Borel transform –
a tool for symmetry-breaking phenomena?

M. Zirnbauer

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- Introduction: spontaneous symmetry breaking, order parameter, collective field methods
- Borel transform for interacting fermions
  example: ferromagnetic order
- Borel transform for random matrices
  SUSY, Voiculescu R-transform, boson-boson sector

Introduction

(Motivation by Perspective)

Symmetry breaking

Statistical mechanics of field $\varphi : \mathbb{Z}^d \to M$

with symmetry group G (global symmetry)

- In infinite volume, the thermal equilibrium state may spontaneously break the symmetry G.

$\to$ Long-range order in correlation functions:

$$\left\langle \mu(\varphi_x)\ \mu(\varphi_y) \right\rangle \xrightarrow{|x-y|\to\infty} \text{const} \neq 0$$

in spite of $\oint_G \mu(g \cdot \varphi_x)\, dg = 0.$

- Example:

the ferromagnetic phase of a spin-magnetic system spontaneously breaks rotation symmetry, $G = \mathrm{SO}_3$.

The textbook example of a scalar field

Scalar field $\varphi(x)$ (e.g., Ising model)

Partition function in the presence of external field:

$$Z[j] = \int \mathrm{e}^{-S[\varphi] + \int d^d x\, j(x)\, \varphi(x)}$$

- Free energy functional $F[j] = -\ln Z[j]$ is generating function for connected n-point functions

- Legendre transform:

$$\Gamma[\phi] = -\int d^d x\, j(x)\, \phi(x) + F[j] \Big|_{\delta F / \delta j(x) = \phi(x)}$$

Why use Legendre transform?

- Perturbation theory :
  graphical expansion of Γ is simpler
  (only 1P-irreducible graphs contribute).

- Spontaneous symmetry breaking signaled by
  appearance of critical points $\frac{\delta\Gamma}{\delta\phi(x)} = 0$ for $\phi \neq 0$.

- Γ may remain analytic even in the
  infinite-volume limit ($\rightarrow$ Landau theory).

Collective-field methods

- Hubbard-Stratonovich transformation

$$V_{\text{int}} = \sum_{x,y\in\Lambda} K_{xy}\, O_x O_y \to \sum (K^{-1})_{xy}\, h_x h_y + \sum h_x O_x$$

- Bosonization

  Dirac fermions in $1+1$ dimensions

  bosonization by gauge forms

- Landau theory of order parameter (phenomenological)

- *! New ?* Borel transformation

Basic notion of Borel transform

The simplest example :

consider some polynomial $Z(p) = \sum_{k=0}^{N} a_k p^k .$

Borel transform $(q \neq 0)$:

$$\hat{Z}(q) = \int_0^\infty Z(p/q)\, \mathrm{e}^{-p} dp = \sum_{k=0}^{N} k!\, a_k q^{-k} .$$

Inverse Borel transform :

$$Z(p) = (2\pi \mathrm{i})^{-1} \oint_{\mathrm{U}_1} \hat{Z}(q)\, \mathrm{e}^{pq} q^{-1} dq = \sum_{k=0}^{N} a_k p^k .$$

Generalization

$M_{\mathbb{C}}$ = complex $n \times n$ matrices,

M_0 = positive Hermitian $n \times n$ matrices,

Lebesgue measure dP, $\int_{M_0} \mathrm{e}^{-\mathrm{Tr}\,P} dP = 1$,

M_1 = unitary $n \times n$ matrices,

Haar measure $d\mu(Q)$, $\int_{M_1} d\mu(Q) = 1$.

Define Borel transform of $f \in \mathrm{Hol}(M_{\mathbb{C}})$ by

$$\hat{f}(Q) = \int_{M_0} f(Q^{-1}P)\, \mathrm{e}^{-\mathrm{Tr}\,P} dP, \qquad Q \in \mathrm{GL}_n(\mathbb{C}).$$

- Thm (Littelmann, MZ): Borel transform has inverse

$$f(P) = \oint_{M_1} \hat{f}(Q)\, \mathrm{e}^{\mathrm{Tr}\,(PQ)} d\mu(Q).$$

Borel Transform: Interacting Fermions

Setting

Quantum system of interacting fermions with
grand canonical partition function $Z = \mathrm{Tr}\, \mathrm{e}^{-\beta(H-\mu n)}$

Use coherent-state path integral representation
in terms of fermion fields $\psi, \bar{\psi}$.

Let $O(\bar{\psi}, \psi)$ be a well motivated 'order parameter'
(e.g., Cooper pairing with d-wave symmetry) which
is expected to acquire a nonzero expectation value.

How to proceed?

Main idea

Augment the Hamiltonian by coupling to an external field P via the order parameter :

$$H \to H + \sum_{j \in \Lambda} \sum_{\alpha} P_{\alpha}(j)\, O_{\alpha}(\overline{\psi}(j), \psi(j))\ .$$

Sum is over lattice Λ of cubes j (coarse graining).

- Borel transform of partition function (schematic) :

$$\hat{Z}[Q] = \int Z[P]\, \mathrm{e}^{-PQ}\, dP.$$

Inverse Borel transform (schematic) :

$$Z[P] = \int \hat{Z}[Q]\, \mathrm{e}^{+PQ}\, dQ.$$

Example: ferromagnet

Order parameter = magnetization (spin) of cube:

$$\mathcal{O} = \begin{pmatrix} \frac{1}{2}(\overline{\psi}_\uparrow \psi_\uparrow - \overline{\psi}_\downarrow \psi_\downarrow) & \overline{\psi}_\uparrow \psi_\downarrow \\ \overline{\psi}_\downarrow \psi_\uparrow & -\frac{1}{2}(\overline{\psi}_\uparrow \psi_\uparrow - \overline{\psi}_\downarrow \psi_\downarrow) \end{pmatrix}.$$

- Ambient space is $\mathfrak{sl}_2(\mathbb{C})$. Let $J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and

$$M_0 = \left\{ P \in \mathfrak{sl}_2(\mathbb{C}) \mid (JP)^* = JP > 0 \right\} \cong \mathrm{GL}_2(\mathbb{R})/\mathrm{O}_2$$
$$M_1 = \left\{ Q \in \mathfrak{sl}_2(\mathbb{C}) \mid Q^* = Q^{-1} \right\} \cong \mathrm{U}_2/\mathrm{O}_2 \ .$$

- $\mathrm{GL}_2(\mathbb{C})$ acts on $\mathfrak{sl}_2(\mathbb{C})$ by twisted conjugation:

$X \mapsto g\, X\, \tau(g^{-1})$ where $\tau(g^{-1}) = J g^T J^{-1}$.

Example: ferromagnet (cont'd)

Introduce coupling: $H \to H + \sum_{j\in\Lambda} \mathrm{Tr}\, P(j)\, \mathcal{O}(j)$.

For each cube $j \in \Lambda$ let $Q(j) \in M_1 = \mathfrak{sl}_2(\mathbb{C}) \bigcap \mathrm{U}_2$

and $P(j) \in M_0 = \mathfrak{sl}_2(\mathbb{C}) \bigcap J^{-1} \cdot \mathrm{Herm}_2^+$

- Borel transform:

$$\hat{Z}[Q] := \int_{M_0^\Lambda} Z[U^{-1} P \tau(U)] \prod_j \mathrm{e}^{-\mathrm{Tr}\, JP(j)}\, dP(j).$$

where $Q = UJ\tau(U^{-1}),\ U \in \mathrm{U}_2$ for each cube.

- Inverse Borel transform:

$$Z[P] = \oint_{M_1^\Lambda} \hat{Z}[Q] \prod_j \mathrm{e}^{\mathrm{Tr}\, P(j)\, Q(j)}\, d\mu(Q(j)).$$

Borel Transform for Random Matrices

Main idea

Spectral correlations are encoded in the

"partition function" $\mathbb{E}\left\{\prod_{j=1}^{n}\frac{\mathrm{Det}\,(p_{1,j}-H)}{\mathrm{Det}\,(p_{0,j}-H)}\right\}.$

- Partition function extends to radial function of supermatrix, $Z(P) := \mathbb{E}\left\{\mathrm{SDet}^{-1}(P\otimes 1_N - 1_n \otimes H)\right\}.$

- Transform (schematic): $\hat{Z}(Q) = \int DP \; \mathrm{e}^{+\mathrm{STr}\,(PQ)}\, Z(P),$

Inverse (schematic): $Z(P) = \int DQ \; \mathrm{e}^{-\mathrm{STr}\,(PQ)}\, \hat{Z}(Q),$

(integrate over Riemannian symmetric superspaces).

Why Borel transform?

What' s the advantage?

- $$Z(p) = \mathbb{E}\left\{\mathrm{Det}^{-1}(p-H)\right\}$$
  $$= \mathbb{E}\left\{\mathrm{e}^{-\mathrm{Tr}\,\ln(p-H)}\right\} \approx \mathrm{e}^{-\mathbb{E}\left\{\mathrm{Tr}\,\ln(p-H)\right\}}$$

  is bad approximation when p near spectrum.

- $$\hat{Z}(q) = \oint \mathbb{E}\left\{\mathrm{Det}^{-1}(p-H)\right\}\mathrm{e}^{pq}\,dp$$
  $$\approx \oint \mathrm{e}^{-\mathbb{E}\left\{\mathrm{Tr}\,\ln(p-H)\right\}}\mathrm{e}^{pq}\,dp$$

  is good approximation if p kept away from spectrum.

- Identity principle ?!

Voiculescu R-transform

Consider the average trace of resolvent :

$$g(z) = \lim_{N\to\infty} N^{-1}\,\mathbb{E}\left\{\mathrm{Tr}\,(z-H)^{-1}\right\}, \quad z \notin \mathbb{R}.$$

Voiculescu's R-transform is defined by inversion :

$$z \mapsto g(z) = q \quad \Leftrightarrow \quad q \mapsto q^{-1} + R(q) = z\,.$$

Linearity : $R_{A+B}(q) = R_A(q) + R_B(q)$ if A, B free

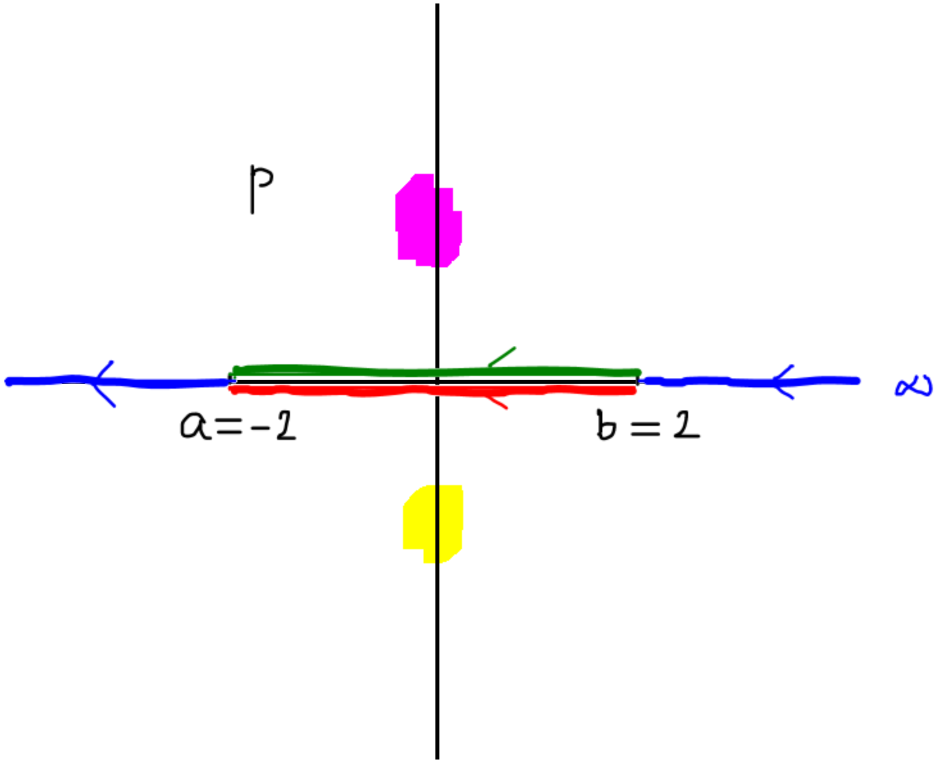
[Hermitian $N \times N$ random matrices A, B are *free* in the limit $N \to \infty$ if the law of $A + UBU^*$ is independent of $U \in \mathrm{U}_N$].

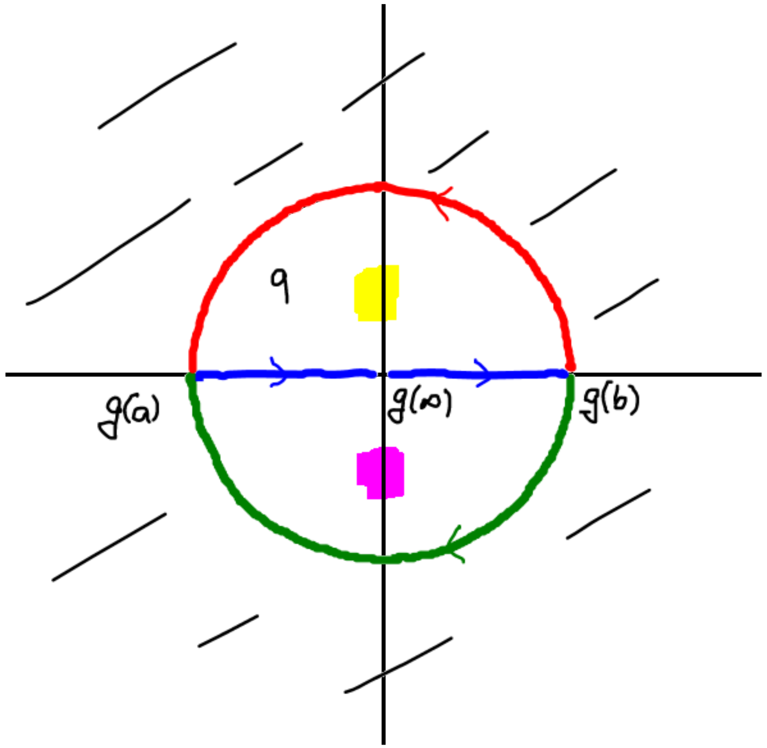
Power series : $R(q) = \sum_{k=1}^{\infty} c_k\, q^{k-1}$ (free cumulants c_k).

Example: Gaussian Unitary Ensemble

$$d\mu_{\mathrm{GUE}}(H) = \exp\left(-N\,\mathrm{Tr}\,H^2/2\right)dH,$$

$$R_{\mathrm{GUE}}(q) = q \quad \Leftrightarrow \quad g_{\mathrm{GUE}}(z) = \tfrac{1}{2}\left(z \pm \sqrt{z^2-4}\right).$$





The main objective

$$\hat{Z}_{\mathrm{GUE}}(Q) = \int DP\, \mathrm{e}^{\mathrm{STr}\, PQ}\, \mathbb{E}_{\mathrm{GUE}} \left\{ \mathrm{SDet}^{-1}(P \otimes 1_N - 1_n \otimes H) \right\}$$
$$= \mathrm{SDet}^{N}(Q)\, \exp\left(\mathrm{STr}\, Q^2 / 2N \right).$$

Grand (sine kernel) universality conjecture (reformulated):

$$\Gamma(Q) := \lim_{N\to\infty} N^{-1} \ln \hat{Z}(NQ) - \mathrm{STr}\, \ln Q$$

stays holomorphic for any 'small' deformation of GUE.

Expect: $\Gamma(Q) = \mathrm{STr} \sum_{k=1}^{\infty} c_k\, Q^k / k$

o.k. for sum of Wigner and unitary ensembles

Erdös et al., Tao & Vu — Mandt & MZ

Random Matrices: Boson-Boson Sector

Small q

Consider $Z(p) = \mathbb{E}\left\{\mathrm{Det}^{-1}(p-H)\right\}$ with

Borel transform $\hat{Z}(q) = (2\pi \mathrm{i})^{-1} \oint Z(p)\, \mathrm{e}^{pq}\, dp$.

If the spectral measure of H for $N \to \infty$ converges weakly to a compactly supported measure, then:

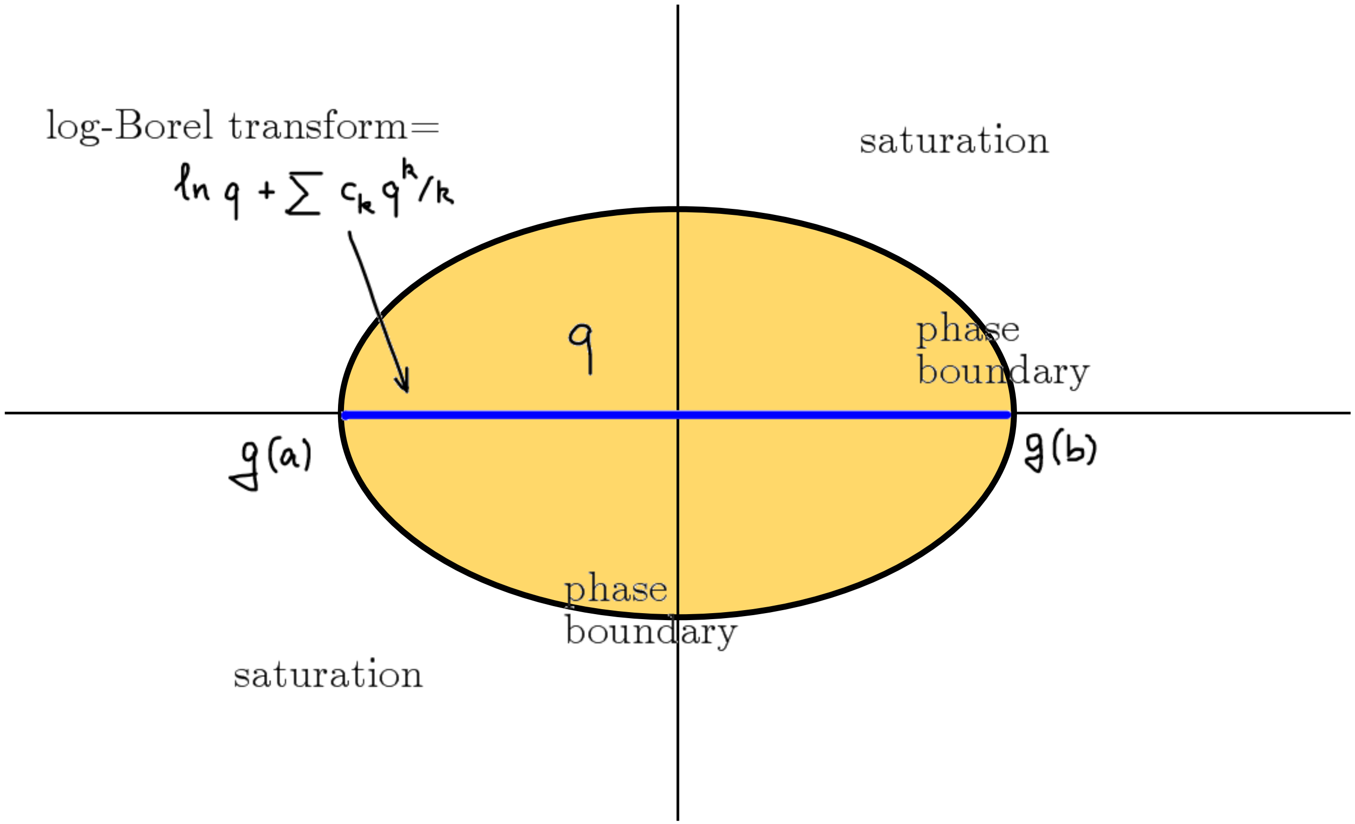
- Thm (Guionnet, Maida; 2004): for $q \in \mathbb{C}$ small enough,

$$\lim_{N\to\infty} N^{-1} \ln \hat{Z}(Nq) = \ln q + \sum_{k=1}^{\infty} c_k\, q^k / k\ .$$

Heuristic: do saddle analysis on

$$\hat{Z}(Nq) \approx (2\pi \mathrm{i})^{-1} \oint \mathrm{e}^{-N\int_a^b \ln(p-x)\, d\nu(x)}\ \mathrm{e}^{Npq}\, dp\,.$$

Guionnet and Maida



Large q

Unitary ensemble $d\mu(H) = \mathrm{e}^{-N\,\mathrm{Tr}\,V(H)}dH.$

$$\hat{Z}(Nq) = (2\pi \mathrm{i})^{-1}\oint_C \mathbb{E}\left\{\mathrm{Det}^{-1}(p-H)\right\}\mathrm{e}^{Nqp}\,dp$$

$$= \mathbb{E}\left\{\sum_{j=1}^{N}\frac{\mathrm{e}^{Nqx_j}}{\prod_{k(\neq j)}(x_j - x_k)}\right\}$$

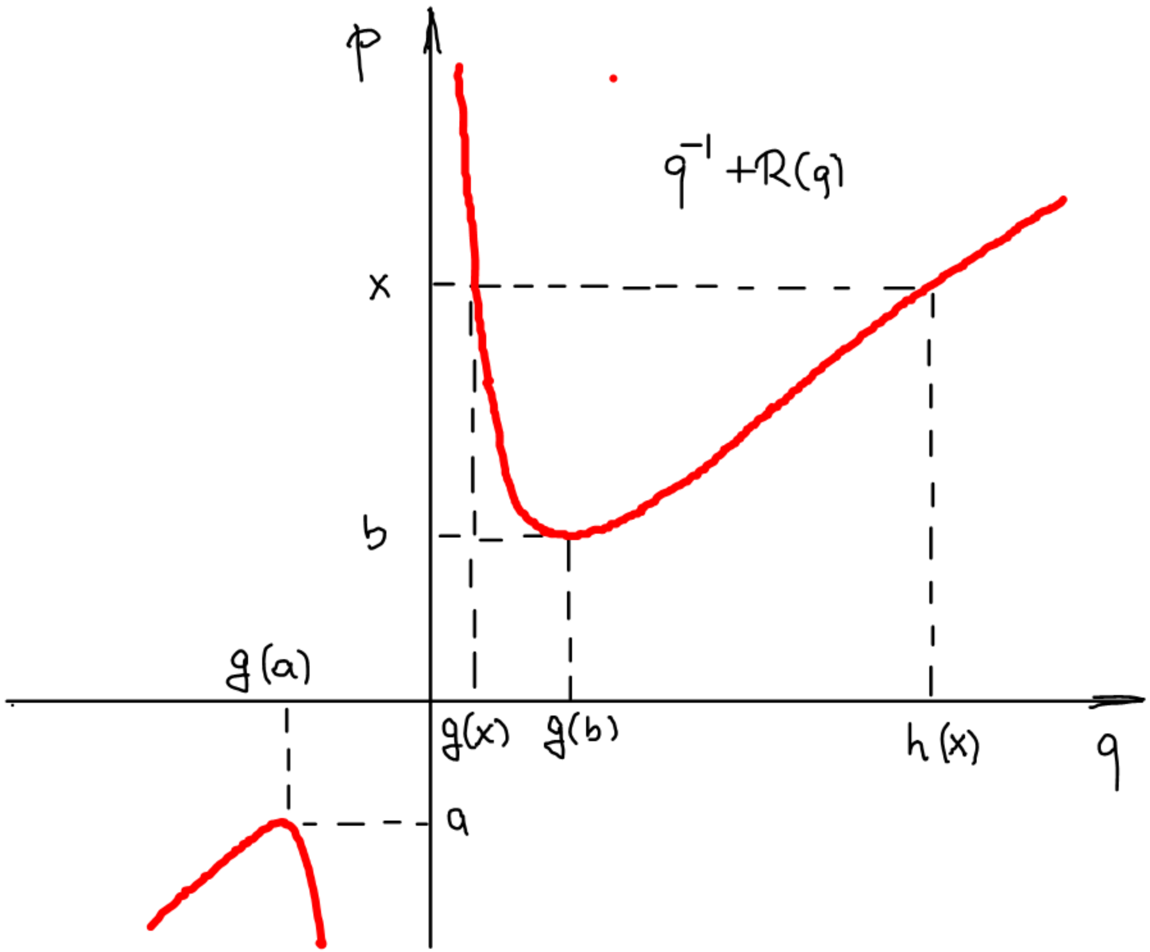
$$= c_N \int_{\mathbb{R}} \mathrm{e}^{Nqx - NV(x)}\pi_{N-1,N}(x)\,dx,$$

$\pi_{N-1,N}$ orthogonal polynomial of $\mathrm{e}^{-NV(x)}$ of degree $N-1$.

For V analytic, uniformly convex, let $q > g(b)$ or $q < g(a)$.

- Mandt, MZ (2009): $\lim N^{-1}\ln \hat{Z}(Nq) = \ln q + \sum_{k=1}^{\infty} c_k\, q^k / k$

Analyticity of Borel transform



Other symmetry classes: AI

- Propn (Bergere, Eynard, 2008; MZ, 2009):

Let H, Q be real symmetric matrices

with $rk(H) - rk(Q) = N - n \in 2\mathbb{N}+1$.

Let $\|Q \otimes H\|_{\mathrm{op}} < 1$. Then with $c_{n,N} = \left(\frac{n}{2}(N-n+1)\right)!$

$$\oint_{\mathrm{SO}_N} \mathrm{e}^{\mathrm{Tr}(Q g^{-1} H g)} d\mu(g) = c_{n,N} \oint_{M_n} \frac{\mathrm{e}^{\mathrm{Tr}\, P} \mathrm{Det}^{(n+1)/2}(P)\, d\nu(P)}{\mathrm{Det}^{1/2}(P \otimes 1_N - Q \otimes H)},$$

where $d\mu(g)$ Haar measure, and $d\nu(P)$ inv. measure on the unitary symmetric matrices $M_n \cong U_n / O_n$.

This result allows to establish Borel transform and its inverse for $Z(P) = \mathbb{E}\left\{\mathrm{Det}^{-1/2}(P \otimes 1_N - 1_n \otimes H)\right\}$.

Summary

- Borel transform looks intriguing as a method to construct effective field theories for symmetry-broken phases of matter (ferromagnetism, superconductivity, ...)

- For certain random matrix ensembles the large-N limit of the SUSY-Borel transform of the partition function is determined by the Voiculescu R-transform.

- (Sine kernel) grand universality conjecture is reformulated as a conjecture of holomorphicity of Borel transform.

- Question : Can SUSY-BT be computed in a controlled way, say for the Anderson model?