The Supersymmetry Method for Random Matrices with Local Gauge Symmetry

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- What random matrices? Which disordered systems?
- Why supersymmetry? What's the idea?
- What's a supersymmetric non-linear sigma model?
- Superbosonization: a new variant of the method for systems with local gauge symmetry

Symmetry Classes of Disordered Fermions

- Heinzner, Huckleberry, MRZ:
  Commun. Math. Phys. 257 (2005) 725
- MRZ: Encylopedia of Mathematical Physics,
  vol.5, 151-160 (Elsevier, 2006)

Symmetry classes: setting & motivation

Consider one – particle Hamiltonians (fermions) :

$$H = \tfrac{1}{2}\sum W_{\alpha\beta}\,(c_\alpha^* c_\beta - c_\beta\, c_\alpha^*) + \tfrac{1}{2}\sum (Z_{\alpha\beta}\, c_\alpha^* c_\beta^* + \overline{Z}_{\alpha\beta}\, c_\beta\, c_\alpha)$$

Canonical anticommutation relations : $c_\alpha^*\, c_\beta + c_\beta\, c_\alpha^* = \delta_{\alpha\beta}$

Applications/examples :

- Hartree – Fock – Bogoliubov theory of superconductors
- Dirac equation for relativistic spin 1/2 particles

Following Dyson, classify such Hamiltonians according to symmetries! What are the irreducible blocks that occur?

Conjecture (Altland & MRZ, 1996) :
Classification by large families of symmetric spaces

Proof of conjecture

Disordered fermion systems with quadratic Hamiltonians:

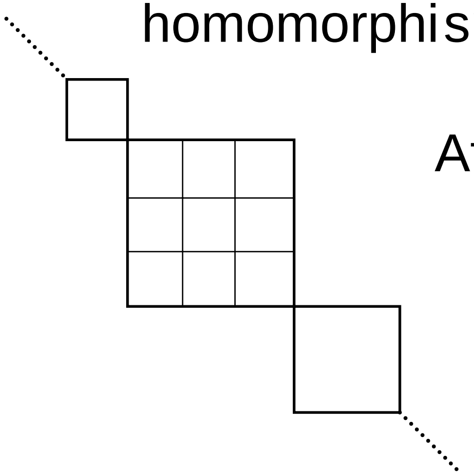
$$H = \tfrac{1}{2}\sum W_{ij}\, c_i^* c_j + \tfrac{1}{2}\sum Z_{ij}\, c_i^* c_j^* + \text{h.c.}$$

$$= \tfrac{1}{2}\begin{pmatrix} c^* & c \end{pmatrix}\begin{pmatrix} W & Z \\ Z^* & -W^{\mathrm{t}} \end{pmatrix}\begin{pmatrix} c \\ c^* \end{pmatrix} \qquad (W^* = W,\ Z = -Z^{\mathrm{t}}).$$

Nambu space $V \oplus V^*$ with symmetric bilinear form by C.A.R.

Unitary and antiunitary symmetries: $G = U \cup TU$ (U compact).

Decompose into irreducible blocks of U – equivariant homomorphisms $R \to V$ and transfer structure (C.A.R. + T)



After transfer, every set of block data specifies a classical irreducible symmetric space, and every classical irreducible symmetric space occurs in this way. (Heinzner, Huckleberry, MRZ; 2005)

The ten large families of symmetric spaces

family	symmetric space	form of $H = \begin{pmatrix} W & Z \\ Z^* & -W^t \end{pmatrix}$
A	$\mathrm{U}(N)$	complex Hermitian
$A\mathrm{I}$	$\mathrm{U}(N)/\mathrm{O}(N)$	real symmetric
$A\mathrm{II}$	$\mathrm{U}(2N)/\mathrm{USp}(2N)$	quaternion self – adjoint
C	$\mathrm{USp}(2N)$	Z complex symmetric, $W = W^*$
$C\mathrm{I}$	$\mathrm{USp}(2N)/\mathrm{U}(N)$	Z complex symmetric, $W = 0$
D	$\mathrm{SO}(2N)$	Z complex skew, $W = W^*$
$D\mathrm{III}$	$\mathrm{SO}(2N)/\mathrm{U}(N)$	Z complex skew, $W = 0$
$A\mathrm{III}$	$\mathrm{U}(p+q)/\mathrm{U}(p)\times\mathrm{U}(q)$	Z complex $p\times q$, $W = 0$
$BD\mathrm{I}$	$\mathrm{SO}(p+q)/\mathrm{SO}(p)\times\mathrm{SO}(q)$	Z real $p\times q$, $W = 0$
$C\mathrm{II}$	$\mathrm{USp}(2p+2q)/\mathrm{USp}(2p)\times\mathrm{USp}(2q)$	Z quaternion $2p\times 2q$, $W = 0$

Physical realizations

AI electrons in a disordered metal with conserved spin and with time reversal invariance invariance

A same as AI, but with time reversal broken by a magnetic field or magnetic impurities

AII same as AI, but with spin-orbit scatterers

CI quasi-particle excitations in a disordered spin-singlet superconductor in the Meissner phase

C same as CI but in the mixed phase with magnetic vortices

DIII disordered spin-triplet superconductor

D spin-triplet superconductor in the vortex phase, or with magnetic impurities

AIII massless Dirac fermions in SU(N) gauge field background (N > 2)

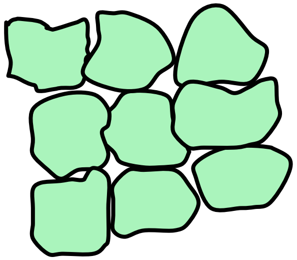
BDI same as AIII but with gauge group SU(2) or Sp(2N)

CII same as AIII but with adjoint fermions, or gauge group SO(N)

Altland, Simons & MRZ: Phys. Rep. 359 (2002) 283

Wegner's N-orbital model (class A)

Hermitian random matrices H for
a lattice Λ with N orbitals per site $i \in \Lambda$.
Hilbert space $V = \oplus_{i\in\Lambda} V_i$, $V_i = \mathbb{C}^N$.
Orthogonal projectors: $\Pi_i : V \to V_i$.



Fourier transform of probability measure $d\mu(H)$:

$$\int \exp(-\mathrm{i}\,\mathrm{Tr}\, HK)\, d\mu(H) = \prod \Omega_{ij}(K_{ij}), \quad K_{ij} = \Pi_i K \Pi_j ,$$

local gauge invariance: $\mathrm{U}(V_1)\times \mathrm{U}(V_2)\times \ldots \times \mathrm{U}(V_{|\Lambda|})$.

Gaussian distribution as a special case:

$$\prod \Omega_{ij}(K_{ij}) = \exp\left(-\tfrac{1}{2N}\sum_{i,j} C_{ij}\,\mathrm{Tr}\,\Pi_i K \Pi_j K\right)$$

$$i = j:\ C_{ii} = \mathcal{O}(N^0) , \qquad i \neq j:\ C_{ij} = \mathcal{O}(N^{-1}).$$

Outline

- Symmetry classes of disordered fermions: 10-fold way
- Riemannian symmetric superspaces
- Superbosonization
- Heuristics from non-linear sigma model
- Connection with free randomness (R-transform)

Supersymmetry method

$H = H^*$ linear operator on Hermitian vector space $\mathbb{C}^N$.

Gaussian integral over commuting variables $\varphi \in \mathbb{C}^N$:

$$\mathrm{Det}^{-1}(z - H) \;=\; \int \exp \mathrm{i}\,(\overline{\varphi}, \varphi\, z - H\varphi)\,, \qquad \mathrm{Im}\, z > 0\,.$$

Gaussian (*Berezin*) integral over anti – commuting variables ψ :

$$\mathrm{Det}\,(w - H) \;=\; \int \exp -\mathrm{i}\,(\overline{\psi}, \psi\, w - H\psi)\,, \qquad w \in \mathbb{C}.$$

$$\left\langle \frac{\mathrm{Det}\,(w - H)}{\mathrm{Det}\,(z - H)} \right\rangle_{\mu} = \int \Omega\big(\varphi \otimes \overline{\varphi} + \psi \otimes \overline{\psi}\big) \exp \mathrm{i}\, z\,(\overline{\varphi}, \varphi) - \mathrm{i}\, w\,(\overline{\psi}, \psi)\,,$$

characteristic function: $\quad \Omega(K) = \int \exp\big(-\mathrm{i}\,\mathrm{Tr}\, HK\big)\, d\mu(H)\,.$

Riemannian symmetric superspaces

MRZ: J. Math. Phys. 37 (1996) 4986

Universal construction of symmetric superspaces

Complex Lie superalgebra: $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ ($\mathbb{Z}_2$ – grading)
(with Cartan involution) $= (\mathfrak{h}_0 \oplus \mathfrak{p}_0) \oplus (\mathfrak{h}_1 \oplus \mathfrak{p}_1)$.

Real Lie groups $G \supset H$ with $\mathrm{Lie}(G) \otimes \mathbb{C} = \mathfrak{g}_0$ and $\mathrm{Lie}(H) \otimes \mathbb{C} = \mathfrak{h}_0$
such that $M := G / H$ is a globally symmetric Riemannian manifold.

H acts on $\mathfrak{p}_1$ by the adjoint representation.
Form the associated vector bundle $E = G \times_H \mathfrak{p}_1 \to M$.

$\mathfrak{g}$ canonically acts on 'superfunctions' $f \in \Gamma(M, \wedge E^*)$.
$\mathfrak{g}$ – invariant Berezin form $\Gamma(M, \wedge E^*) \xrightarrow{\int_F} \Gamma(M, \wedge^{\mathrm{top}} T^* M)$.

Symmetric supermanifolds: an example

Hermitian vector space $U = \mathbb{C}^{p+q}$.

The space of all orthogonal decompositions

$$U = U^+ \oplus U^- \cong \mathbb{C}^p \oplus \mathbb{C}^q$$

is a Grassmann manifold $\mathrm{U}_{p+q}/(\mathrm{U}_p \times \mathrm{U}_q) = M_1$.

Pseudo – hermitian vector space $V = \mathbb{C}^{p+q}$ of signature (p,q).

The pseudo – orthogonal decompositions

$$V = V^+ \oplus V^- \cong \mathbb{C}^p \oplus \mathbb{C}^q$$

form a non – compact Grassmannian $\mathrm{U}_{p,q}/(\mathrm{U}_p \times \mathrm{U}_q) = M_0$.

Globally symmetric Riemannian manifold : $M = M_1 \times M_0$

(of type AIII)

Example (cont'd)

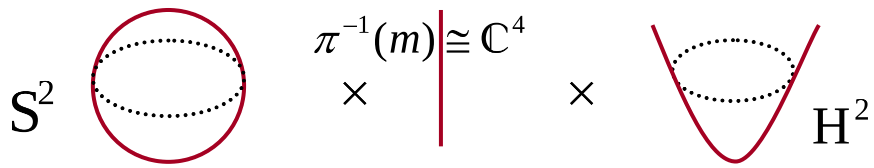
Vector bundle $E \xrightarrow{\pi} M$.

A point $m \in M$ determines $U = U^+ \oplus U^-$, $V = V^+ \oplus V^-$.

Fibre $\pi^{-1}(m) = \mathrm{Hom}(U^+, V^-) \oplus \mathrm{Hom}(U^-, V^+)$

$$\oplus \mathrm{Hom}(V^-, U^+) \oplus \mathrm{Hom}(V^+, U^-)\,.$$

Minimal case:



The algebra of sections $\Gamma(M, \wedge E^*)$ carries a canonical action of the Lie superalgebra $\mathfrak{g} = \mathfrak{gl}(U \oplus V) \cong \mathfrak{gl}_{p+q|p+q}$.

Riemannian symmetric superspace $(M, \wedge E^*, \mathfrak{g})$.

The 10-Way Table (MRZ, 1996)

A functional integral of maps $Q\colon \mathbb{R}^d \supset U \to E$,

$$\mathcal{Z} = \int \exp(-S/t), \quad S = \int_U \|DQ\|^2 d^d x,$$

into a Riemannian symmetric superspace is called a supersymmetric nonlinear sigma model (NLsM).

Correspondence between gauge-invariant (Wegner-type) random matrix models and supersymmetric nonlinear sigma models:

RME	A	AI	AII	C	CI	D	DIII	AIII	BDI	CII
noncomp.	AIII	BDI	CII	DIII	D	CI	C	A	AI	AII
susy NLsM compact	AIII	CII	BDI	CI	C	DIII	D	A	AII	AI

For small t constant maps dominate $\to$ universality.

Superbosonization

- P. Littelmann, H.-J. Sommers, and MRZ: Commun. Math. Phys. (in press)
- J.E. Bunder, K.B. Efetov, V.E. Kravtsov, O.M. Yevtushenko, MRZ: J. Stat. Phys. 129 (2007) 809

Reminder: supersymmetry method

Central object of the theory is the characteristic function of the disorder distribution, $\Omega(K) = \int \mathrm{e}^{-\mathrm{i}\,\mathrm{Tr}\,HK} d\mu(H)$, where ($\varphi$ commuting, ψ anticommuting variables): $K_{ab} = \sum_{i=1}^{p} \varphi_{a,i}\,\tilde{\varphi}_{i,b} + \sum_{j=1}^{q} \psi_{a,j}\,\tilde{\psi}_{j,b}$.

Generating function for spectral correlation functions:

$$\int_{\tilde{\varphi}=\varphi^*} D_{\varphi,\tilde{\varphi};\psi,\tilde{\psi}}\, f(\varphi,\tilde{\varphi};\psi,\tilde{\psi}) \equiv \int f \quad \text{where}$$

$$f(\varphi,\tilde{\varphi};\psi,\tilde{\psi}) = \Omega(K)\,\exp\left(\mathrm{i}\sum_{i,a}\varphi_{a,i}\, z_i\, \tilde{\varphi}_{i,a} + \mathrm{i}\sum_{j,b}\psi_{b,j} w_j\, \tilde{\psi}_{j,b}\right).$$

Assume $d\mu(H)$ invariant by some group G acting by conjugation $H \mapsto gHg^{-1}$. Then $f(\varphi,\tilde{\varphi};\psi,\tilde{\psi}) = f(g\varphi,\tilde{\varphi}\,g^{-1};g\psi,\tilde{\psi}\,g^{-1})$.

Special case: commuting variables only

Let $p=1,\ q=0$ and consider GL_N −invariant holomorphic function $f:\mathbb{C}^N\times(\mathbb{C}^N)^*\to\mathbb{C},\ \ f(\varphi,\tilde{\varphi}) = f(g\varphi,\tilde{\varphi}\,g^{-1}),\ \ g\in\mathrm{GL}_N$.

Fact (from invariant theory): there exists a holomorphic function $F:\ \mathbb{C}\to\mathbb{C}$ such that $F(\tilde{\varphi}\cdot\varphi)=f(\varphi,\tilde{\varphi})$.

By push forward of the integral one has

$$\int_{\mathbb{C}^N} f(\varphi,\varphi^*)\,d^{2N}\varphi = c_N\int_{\mathbb{R}_+} F(r)\,r^{N-1}dr$$ (if the integral exists).

generalization: see Fyodorov, Nucl. Phys. B **621** (2002) 643

Special case: Grassmann variables only

Let $F:\mathbb{C}\to\mathbb{C}$ be a holomorphic function. For a vector ψ of anticommuting variables $\psi_1,\dots,\psi_N$ consider $F(\tilde{\psi}\cdot\psi)$.

Berezin integral $\int F(\tilde{\psi}\cdot\psi)\, d\tilde{\psi}\, d\psi \;:=$

$$= \frac{\partial^2}{\partial\psi_1\partial\tilde{\psi}_1}\cdots\frac{\partial^2}{\partial\psi_N\partial\tilde{\psi}_N} F(\tilde{\psi}_1\psi_1+\dots+\tilde{\psi}_N\psi_N)$$

$= F^{(N)}(0)$ (the N^{th} derivative at the origin)

$$= N!\oint_{\mathrm{U}(1)} F(z)\, z^{-N-1} dz/2\pi\mathrm{i}.$$

Kawamoto and Smit, Nucl. Phys. B **192** (1981) 100

A statement of result

Recall $f(\varphi,\tilde{\varphi};\psi,\tilde{\psi}) \coloneqq f(g\varphi,\tilde{\varphi}\,g^{-1};g\psi,\tilde{\psi}\,g^{-1})$

for $g \in G$. Let $G = \mathrm{GL}_N$ or $G = \mathrm{O}_N$ or $G = \mathrm{Sp}_N$.

Superbosonization exploits this symmetry

to make a step of reduction.

Example : $G = \mathrm{O}_N$

$$f(\varphi,\tilde{\varphi};\psi,\tilde{\psi}) = F\begin{pmatrix} \tilde{\varphi}\varphi & \tilde{\varphi}\tilde{\varphi}^{\mathrm{t}} & \tilde{\varphi}\psi & -\tilde{\varphi}\tilde{\psi}^{\mathrm{t}} \\ \varphi^{\mathrm{t}}\varphi & \varphi^{\mathrm{t}}\tilde{\varphi}^{\mathrm{t}} & \varphi^{\mathrm{t}}\psi & -\varphi^{\mathrm{t}}\tilde{\psi}^{\mathrm{t}} \\ \tilde{\psi}\varphi & \tilde{\psi}\tilde{\varphi}^{\mathrm{t}} & \tilde{\psi}\psi & -\tilde{\psi}\tilde{\psi}^{\mathrm{t}} \\ \psi^{\mathrm{t}}\varphi & \psi^{\mathrm{t}}\tilde{\varphi}^{\mathrm{t}} & \psi^{\mathrm{t}}\psi & -\psi^{\mathrm{t}}\tilde{\psi}^{\mathrm{t}} \end{pmatrix}$$

A statement of result (continued)

Theorem (LSZ). If $N \geq 2p$ and f Schwartz function along $\tilde{\varphi} = \varphi^*$, then

$$\int_{\tilde{\varphi}=\varphi^*} f \ = \int_M DQ \ \mathrm{SDet}^{N/2}(Q) \ F(Q)$$

with integration domain $M \cong (\mathrm{GL}_{2p}(\mathbb{R}) / \mathrm{O}_{2p}) \times (\mathrm{U}_{2q} / \mathrm{USp}_{2q})$ and $\mathfrak{gl}-$invariant Berezin integration form DQ.

Remark: $(M, \wedge E^*, \mathfrak{gl})$ is a Riemannian symmetric superspace.

Application: Wegner's model (Gaussian, class A)

Hilbert space $V = \oplus_{i\in\Lambda} V_i$, orthogonal projector $\Pi_i : V \to V_i$

Characteristic function: $\int \mathrm{e}^{-\mathrm{i}\,\mathrm{Tr}\,HK} d\mu(H) = \mathrm{e}^{-(1/2N)\sum_{i,j} C_{ij}\,\mathrm{Tr}\,K\Pi_i K \Pi_j}$

Average ratio of characteristic polynomials:

$$R(E_0, E_1) = \int \frac{\mathrm{Det}(E_1 - H)}{\mathrm{Det}(E_0 - H)}\, d\mu(H) \qquad (\mathrm{Im}\,E_0 > 0).$$

Hubbard–Stratonovich transformation ($x_k \in \mathbb{R}$, $y_k \in \mathrm{i}\mathbb{R}$):

$$R(E_0, E_1) = \int \mathrm{e}^{-(N/2)\sum_{i,j}(C^{-1})_{ij}(x_i x_j - y_i y_j)} D_{\mathrm{HS}}(x,y) \prod_{k\in\Lambda} \left(\frac{y_k - E_1}{x_k - E_0} \right)^N \frac{dx_k dy_k}{2\pi \mathrm{i}}$$

Superbosonization ($x_k \in \mathbb{R}_+$, $y_k \in \mathrm{U}_1$):

$$R(E_0, E_1) = \int \mathrm{e}^{-(N/2)\sum_{i,j} C_{ij}(x_i x_j - y_i y_j)} D_{\mathrm{SB}}(x,y) \prod_{k\in\Lambda} \left(\frac{x_k \mathrm{e}^{\mathrm{i}E_0 x_k}}{y_k \mathrm{e}^{\mathrm{i}E_1 y_k}} \right)^N \frac{dx_k dy_k}{2\pi \mathrm{i}\, x_k y_k}$$

Heuristics from NLsM (Gaussian case)

Controlled (for $N \to \infty$) reduction to NLsM of maps $Q: \mathbb{R}^d \supset U \to E$ (vector bundle of RSS),

$$\mathcal{Z} = \int \exp(-S/t), \quad S = \int_U \|DQ\|^2 d^d x.$$

Coupling constant : $t^{-1} = \sum_j C_{ij} |i-j|^2.$

Perturbative renormalization group flow :

$$\frac{d\,t}{d \log a} = 2 - d + \beta(t), \qquad \text{(cutoff } a).$$

Expectations for d = 2, small t

Friedan (1985): $\beta(t) = b\,t + \mathcal{O}(t^2)$,

Riemann tensor: $R(X,Y) = \nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]}$,

Ricci curvature: $\mathrm{Ric}(X,Y) = \left\langle e^i, R(e_i, X)Y \right\rangle$,

Symmetric space $\Rightarrow$ $\mathrm{Ric} = b\,g$ (metric tensor g).

Class	AI	A	AII
Curvature	$b > 0$	$b = 0$	$b < 0$
Loc. length	$\exp(1/t)$	$\exp(1/t^2)$	∞

Voiculescu R-transform

Stieltjes transform of scaled counting measure,

$$g(z) = \lim_{N\to\infty} N^{-1} \int \mathrm{Tr}\,(z-H)^{-1} d\mu_N(H)\ ,$$

solves the equation $g(z) = \dfrac{1}{z-R(g(z))}$,

where Voiculescu's R - transform

$R(\lambda) = \sum_{n\geq 0} c_{n+1}\,\lambda^n$ is the function which

generates the non – crossing cumulants c_n .

Connection with R-transform

Let $\hat{\Omega}(Q)$ be a lift of the Fourier transform

$$\Omega(g\varphi,\tilde{\varphi}\, g^{-1}; g\psi,\tilde{\psi}\, g^{-1}) = \Omega(\varphi,\tilde{\varphi};\psi,\tilde{\psi}) .$$

Being $\mathfrak{g}-$radial, $\frac{d}{d\tau}\hat{\Omega}(\mathrm{e}^{-\tau X} Q\, \mathrm{e}^{\tau X})\Big|_{\tau=0} = 0$,

$\hat{\Omega}$ is determined by its numerical values on a maximal torus.

Suppose $\omega(-\mathrm{i}Q) = \lim_{N\to\infty} N^{-1} \log \hat{\Omega}(NQ)$

exists. All partial derivatives at unity are the same and are given by Voiculescu's R$-$transform :

$$\pm \frac{\partial}{\partial \lambda_i} \omega(\lambda,\ldots,\lambda) = R(\lambda) .$$