stratonovich transformation) limited to *Gaussian* random variables.

Recent variant called 'superbosonization' allows to treat much wider class of distributions.

Introduction: supersymmetry method (first steps)

$H = H^*$ linear operator on Hermitian vector space $\mathbb{C}^N$.

Gaussian integral over commuting variables $\varphi \in \mathbb{C}^N$:

$$\mathrm{Det}^{-1}(z - H) \;=\; \int \mathrm{e}^{-(\tilde{\varphi},\, \varphi z - H\varphi)}, \;\; \tilde{\varphi} = -\mathrm{i}\,\mathrm{sign}(\mathrm{Im}\, z)\; \overline{\varphi} \in (\mathbb{C}^N)^*$$

Gaussian ($Berezin$) integral over anti-commuting

variables ψ : $\quad \mathrm{Det}\,(w - H) \;=\; \int \mathrm{e}^{(\tilde{\psi},\, \psi w - H\psi)}$.

$$\left\langle \frac{\mathrm{Det}\,(w - H)}{\mathrm{Det}\,(z - H)} \right\rangle_{\mu} = \int \Omega\big(\varphi \otimes \tilde{\varphi} + \psi \otimes \tilde{\psi}\big)\, \mathrm{e}^{-z(\tilde{\varphi},\varphi) + w(\tilde{\psi},\psi)},$$

Characteristic function: $\quad \Omega(K) = \int \exp\big(\mathrm{Tr}\, HK\big)\, d\mu(H)$.

From moments to cumulants

From moments to cumulants (I)

Commutative case $(N=1)$:

Moments $m_n = \int x^n d\mu(x)$ are generated by the

characteristic function $\Omega(k) = \int \mathrm{e}^{kx} d\mu(x) = \sum_{n=0}^{\infty} m_n \frac{k^n}{n!}$

The logarithm $\omega(k) = \ln \Omega(k) = \sum_{n=1}^{\infty} c_n \frac{k^n}{n!}$

generates the cumulants $c_n = \frac{d^n}{dk^n} \omega(k) \Big|_{k=0}$

From moments to cumulants (II): combinatorial description

Moments are expressed in terms of cumulants,

$$m_n = \frac{d^n}{dk^n}\Omega(k)\bigg|_{k=0} = \frac{d^n}{dk^n}\mathrm{e}^{\omega(k)}\bigg|_{k=0} = \sum_{p\in\Pi(n)}\prod_{l} c_l^{\nu_l(p)} \,,$$

by summing over partitions $p \in \Pi(n): \; \sum_{l\geq 1} l\,\nu_l(p) = n$

where $\nu_l(p)$ is the number of blocks of length l.

Example $(n=8)$: $p = \{136\}\cup\{28\}\cup\{45\}\cup\{7\}$

$\nu_1(p)=1, \;\; \nu_2(p)=2, \;\; \nu_3(p)=1, \;\; \nu_4(p)=\ldots=0.$

From moments to cumulants (III): R-transform

Probability measure $d\mu_N(H)$ for $N \times N$ matrices $H = H^*$

Characteristic function: $\Omega(K) = \int \mathrm{e}^{\mathrm{Tr}\, HK}\, d\mu_N(H)$

Moments $m_{n,N} = N^{-1} \int \mathrm{Tr}(H^n)\, d\mu_N(H)$ are generated by

differentiation: $$m_{n,N} = N^{-1} \sum_{i_1,\dots,i_n} \frac{\partial^n}{\partial K_{i_n i_{n-1}} \cdots \partial K_{i_2 i_1} \partial K_{i_1 i_n}} \Omega(K) \Big|_{K=0}$$

Let $m_n := \lim_{N\to\infty} m_{n,N}$

By definition, $z \mapsto g(z) = \sum_{n=0}^{\infty} m_n\, z^{-n-1}$ is inverted by

$k \mapsto k^{-1} + R(k)$ where $R(k) = \sum_{n=0}^{\infty} c_{n+1}\, k^n$ (R-transform).

R-transform (examples)

$$z = k^{-1} + R(k) \quad \Leftrightarrow \quad k = g(z)$$

Example 1: $R(k) = k$ (GUE)

$$g(z) = \frac{z}{2}\left(1 - \sqrt{1 - \frac{4}{z^2}}\right)$$ (Wigner semicircle)

Example 2: $R(k) = \frac{k}{1-k^2}$

$$g(z) = \mathrm{i}\left(\sqrt{\frac{1}{27} - \frac{1}{4z^2}} - \frac{\mathrm{i}}{2z}\right)^{1/3} - \mathrm{i}\left(\sqrt{\frac{1}{27} - \frac{1}{4z^2}} + \frac{\mathrm{i}}{2z}\right)^{1/3}$$

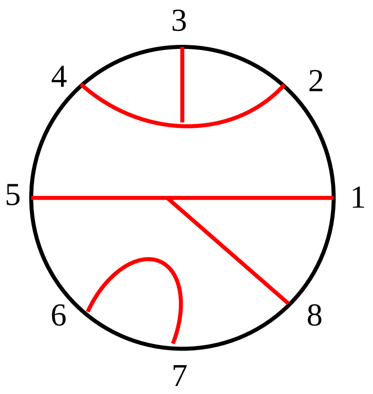
Lueck, Sommers, MZ: J. Math. Phys. **47** (2006) 103304

Recall $R(k) = \sum c_{n+1} k^n$.

R. Speicher (1994): Combinatorial definition of ***free*** cumulants c_n by $m_n = \sum_{p \in \mathrm{NC}(n)} \prod_l c_l^{\nu_l(p)}$ where

the sum runs over ***non-crossing*** partitions $p \in \mathrm{NC}(n)$.

Example $(n=8)$: $p = \{158\} \cup \{234\} \cup \{67\}$



Free cumulants ***add*** under convolution of measures (or addition of independent random matrices).

Large-N characteristic function

by free probability

Note: $\omega(K) = \lim_{N\to\infty} N^{-1} \ln \Omega(NK)$ is additive under addition of independent random matrices.

Assume $d\mu_N(H) = \mathrm{e}^{-N \operatorname{Tr} V(H)} dH$. Then $\omega(K) = \omega(g^{-1}Kg)$ for $g \in \mathrm{GL}_N$ and cumulants must be of the general form [where $\gamma_n(\pi)$ is constant on conjugacy classes]

$$\frac{\partial^n}{\partial K_{i_1 j_1}\, \partial K_{i_2 j_2} \cdots \partial K_{i_n j_n}}\, \omega(K)\Bigg|_{K=0} = \sum_{\pi \in S_n} \gamma_n(\pi) \prod_l \delta_{i_l, j_{\pi(l)}}$$

Graphical methods suggest large-N hypothesis:
$\gamma_n(\pi) = c_n$ if π irreducible cycle, and $\gamma_n(\pi) = 0$ else.

Heuristics from planar graphs (I)

Recall $\Omega(NK) = \int e^{-N\,\mathrm{Tr}\,V(H) + N\,\mathrm{Tr}\,HK}\,dH$.

Perturbation theory for $\ln \Omega(NK)$ (connected graphs)

leads to topol. expansion: $\ln \Omega(NK) = \sum_{\chi=1,0,-1,\ldots} N^{\chi}\omega_{\chi}(K)$.

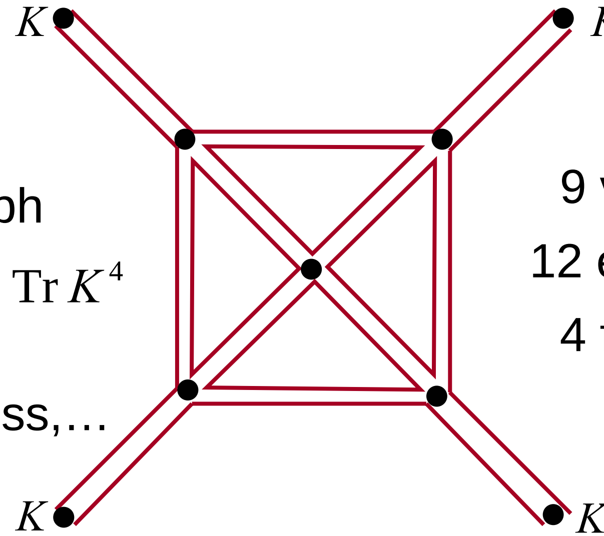
Leading contribution comes from summing

all planar graphs (Euler characteristic $\chi = 1$).

Example: graph

contributing to $\mathrm{Tr}\,K^4$

t'Hooft, Witten, Gross,…

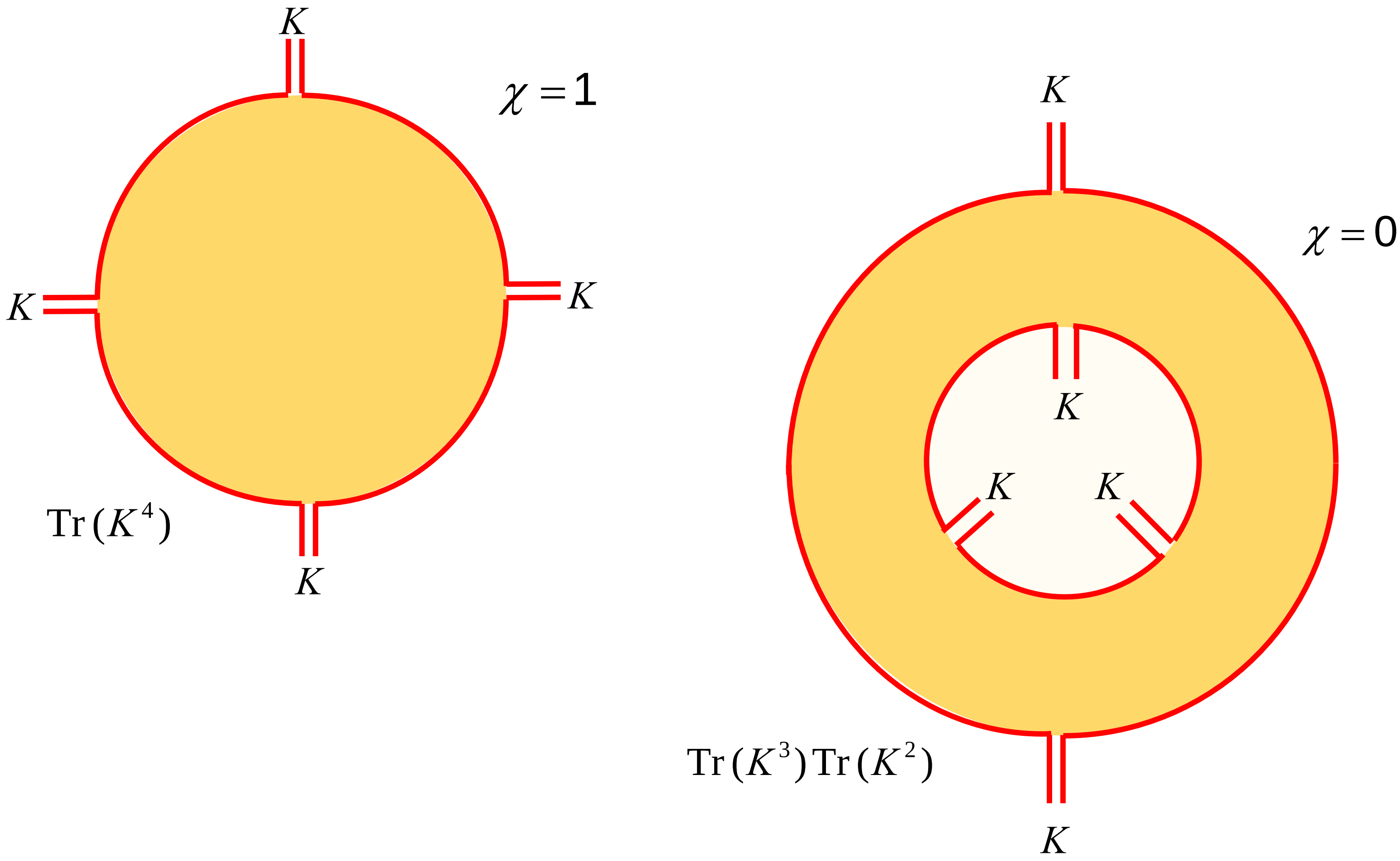


9 vertices: N^9

12 edges: N^{-12}

4 faces: N^4

Heuristics from planar graphs (II)



Large-N characteristic function from free probability (I)

Recall the large-N scenario from planar graphs:

$$\frac{\partial^n}{\partial K_{i_1 j_1} \cdots \partial K_{i_n j_n}} N^{-1} \ln \Omega(NK) \Big|_{K=0} \xrightarrow{N\to\infty} c_n \sum_{\pi \in [\mathrm{irr}]} \prod_l \delta_{i_l, j_{\pi(l)}}$$

By Speicher's combinatorial description the numbers c_n are identified as the free cumulants. In fact, taking $N \to \infty$ in the formula for the moment, we have $m_{n,N} =$

$$\sum_{i_1,\ldots,i_n} \frac{N^{-n-1}\partial^n}{\partial K_{i_n i_{n-1}} \cdots \partial K_{i_1 i_n}} \mathrm{e}^{\ln \Omega(NK)} \Big|_{K=0} \xrightarrow{N\to\infty} \sum_{p \in \mathrm{NC}(n)} \prod_l c_l^{\,\nu_l(p)}$$

Summary.

$\omega(K) := \lim_{N\to\infty} N^{-1} \ln \Omega(NK)$.

For ensembles with GL_N-symmetry

we have $\omega(K) = \sum_{n=0}^{\infty} c_{n+1} \frac{\mathrm{Tr}(K^{n+1})}{n+1}$.

Note: the derivative of $\omega(K)$ for $K = k\Pi$

(rank-1 projector Π) is the R-transform:

$$\frac{d}{dk}\omega(k\Pi) = \sum_{n=0}^{\infty} c_{n+1} k^n = R(k)$$

Non-perturbative argument: reduction to 1D integral

Let $K = Nk\Pi$ ($k \in \mathbb{C}$, rank-one projector Π).

Recall: $\Omega(Nk\Pi) = \int \mathrm{e}^{Nk\,\mathrm{Tr}\,(\Pi H) - N\,\mathrm{Tr}\,V(H)} dH.$

Diagonalize $H = g\,\mathrm{diag}(\lambda_1, \dots, \lambda_N)\, g^{-1}.$

Do integral over $g \in \mathrm{U}_N$ using HCIZ formula:

$$\Omega(Nk\Pi) = c_N\, k^{-N+1} \int_{\mathbb{R}} \mathrm{e}^{Nkx - N V(x)} p_{N-1,N}(x)\, dx$$

where $p_{N-1,N}(x)$ orthogonal polynomial of degree $N-1$ for $\mathrm{e}^{-NV(x)}$.

Note: $p_{N-1,N}$ has $N-1$ zeroes in $[a,b] \subset \mathbb{R}$.

Asymptotics for k large

Saddle point of x-integral lies outside of $[a,b]$.

Large-N asymptotics of $p_{N-1,N}$ when $x \notin [a,b]$:

$$p_{N-1,N}(x) \approx \mathrm{e}^{N\int_a^b \ln(x-y)\, d\nu(y)} = \mathrm{e}^{NG(x)} .$$

Integral for scaled logarithm of char. function

$$\omega(k\Pi) = c - \ln k + \lim_{N\to\infty} N^{-1} \ln \int_{\mathbb{R}} \mathrm{e}^{Nkx - NV(x) + NG(x)}\, dx$$

has good saddle point : $0 = k - V'(x) + g(x)$.

Final result (Dyson Coulomb gas) : $\omega(k\Pi) = \int_0^k R(t)\, dt$.

Asymptotics for k small

No good saddle point for previous integral (saddle wants to be in oscillatory region).

Use known asymptotics of $p_{N-1,N}$ to switch to new representation (contour integral):

$$\omega(k\Pi) = -1 + \ln k + \lim_{N\to\infty} N^{-1} \ln \oint_C \mathrm{e}^{Nkz - NG(z)}\, dz$$

with saddle point equation $k = g(z)$.

Final result (again): $\omega(k\Pi) = \int_0^k R(t)\, dt$.

Guionnet & Maida (2004)

Superbosonization

Hackenbroich, Weidenmüller (95)

Lehmann, Saher, Sokolov, Sommers (95)

Barruto, Brower, Svetitsky (01)

Efetov, Schwiete, Takahashi (04)

Guhr (06); Basile, Akemann (07)

Bunder, Efetov, Kravtsov, Yevtushenko, MZ (07)

Littelmann, Sommers, MZ (08)

Reminder: supersymmetry method

Characteristic function $\Omega(K) = \int \mathrm{e}^{\,\mathrm{Tr}\, HK} d\mu(H)$

is evaluated on $K_{ij} = \sum_{a=1}^{p} \varphi_{i,a}\,\tilde{\varphi}_{a,j} + \sum_{b=1}^{q} \psi_{i,b}\,\tilde{\psi}_{b,j}$.

Generating function for spectral correlation functions:

$$\int D_{\varphi,\tilde{\varphi};\psi,\tilde{\psi}}\, f(\varphi,\tilde{\varphi};\psi,\tilde{\psi}) \equiv \int f$$

where the integral is along $\tilde{\varphi}_{a,j} = -\mathrm{i}\,\mathrm{sign}\,(\mathrm{Im}\, z_a)\,\overline{\varphi}_{j,a}$ and

$$f(\varphi,\tilde{\varphi};\psi,\tilde{\psi}) = \Omega(K)\,\exp\left(-\sum_i \left(\sum_a \varphi_{i,a}\, z_a\,\tilde{\varphi}_{a,i} + \sum_b \psi_{i,b} w_b\,\tilde{\psi}_{b,i}\right)\right).$$

If $d\mu(H)$ invariant by some group G acting by conjugation $H \mapsto gHg^{-1}$, then $f(\varphi,\tilde{\varphi};\psi,\tilde{\psi}) = f(g\varphi,\tilde{\varphi}\,g^{-1};g\psi,\tilde{\psi}\,g^{-1})$.

Superbosonization (special case: commuting variables only)

Let $p=1,\ q=0$ and consider GL_N-invariant holom. fctn

$$f:\mathbb{C}^N\times(\mathbb{C}^N)^*\to\mathbb{C},\ \ f(\varphi,\tilde{\varphi})=f(g\varphi,\tilde{\varphi}\,g^{-1}),\ \ g\in\mathrm{GL}_N\,.$$

Fact (invariant theory): there exists a holomorphic function $F:\ \mathbb{C}\to\mathbb{C}$ such that $F(\langle\tilde{\varphi},\varphi\rangle)=f(\varphi,\tilde{\varphi})$.

By push forward of the integral one has

$$\int_{\mathbb{C}^N} f(\varphi,\varphi^*)\,d^{2N}\varphi=c_N\int_{\mathbb{R}_+}F(r)\,r^{N-1}dr$$ (if integral exists).

generalization to p > 1: see Fyodorov, Nucl. Phys. B **621** (2002) 643

Superbosonization (special case: Grassmann variables only)

$p=0,\ q=1.$ Let $F:\mathbb{C}\to\mathbb{C}$ be holomorphic.

Anticommuting variables $\psi=(\psi_1,\ldots,\psi_N)$.

Berezin integral $\int F(\langle\tilde{\psi},\psi\rangle)\,d\tilde{\psi}\,d\psi\ :=$

$$= \frac{\partial^2}{\partial\psi_1\partial\tilde{\psi}_1}\cdots\frac{\partial^2}{\partial\psi_N\partial\tilde{\psi}_N}\,F(\tilde{\psi}_1\psi_1+\ldots+\tilde{\psi}_N\psi_N)$$

$= F^{(N)}(0)$ (the N^{th} derivative at the origin)

$$= N!\oint_{\mathrm{U}(1)} F(z)\,z^{-N-1}dz/2\pi\mathrm{i}.$$

q > 1: Kawamoto and Smit, Nucl. Phys. B **192** (1981) 100

The idea of superbosonization

Recall $f(\varphi,\tilde{\varphi};\psi,\tilde{\psi}) = f(g\varphi,\tilde{\varphi}\,g^{-1};g\psi,\tilde{\psi}\,g^{-1})$

for $g \in G$. Let $G = \mathrm{GL}_N$ or $G = \mathrm{O}_N$ or $G = \mathrm{Sp}_N$.

Superbosonization exploits this symmetry to make a step of *reduction* :

The integral over $\varphi,\tilde{\varphi},\psi,\tilde{\psi}$ of the G-invariant function f is converted to an integral over a Riemannian symmetric superspace.

(The large number N of variables $\varphi,\tilde{\varphi},\psi,\tilde{\psi}$ then becomes a *parameter* of the integral.)

Superbosonization: unitary symmetry

Let $G = \mathrm{GL}_N$.

Lift $f(\varphi, \tilde{\varphi}; \psi, \tilde{\psi})$ to $F(Q)$:

$$f(\varphi, \tilde{\varphi}; \psi, \tilde{\psi}) = F\begin{pmatrix} \langle \tilde{\varphi}, \varphi \rangle & \langle \tilde{\varphi}, \psi \rangle \\ \langle \tilde{\psi}, \varphi \rangle & \langle \tilde{\psi}, \psi \rangle \end{pmatrix}.$$

Theorem (Littelmann, Sommers, MZ). If $N \geq p$ and f holomorphic and Schwartz along $\tilde{\varphi} = -\mathrm{i}\overline{\varphi}^{\mathrm{t}}$, then

$$\int_{\tilde{\varphi} = -\mathrm{i}\overline{\varphi}^{\mathrm{t}}} f = \int_M DQ \; \mathrm{SDet}^N(Q) \; F(Q)$$

with integration domain $M \cong (\mathrm{GL}_p(\mathbb{C}) / \mathrm{U}_p) \times \mathrm{U}_q$

and $\mathfrak{gl}$-invariant Berezin integration form DQ.

Application to disordered scattering

The setting

N internal states (random matrix Hamiltonian H)
coupled to M scattering channels.
Heidelberg approximation to scattering matrix :

$$S(E) = \mathrm{Id}_M - 2\mathrm{i}W^* \frac{1}{E - H + \mathrm{i}WW^*} W.$$

To compute correlations,

$$C_{ab,cd}(E_1, E_2) = \left\langle (S_{ab}(E_1) - \delta_{ab}) (\overline{S_{cd}(E_2)} - \delta_{cd}) \right\rangle,$$

use generating function :

$$Z(X,Y) = \left\langle \frac{\mathrm{Det}(E_1 - H + \mathrm{i}WXW^*)\,\mathrm{Det}(E_2 - H - \mathrm{i}WW^*)}{\mathrm{Det}(E_1 - H + \mathrm{i}WW^*)\,\mathrm{Det}(E_2 - H + \mathrm{i}WYW^*)} \right\rangle$$

Averaging trick

Problem: can't use superbosonization directly, since presence of WW^* breaks U_N-symmetry.

Use trick of averaging integrand over U_N to enforce (!) U_N-symmetry.

For large N we have formula

$$\lim_{N\to\infty} \int_{\mathrm{U}_N} \mathrm{e}^{N\,\mathrm{Tr}\,(AgBg^{-1})} dg = \mathrm{Det}^{-1}(\mathrm{Id} - A \otimes B)$$

if rank of both A and B is kept fixed. (Entries of $g \in \mathrm{U}_N$ become Gaussian random variables.)

Universality of correlation function

Correlation function by superbosonization:

$$C_{ab,cd}(E_1, E_2) = \lim_{N\to\infty} \int_M DQ \; \mathrm{e}^{N(\mathrm{STr}\ln Q + \omega(Q) - z\,\mathrm{STr}\,Q)}$$

$$\times \mathrm{e}^{(N/2)(E_1 - E_2)\,\mathrm{STr}\,\Lambda Q} F_{ab,cd}(Q).$$

Taking $N \to \infty$ gives saddle-point equation for Q:

$Q^{-1} + R(Q) = z = \frac{1}{2}(E_1 + E_2).$

Solution (saddle-point manifold) is determined by symmetry and hence universal (up to scale factor).

Conclusion: Correlations of S-matrix elements are universal (i.e., independent of the choice of random matrix ensemble) in the large-N limit.

Conclusions

Free probability theory provides the proper framework in which to take the large-N limit of the density of states for ensembles which are invariant but non-Gaussian.

Free cumulants are the Taylor coefficients of the (logarithm of the) characteristic function which is encountered when using superbosonization.

The group of supersymmetries determines the critical integration manifold (saddle points).

Our formalism establishes random matrix universality of spectral correlations as well as transport observables.