Bott periodicity for $\mathbb{Z}_2$ symmetric ground states of gapped free-fermion systems with disorder

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Periodic Table of topological insulators/superconductors

from Hasan & Kane, Rev. Mod. Phys. (2011):

	Symmetry			d							
AZ	Θ	Ξ	Π	1	2	3	4	5	6	7	8
A	0	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$
AIII	0	0	1	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0
AI	1	0	0	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$
BDI	1	1	1	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$
D	0	1	0	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$
DIII	-1	1	1	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0
AII	-1	0	0	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$
CII	-1	-1	1	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0
C	0	-1	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0
CI	1	-1	1	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0

TABLE I Periodic table of topological insulators and superconductors. The 10 symmetry classes are labeled using the notation of Altland and Zirnbauer (1997)

Quantum Hall Effect

He-3 (B phase)

QSHI: HgTe

Majorana

Bi_2 Se_3

Schnyder, Ryu, Furusaki, Ludwig (2008); Kitaev (2009); Teo & Kane (2010); Stone, Chiu, Roy (2011); Freedman, Hastings, Nayak, Qi, Walker, Wang (2011); Abramovici & Kalugin (2012); Freed & Moore (2013)

Question: does there exist a ``diagonal map''?

from Hasan & Kane, Rev. Mod. Phys. (2011):

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AI	1	0	0	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$
BDI	1	1	1	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$
D	0	1	0	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$
DIII	-1	1	1	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0
AII	-1	0	0	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$
CII	-1	-1	1	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0
C	0	-1	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0
CI	1	-1	1	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0

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Quantum Hall Effect

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Majorana

$Bi_2 Se_3$

Teo & Kane:

$$\mathcal{H}_{nc}(\mathbf{k},\mathbf{r},\theta) = \cos\theta \mathcal{H}_c(\mathbf{k},\mathbf{r}) + \sin\theta \Pi \qquad (A1)$$

$$\mathcal{H}_c(\mathbf{k},\mathbf{r},\theta) = \cos\theta \mathcal{H}_{nc}(\mathbf{k},\mathbf{r}) \otimes \tau_z + \sin\theta \mathbb{1} \otimes \tau_a, \qquad (A3)$$

We ask a more ambitious question …

	Symmetry			d							
AZ	Θ	Ξ	Π	1	2	3	4	5	6	7	8
A	0	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$
AIII	0	0	1	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0
AI	1	0	0	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$
BDI	1	1	1	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$
D	0	1	0	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$
DIII	-1	1	1	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0
AII	-1	0	0	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	[illegible]	0	0	$\mathbb{Z}$
CII	-1	-1	1	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0
C	0	-1	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0
CI	1	-1	1	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0

? ?

- "Master" diagonal map (from single, universal principle) ?
- Make convincing argument for bijection between SPT phases ?
- Delineate limits of validity (stable vs. non-stable regime) ?

Distinctive features of our approach

(joint work with R. Kennedy)

- Revisit the Kitaev sequence of pseudo-symmetries

  A symmetry is an (anti-)unitary operation that commutes with H

- Universal model for free-fermion ground states

  Avoid “spectral flattening” of the Hamiltonian

- Topological classification by homotopy theory

  Do not make the approximation of K-theory

- Use the standard framework of Hermitian Q.M. over $\mathbb{C}$

  Keep the $\mathbb{R}$-structure flexible

[Overview:]

I. Universal Model for Free Fermion Ground States

of gapped systems with symmetries

II. The Diagonal Map

Ground states as vector bundles

Notation: Fock operators $c^\dagger$ (creation), c (annihilation)
Momentum space M, momentum k
Assume translation invariance (for disorder: see later)

Fact. Free-fermion ground state $|\text{g.s.}\rangle \overset{1:1}{\leftrightarrow} \{A_k\}_{k\in M}$

where $A_k = \text{span}_{\mathbb{C}}\{\widetilde{c}_1(k), \ldots, \widetilde{c}_n(k)\}$

and $\widetilde{c}_j(k)$ q.p. annihilation ops. removing momentum k:

$\widetilde{c}_j(k)|\text{g.s.}\rangle = 0 \quad (j = 1, \ldots, n)$

Example. Charge conserved, two bands:

$$A_k = \text{span}_{\mathbb{C}}\{c_p(k), c_h^\dagger(-k)\}$$

Insulators. Gapped system $\curvearrowright$ $\{A_k\}_{k\in M}$ complex vector bundle

Fermi constraint: $\{A_k, A_{-k}\} = 0$

Fermi constraint

$$\{A_k\,, A_{-k}\} = 0$$

Symmetry				d							
AZ	Θ	Ξ	Π	1	2	3	4	5	6	7	8
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AI	1	0	0	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$
BDI	1	1	1	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$
D	0	1	0	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$
DIII	-1	1	1	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0
AII	-1	0	0	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$
CII	-1	-1	1	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0
C	0	-1	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0
CI	1	-1	1	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0

Universal Model (including symmetries)

Clifford algebra of pseudo-symmetries:

$$J_l J_m + J_m J_l = -2\delta_{lm}\mathbf{1}$$

where $J_1, \ldots, J_s$ are $\mathbb{C}$-linear operators on $A_k \oplus A_k^{\mathrm{c}} \equiv \mathbb{C}^{2n}$

Definition. A translation-invariant ground state of a gapped system of symmetry class s is a rank-n complex vector sub-bundle $\{A_k\}_{k \in M}$ with fibers $A_k \subset \mathbb{C}^{2n}$ subject to (for all $k \in M$):

- Fermi constraint: $\{A_k \, , A_{-k}\} = 0$
- pseudo-symmetries: $J_1 A_k = \ldots = J_s A_k = A_k^{\mathrm{c}}$

Example. Time-reversal symmetry: $T A_k = A_{-k}$

$\curvearrowright J_1 = \gamma \circ T$ ($\gamma :\ c \leftrightarrow c^\dagger$ Hermitian conj.)

Kitaev Sequence ("real" classes)

class	symmetries	s	pseudo-syms
D	none	0	**Fermi constraint**
DIII	T (time reversal)	1	$J_1 = \gamma T$
AII	T, Q (charge)	2	$J_2 = \mathrm{i}\gamma T Q$
CII	T, Q, C (ph-conj)	3	$J_3 = \mathrm{i}\gamma C Q$
C	S_1, S_2, S_3 (spin rot)	4	see below
CI	S_1, S_2, S_3, T	5	
AI	S_1, S_2, S_3, T, Q	6	
BDI	S_1, S_2, S_3, T, Q, C	7	

Q: Why do 3 spin generators amount to 4 pseudo-symmetries?

Tool: (1,1) doubling isomorphism.

$$C_s(n) := \{A \in \mathrm{Gr}_n(\mathbb{C}^{2n}) \mid J_1 A = \ldots = J_s A = A^{\mathrm{c}}\}$$
$$R_s(n) := \{A \in C_s(n) \mid \{A, A\} = 0\}$$

Double the band number ($\mathbb{C}^{2n}$ to $\mathbb{C}^{2n} \oplus \mathbb{C}^{2n}$) and let

$$I = \begin{pmatrix} 0 & \mathbf{1}_{2n} \\ -\mathbf{1}_{2n} & 0 \end{pmatrix}, \quad K = \mathrm{i}\begin{pmatrix} \mathbf{1}_{2n} & 0 \\ 0 & -\mathbf{1}_{2n} \end{pmatrix}, \quad \widetilde{J}_l = \begin{pmatrix} 0 & J_l \\ J_l & 0 \end{pmatrix} \quad (l = 1, \ldots, s)$$

Note: K is "imaginary": $\{Kw, Kw'\} = -\{w, w'\}$

Lemma. $C_s(n) \simeq C_{s+2}(2n), \quad R_s(n) \simeq R_{s+1,1}(2n)$

Corollary. Let (note $K = \mathrm{i}\widetilde{J}_1 \widetilde{J}_2 \widetilde{J}_3$)

$$\widetilde{J}_l := \begin{pmatrix} \mathrm{i}S_l & 0 \\ 0 & -\mathrm{i}S_l \end{pmatrix} \quad (l \leq 3), \quad \widetilde{J}_4 := I, \quad \widetilde{J}_l := \begin{pmatrix} 0 & J_l \\ J_l & 0 \end{pmatrix} \quad (l \geq 5)$$

Then (1,1) doubling isomorphism $\curvearrowright$ s pseudo-symmetries equivalent to $s-4$ pseudo-syms. plus 3 spin rotation symmetries.

Kitaev Sequence ("real" classes)

class	symmetries	s	pseudo-syms
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Disorder

From the work on QHE, one knows how to proceed:

Replace momentum k by twist parameters θ changing the boundary conditions. Study vector bundles $\{A_\theta\}_{\theta \in \mathrm{T}^d}$ over parameter space T^d. (The diagonal map still exists ...)

Q: Why are there exactly ten classes?

A: See MZ (Tenfold Way), arXiv:1001.0722, www.thp.uni-koeln.de/zirn/

"Complex" classes:

class	symmetries	pseudo-syms
A	Q	none
AIII	Q, C	$J_1 = \mathrm{i}\gamma C$

~~Fermi constraint~~

The Diagonal Map

How to classify?

Notions of topological equivalence for vector bundles:

1. Homotopy classes
2. Isomorphism classes

   (okay for "many conduction bands")

3. Stable equivalence (K-theory)

   (okay for "many conduction & many valence bands")

Example. Class D in 2+1 dimensions (defect).

K-theory (Teo & Kane) predicts a $\mathbb{Z}_2$-classification,

but for $n = 1$ the homotopy group is actually $\mathbb{Z}$.

4.1 Homotopy Groups

Perhaps the simplest noncontractible spaces are spheres, so to get a glimpse of the subtlety inherent in homotopy groups let us look at some of the calculations of the groups $\pi_i(S^n)$ that have been made. A small sample is shown in the table below, extracted from [Toda 1962].

$\pi_i(S^n)$

$n \downarrow$ \ $i \longrightarrow$	1	2	3	4	5	6	7	8	9	10	11	12
1	$\mathbb{Z}$	0	0	0	0	0	0	0	0	0	0	0
2	0	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{12}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_3$	$\mathbb{Z}_{15}$	$\mathbb{Z}_2$	$\mathbb{Z}_2 \times \mathbb{Z}_2$
3	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{12}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_3$	$\mathbb{Z}_{15}$	$\mathbb{Z}_2$	$\mathbb{Z}_2 \times \mathbb{Z}_2$
4	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z} \times \mathbb{Z}_{12}$	$\mathbb{Z}_2 \times \mathbb{Z}_2$	$\mathbb{Z}_2 \times \mathbb{Z}_2$	$\mathbb{Z}_{24} \times \mathbb{Z}_3$	$\mathbb{Z}_{15}$	$\mathbb{Z}_2$
5	0	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{24}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{30}$
6	0	0	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{24}$	0	$\mathbb{Z}$	$\mathbb{Z}_2$
7	0	0	0	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{24}$	0	0
8	0	0	0	0	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{24}$	0

from A. Hatcher, Algebraic Topology

Diagonal Map $(d,s) \to (d+1, s+1)$

Starting point: $\widetilde{J}_1, \ldots, \widetilde{J}_s$ and $\{\widetilde{A}_k\}_{k \in M_d}$

Preparatory step: jack up by (1,1) doubling isomorphism

↷ New starting point: $\widetilde{J}_1, \ldots, \widetilde{J}_s; I, K$ and $\{A_k\}_{k \in M_d}$

Define:

$$\boxed{A_{k,t} := \mathrm{e}^{(t/2)KJ(A_k)} \cdot A_k} \qquad J(A) = \mathrm{i}(\Pi_A - \Pi_{A^c})$$

Note:

1. Fermi constraint: $\{A_{k,t}, A_{-k,-t}\} = 0$ ✓
2. Pseudo-syms: $J_1 A_{k,t} = \ldots = J_s A_{k,t} = A^c_{k,t} = I A_{k,t}$ ✓
3. Periodicity: $A_{k,t+2\pi} = A_{k,t}$ ✓

Outcome: V.B. $\{A_{k,t}\}_{(k,t) \in M_{d+1}}$ in class $s+1$ on $M_{d+1} = M_d \times \mathrm{S}^1$ ☺

D (none) **DIII (*T*)**

Example 1. $(d,s) = (0,0)$ to $(d,s) = (1,1)$.

$$n = 1: \ R_0(1) = \{\mathbb{C}\cdot c, \mathbb{C}\cdot c^\dagger\}$$

(1,1) doubling $\curvearrowright$ tensor with $(\mathbb{C}^2)_{\mathrm{spin}}$ and let $K = \mathrm{i}(\sigma_1)_{\mathrm{BdG}} \otimes (\sigma_1)_{\mathrm{spin}}$,

$I \equiv J_1 = \gamma T = (\sigma_1)_{\mathrm{BdG}} \otimes (\mathrm{i}\sigma_2)_{\mathrm{spin}}$, $A = \mathrm{span}_{\mathbb{C}}\{c_\uparrow^\dagger, c_\downarrow^\dagger\} \cong |\mathrm{full}\rangle$.

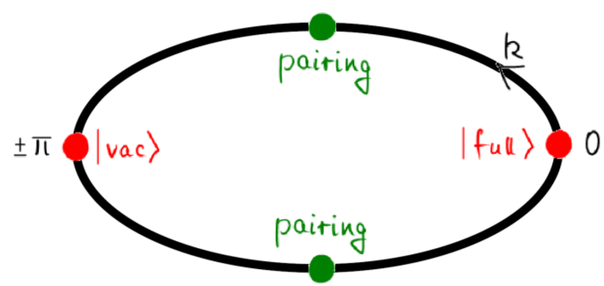
$$A_k = \mathrm{e}^{(k/2)KJ(A)} \cdot A = \mathrm{span}_{\mathbb{C}} \left\{ c_\sigma^\dagger(-k)\cos(k/2) - c_{-\sigma}(k)\sin(k/2) \right\}_{\sigma=\uparrow,\downarrow}$$

In BCS form: $|\mathrm{g.s.}\rangle = \mathrm{e}^{\sum_k \cot(k/2) P_k} |\mathrm{vac}\rangle$ where $P_k = c_\uparrow^\dagger(k) c_\downarrow^\dagger(-k)$.

For more general $K = K(\alpha)$:

$$P_k = c_\uparrow^\dagger(k) c_\downarrow^\dagger(-k)\, \cos\alpha + \left(c_\uparrow^\dagger(k) c_\uparrow^\dagger(-k) - c_\downarrow^\dagger(k) c_\downarrow^\dagger(-k)\right)\, \sin\alpha$$

Topological 1d superconductor with spin-triplet pairing and *T*-invariance



DIII (T) AII (T, Q)

Example 2. $(d,s)=(1,1)$ to $(d,s)=(2,2)$

(1,1) doubling $\curvearrowright$ tensor with band space $(\mathbb{C}^2)_{\mathrm{ph}}$

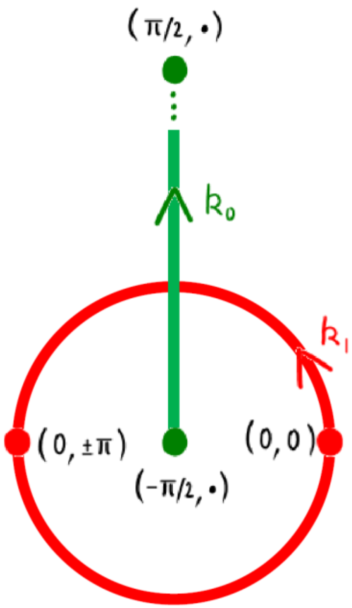
Topological 1d class-DIII superconductor (after ph-transformation):

$$A_{-k_1}=\mathrm{span}_{\mathbb{C}}\left\{c_{h\uparrow}^{\dagger}(k_1)\cos(k_1/2)+\mathrm{i}c_{p\downarrow}^{\dagger}(k_1)\sin(k_1/2);\ \&\uparrow\leftrightarrow\downarrow,\mathrm{i}\to-\mathrm{i}\right\}$$

Apply 1-par. group: $A_{-k}=\mathrm{e}^{-(k_0/2)KJ(A_{-k_1})}\cdot A_{-k_1}=\mathrm{span}_{\mathbb{C}}\Big\{\&\uparrow\leftrightarrow\downarrow,\mathrm{i}\to-\mathrm{i};$

$$\left(c_{h\uparrow}^{\dagger}(k)\cos(k_1/2)+\mathrm{i}c_{p\downarrow}^{\dagger}(k)\sin(k_1/2)\right)\cos(k_0/2)$$

$$+\left(c_{p\downarrow}^{\dagger}(k)\cos(k_1/2)+\mathrm{i}c_{h\uparrow}^{\dagger}(k)\sin(k_1/2)\right)\sin(k_0/2)\Big\}$$



Note: $A_{\pm\pi/2,k_1}=\mathrm{span}_{\mathbb{C}}\left\{c_{h\uparrow}^{\dagger}\pm c_{p\downarrow}^{\dagger},\,c_{h\downarrow}^{\dagger}\pm c_{p\uparrow}^{\dagger}\right\}$

$\curvearrowright$ $k_0=\pm\pi/2$ are isolated zeros of Kane-Mele Pfaffian

$\curvearrowright$ Kane-Mele invariant non-trivial (QSHI)

Next SPT phases in line: 3d insulator (CII), 4d superconductor (C), etc.

- "Master" diagonal map (from single, universal principle) ✓

Symmetry				d							
AZ	Θ	Ξ	Π	1	2	3	4	5	6	7	8
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AIII	0	0	1	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0
AI	1	0	0	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$
BDI	1	1	1	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$
D	0	1	0	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$
DIII	-1	1	1	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0
AII	-1	0	0	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	[illegible]	0	0	$\mathbb{Z}$
CII	-1	-1	1	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0
C	0	-1	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0
CI	1	-1	1	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0

- Make convincing argument for bijection between SPT phases (✓)
- Delineate limits of validity (stable vs. non-stable regime) (✓)

Summary & Outlook

- Kitaev sequence of symmetry classes firmly established.
- New perspective from master diagonal map:
  iterative construction of free-fermion SPT phases
  with high (*d*,*s*) from those with low (*d*,*s*).

- Homotopy-theoretic proof of “Periodic Table” is forthcoming.
- Homotopy theory gives precise bounds on the range of stable equivalence.
- Our method also applies to topological crystalline insulators.

Thank you!