Particle-Hole Symmetries in Condensed Matter

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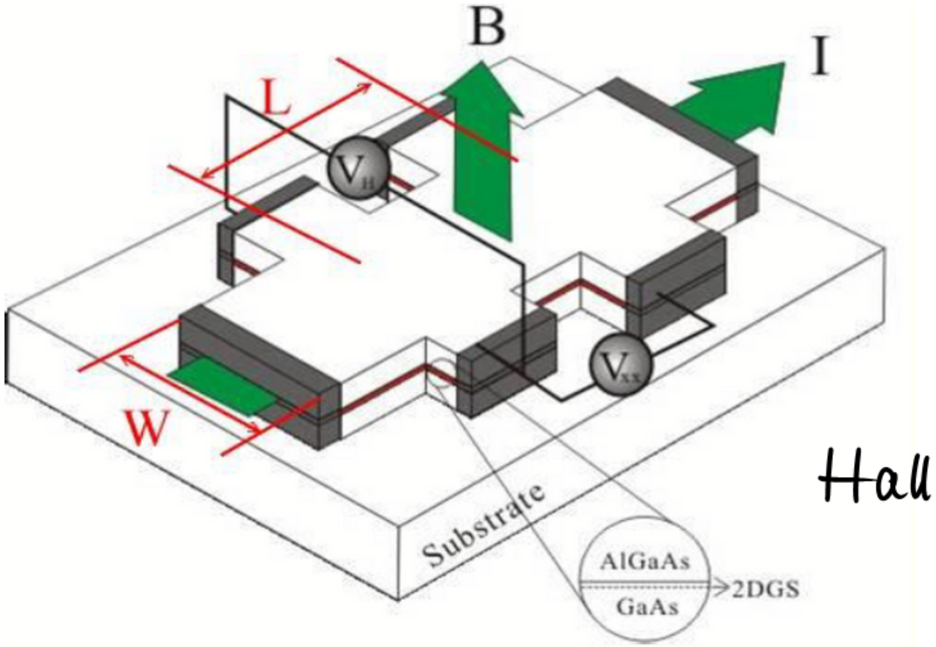
TU München (20. März 2019)

Quantum Hall Effect



Two-dimensional electron gas
at low temperature and
in a strong magnetic field.

Hall resistance exhibits plateaus: $R_H = \frac{h}{\nu e^2}$.

$N_\phi = \frac{e}{h} \iint B = \#$ of states in lowest Landau level (LLL).

$\nu = N_{el}/N_\phi$ filling fraction; $\nu = 1/2$ half-filled LLL.

Particle-hole symmetry: $\sigma_{xx}(\nu) = \sigma_{xx}(1-\nu)$
is exact in the limit $\hbar\omega_c \longrightarrow \infty$ and if interactions are two-body.

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Particle-hole symmetry in the anomalous quantum Hall effect

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(Received 27 February 1984)

This paper explores the uses of particle-hole symmetry in the study of the anomalous quantum Hall effect. A rigorous algorithm is presented for generating the particle-hole dual of any state. This is used to derive Laughlin's quasihole state from first principles and to show that this state is exact in the limit $\nu \to 1$, where ν is the Landau-level filling factor. It is also rigorously demonstrated that the creation of m quasiholes in Laughlin's state with $\nu = 1/m$ is precisely equivalent to creation of one true hole. The charge-conjugation procedure is also generalized to obtain an algorithm for the generation of a hierarchy of states of arbitrary rational filling factors.

I. INTRODUCTION

The anomalous quantum Hall effect[1,2] is one of the most striking many-body phenomena discovered in recent years. The Hall resistivity of a two-dimensional electron gas (inversion layer) in a high magnetic field at low temperatures exhibits quantized plateau values of the form $\rho_{xy} = h/e^2 i$, where i is a rational number $i = p/q$ with q odd. Associated with this quantization of the Hall resistivity is a marked decrease in the dissipation ($\rho_{xx} \to 0$). The latter suggests the

where the exponential factors have been lumped into the measure

$$d\mu(z) = \frac{dx\,dy}{2\pi l^2} e^{-|z|^2/2} \ . \tag{4}$$

Within (the N-particle version of) this space the variational wave functions proposed by Laughlin[3] may be written

$$\psi_m(z_1, \ldots, z_N) = \prod_{i<j} (z_i - z_j)^m \ , \tag{5}$$

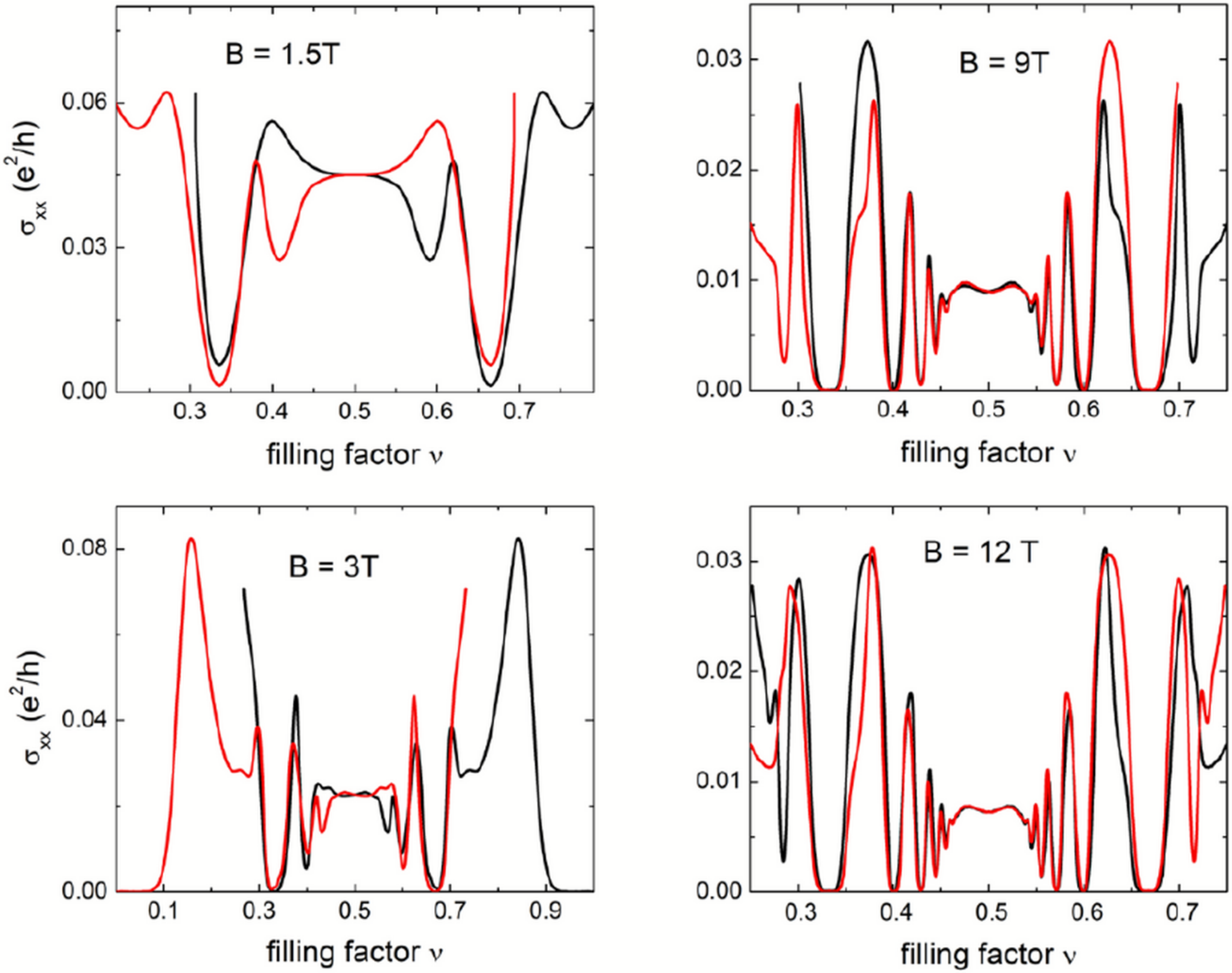

Figure 4. Examination of reflection symmetry in σ_{xx} over a large range of magnetic fields, from 1.5T to 12T. **Pan, Kang, Lilly, Reno, Baldwin, West, Pfeiffer, Tsui (March 2019)**

PHYSICAL REVIEW X **5,** 031027 (2015)

Is the Composite Fermion a Dirac Particle?

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(Received 19 February 2015; published 2 September 2015)

We propose a particle-hole symmetric theory of the Fermi-liquid ground state of a half-filled Landau level. This theory should be applicable for a Dirac fermion in the magnetic field at charge neutrality, as well as for the $\nu = \frac{1}{2}$ quantum Hall ground state of nonrelativistic fermions in the limit of negligible inter-Landau-level mixing. We argue that when particle-hole symmetry is exact, the composite fermion is a massless Dirac fermion, characterized by a Berry phase of π around the Fermi circle. We write down a tentative effective field theory of such a fermion and discuss the discrete symmetries, in particular, $\mathcal{CP}$. The Dirac composite fermions interact through a gauge, but non-Chern-Simons, interaction. The particle-hole conjugate pair of Jain-sequence states at filling factors $n/(2n+1)$ and $(n+1)/(2n+1)$, which in the conventional composite fermion picture corresponds to integer quantum Hall states with different filling factors, n and $n+1$, is now mapped to the same half-integer filling factor $n+\frac{1}{2}$ of the Dirac composite fermion. The Pfaffian and anti-Pfaffian states are interpreted as d-wave Bardeen-Cooper-Schrieffer paired states of the Dirac fermion with orbital angular momentum of opposite signs, while s-wave pairing would give rise to a particle-hole symmetric non-Abelian gapped phase. When particle-hole symmetry is not exact, the Dirac fermion has a $\mathcal{CP}$-breaking mass. The conventional fermionic Chern-Simons theory is shown to emerge in the nonrelativistic limit of the massive theory.

DOI: 10.1103/PhysRevX.5.031027 Subject Areas: Condensed Matter Physics

OUTLINE

0. Introduction

I. Particle-Hole Symmetry

II. Free fermions $\rightsquigarrow$ Haldane phase

III. Half-filled lowest Landau level

I. On the notion of Particle-Hole Symmetry

Motivation: Tenfold Way (symmetry classes of disordered free fermions)

Altland & Z. (1996); Heinzner, Huckleberry & Z. (2004)

$$\underset{\text{symmetry group}}{G} \times \underset{\text{Fock space}}{\mathcal{F}} \longrightarrow \mathcal{F} = \underset{\text{fermions}}{\wedge(V)}$$

where $G_0 \subseteq G$ arbitrary group of unitary symmetries together with two distinguished anti-unitary generators:

- time-reversal symmetry $\quad T: \wedge^n(V) \longrightarrow \wedge^n(V)$
- particle-hole symmetry $\quad C: \wedge^{\mathrm{half}+q}(V) \longrightarrow \wedge^{\mathrm{half}-q}(V)$

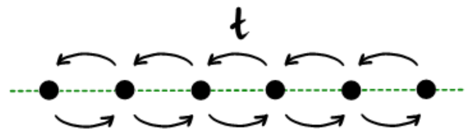
Goal: Classify G-symmetric quadratic Hamiltonians (Hartree-Fock-Bogoliubov mean-field approximation, or "free fermions")

Thm (HHZ). G_0-reduced block data $\overset{10 \longleftrightarrow 10}{\longleftrightarrow}$ classical irred. symmetric spaces

A, AI, AII, AIII, BDI, C, CI, CII, D, DIII

Cor. $H(k)$ (Fourier-Bloch) is invariably of one of the 10 types

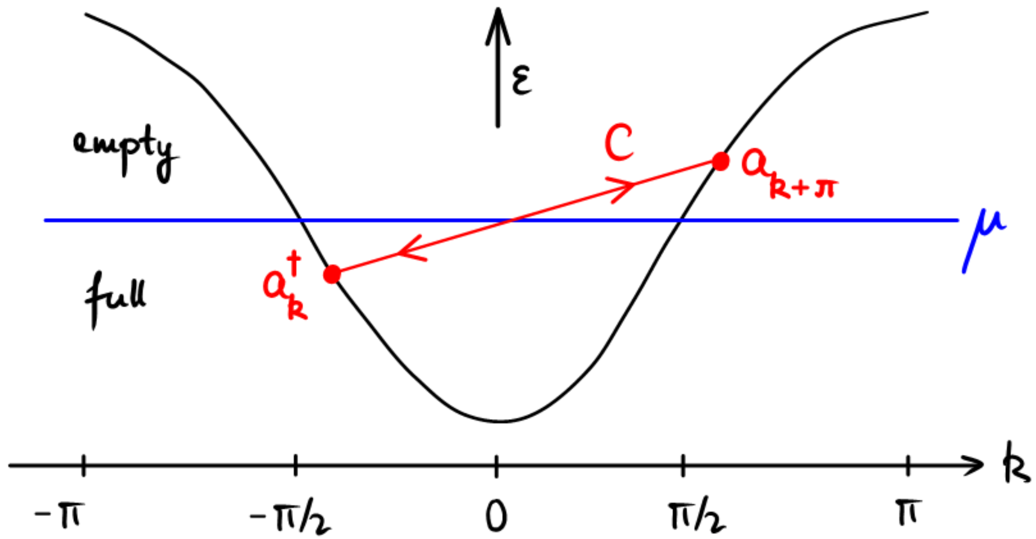
Example 1: cosine band



$\mathbb{Z}$

$$H-\mu N=\sum_k \varepsilon(k)\, a_k^\dagger a_k\,,\qquad \varepsilon(k)=-t\cos k\,,$$

has the particle-hole symmetry $a_k^\dagger \overset{C}{\longleftrightarrow} a_{k+\pi}$.



More generally, any band $\varepsilon(k+\pi)=-\varepsilon(k)$ with $\int \varepsilon(k)\,dk=0$ has the same particle-hole symmetry.

Note. C is complex anti-linear: $C\,i\,C^{-1}=-i$.

Example 2: Hubbard model at half filling

$\mathcal{U} > 0$

Hamiltonian $$H = -\sum_{n\in\mathbb{Z}} \sum_{\sigma=\pm 1/2} \left(t\, a_\sigma^\dagger(n)\, a_\sigma(n+1) + h.c. \right) + \mathcal{U} \sum_n Q^2(n)$$

Charge (normal-ordered) at site n: $$Q(n) = \frac{1}{2} \sum_{\sigma=\pm 1/2} \left(a_\sigma^\dagger(n)\, a_\sigma(n) - a_\sigma(n)\, a_\sigma^\dagger(n) \right)$$

Particle-hole transformation $C\, a_\sigma^\dagger(n)\, C^{-1} = (-1)^n a_\sigma(n)$

is a symmetry: $$\left.\begin{aligned} C\, Q(n)\, C^{-1} &= -\, Q(n) \\ C\, t\, a_\sigma^\dagger(n)\, a_\sigma(n+1)\, C^{-1} &= -\, \bar{t}\, a_\sigma(n)\, a_\sigma^\dagger(n+1) \end{aligned}\right\} \quad C H C^{-1} = H$$

and leaves the ground state (at half filling) invariant.

Cor. Heisenberg antiferromagnetic quantum spin chain is particle-hole symmetric. Note: $C\, S^j(n)\, C^{-1} = -\, S^j(n)$
(spin operators)

Dirac fermions

First quantization.

Dirac operator $H = mc^2\beta + c\sum_\ell \alpha_\ell p_\ell\,, \quad p_\ell = \frac{\hbar}{i}\frac{\partial}{\partial x_\ell}\,,$

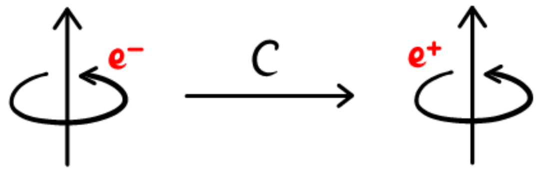
satisfies $KHK^{-1} = -H$ for charge conjugation $K\psi = i\alpha_2\overline{\psi}$ anti-linear

(i.e. K inverts spectrum: $H\psi = E\psi \Rightarrow H\,K\psi = -E\,K\psi$).

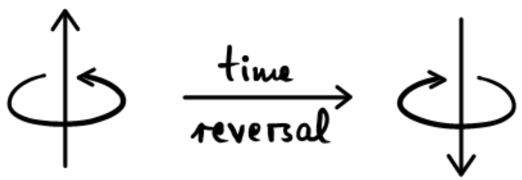
Second quantization.

Charge conjugation is a unitary symmetry, $CHC^{-1} = H$,

mapping electrons to positrons: $C\,i\,C^{-1} = +i$



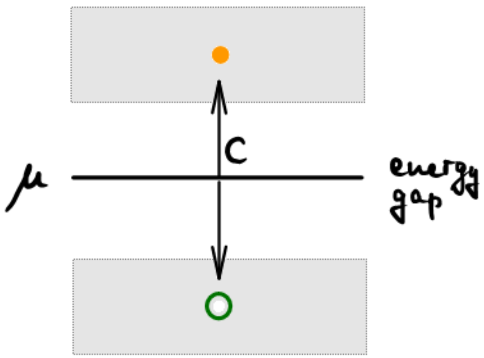
Remark.



Gapped systems (insulators)

- μ : chemical potential (or Fermi energy)
  V_+ : positive-energy states $(E > \mu)$, conduction bands
  V_- : negative-energy states $(E < \mu)$, valence bands

- Fock space $\mathcal{F} = \bigwedge(V_+ \oplus V_-^*)$
  single-particle excitations (V_+)
  single-hole excitations (V_-^*)



- Dirac ket-to-bra bijection $\gamma : V_\pm \to V_\pm^*$, $|v\rangle \mapsto \langle v|$
  (math: Fréchet–Riesz isomorphism $v \mapsto \langle v, \cdot \rangle$)

- Given an isomorphism $V_+ \overset{K}{\longleftrightarrow} V_-$ one has a mapping

$$C : \mathcal{F} \xrightarrow{K} \bigwedge(V_- \oplus V_+^*) \xrightarrow{\gamma} \mathcal{F}$$

Defn. C is called a particle-hole symmetry if $CH = HC$.

Remark. K linear (anti-linear) $\Longleftrightarrow$ C anti-linear (linear)

(cf. Dirac fermion) cosine band $a_k^\dagger \overset{C}{\longleftrightarrow} a_{k+\pi}$

Gapless systems

- What if the energy excitation spectrum is gapless ?

  $V = V_+ \oplus V_0 \oplus V_-$

  fermionic zero modes (e.g. lowest Landau level; boundary zero modes of spin chain)

- Assume $V_0 \cong \mathbb{C}^N$ (finite dimension N)

Defn. Let $\mathcal{A}$ = algebra of (bounded) many-body operators.

Define particle-hole conjugation := anti-linear automorphism of $\mathcal{A}$ determined by $a_j \longleftrightarrow a_j^\dagger$

[where $a_j^\dagger \equiv \varepsilon(v)$, $v \in V_0$ and $a_j \equiv \iota(\varphi)$, $\varphi = \gamma v \in V_0^*$].

Question: does particle-hole conjugation lift to Fock space, i.e. does there exist $\Xi : \Lambda^n(V_0) \to \Lambda^{N-n}(V_0)$ such that $\Xi \, a_j \, \Xi^{-1} = a_j^\dagger$?

Particle-Hole Conjugation Lifted

Recall $V_0 \equiv V \cong \mathbb{C}^N$ with $N < \infty$.

- Fréchet-Riesz isomorphism $\gamma : V \to V^*$, $\quad v \mapsto \langle v, \cdot \rangle$
  on Fock space $\gamma_n : \Lambda^n(V) \to \Lambda^n(V^*) \cong \Lambda^n(V)^*$.

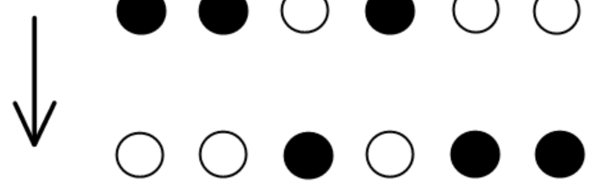
- A choice of generator $\Omega \in \Lambda^{\text{top}} = \Lambda^N(V)$ determines a
  "wedge" isomorphism $\omega_n : \Lambda^n(V)^* \to \Lambda^{N-n}(V)$, $\quad \Phi \mapsto \omega_n \Phi$,

  $$\text{by } (\omega_n \Phi) \wedge \Psi = \Phi(\Psi)\, \Omega_n \,,$$

  $$\Omega_0 \equiv \Omega, \quad \Omega_n = (-1)^{N-n} \Omega_{n-1} \quad (n = 1, 2, \ldots, N).$$

- Particle-hole conjugation $\Xi : \Lambda^n(V) \xrightarrow{\gamma_n} \Lambda^n(V)^* \xrightarrow{\omega_n} \Lambda^{N-n}(V)$
  is a complex anti-linear involution.

Pictorially: Ξ



$$\Xi\, a_j\, \Xi^{-1} \overset{\checkmark}{=} a_j^\dagger$$

Two Facts

Lemma 1. Let X be any self-adjoint and Weyl-ordered one-body operator on $\Lambda(V)$,

i.e. $X = \sum_{ij} X_{ij}(a_i^\dagger a_j - a_j a_i^\dagger) + \sum_{i<j}(Y_{ij} a_i^\dagger a_j^\dagger + \overline{Y}_{ij} a_j a_i)$, $X_{ij} = \overline{X}_{ji}$.

Then X is odd under particle-hole conjugation: $\Xi X \Xi^{-1} = -X$.

Idea of proof.

- $Op \longrightarrow \Xi \, Op \, \Xi^{-1}$ is an algebra automorphism.
- If $\gamma v = \langle v, \cdot \rangle \equiv \varphi$, then $\Xi a_v^\dagger \Xi^{-1} = a_\varphi$.
- Use CAR (canonical anti-commutation relations).

Remark. Ξ can never be a symmetry of any Fermi liquid.

Need spectrum-inverting isomorphism in addition: $C = \Xi K$.

Lemma 2. $\Xi^2 = (-1)^{N(N-1)/2} \, \mathrm{Id}_{\Lambda(V)}$, $N = \dim(V)$.

[cf. time reversal $T^2 = (-1)^n \, \mathrm{Id}_{\Lambda(V)}$]

Symmetry Protection of Zero Modes

Note. Zero modes typically killed by interactions.

Defn. A Hamiltonian H is said to be of class $\mathrm{A\,III}$ if it commutes with a group of symmetries $G = \underset{\substack{\uparrow \\ \text{charge conservation}}}{U(1)_Q} \times \underset{\substack{\uparrow \\ \text{particle-hole transformation}}}{\mathbb{Z}_2^C}$

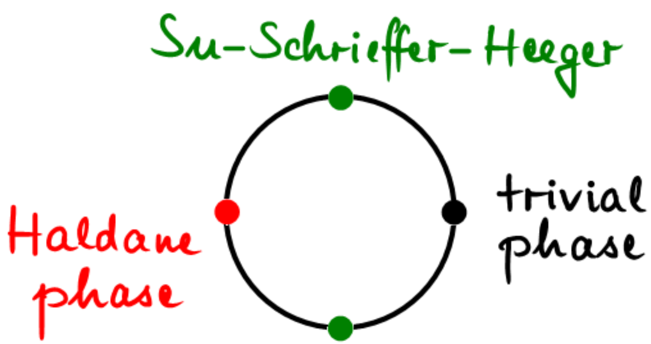
Q: What scenarios (in class $\mathrm{A\,III}$) for protected zero modes?

A: ① Unique ground state $\Psi = C\Psi$ $\quad (C^2 = +1)$

② Kramers pair $\Psi \xleftrightarrow{C} \Psi'$ $(C^2 = -1)$; Ψ, Ψ' same fermion parity

③④ C acts as a supersymmetry $(C^2 = \pm 1)$; Ψ, Ψ' opposite fermion parity

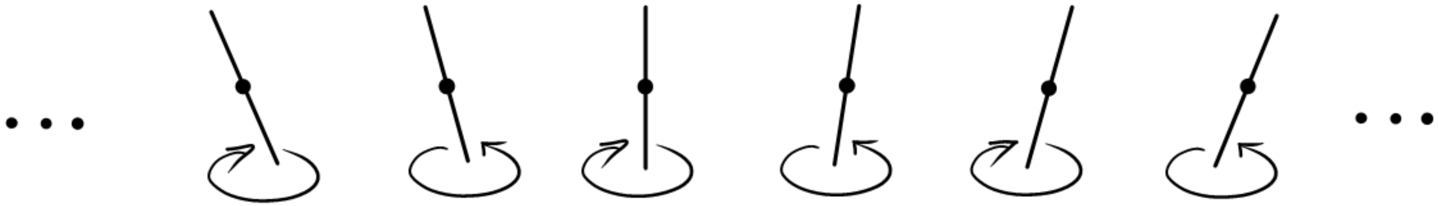
Remark. $\mathbb{Z}_4$ classification of $\mathrm{A\,III}$-protected topological phases in one space dimension:



II. Free fermions

$\rightsquigarrow$ Haldane phase

Anti-ferromagnetic quantum spin chains



$$H = J \sum_n S_n \cdot S_{n+1} \quad (J > 0)$$

Spin 1/2 (Heisenberg chain, Bethe Ansatz):

gapless Lieb–Schultz–Mattis 1961

Spin 1: Haldane (1983) predicts excitation gap from O(3) nonlinear sigma model with topological angle $\Theta = 2\pi|S|$

F.D.M. Haldane
Nobel Prize Physics 2016

— neutron scattering experiments on $CsNiCl_3$

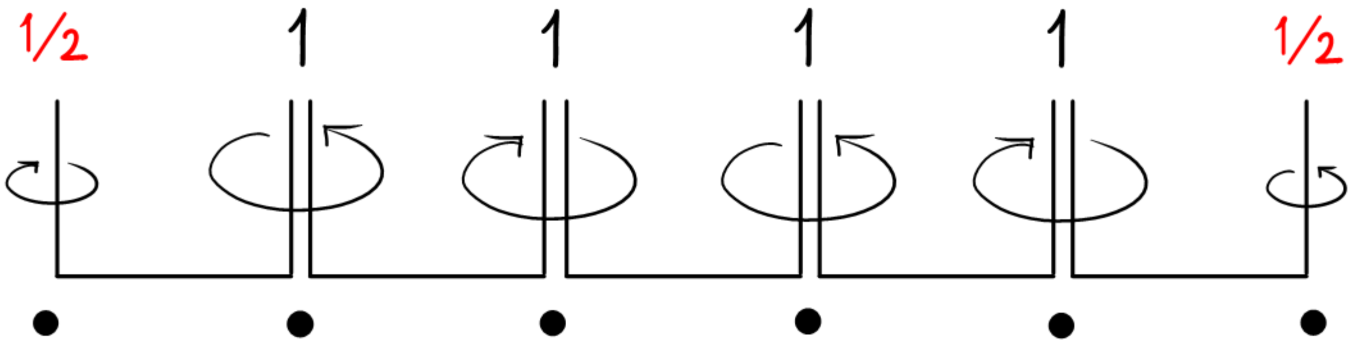
Haldane phase (spin 1)

exponential decay of correlations ($\sim$ mass gap), BUT

hidden topological order: edge excitations of spin 1/2 (!)

$\rightarrow$ "fractionalization"

Example: Affleck–Kennedy–Lieb–Tasaki (AKLT)



matrix–product state

Haldane phase new paradigm (beyond Landau–Ginzburg–Wilson) for (short-range) topological order

Haldane phase as an SPT phase

symmetry-protected topological

Q: protection by what symmetry ?

A1 (Pollmann–Berg–Turner–Oshikawa, 2010):

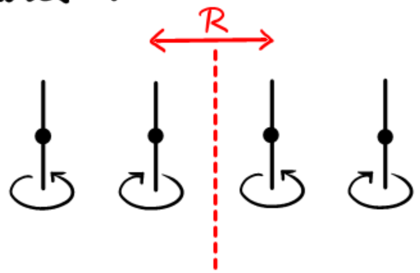
T (time reversal)

OR $\mathbb{Z}_2 \times \mathbb{Z}_2$ (dihedral group)

OR R (space reflection)

~~A1~~ (Anfuso–Rosch, 2007): local charge fluctuations !

A 2 (Moudgalya–Pollmann, 2015): bond inversion



disorder ?

A3 (MRZ, 2016; Verresen–Moessner–Pollmann, 2017): particle-hole symmetry

Def. Two Hamiltonians H_0 and H_1 are said to be in the same topological phase (or topologically equivalent, $H_0 \sim H_1$) if there exists a homotopy $[0,1] \ni t \mapsto H(t)$, $H(0) = H_0$, $H(1) = H_1$ such that $H(t)$ has a unique ground state with a finite energy gap for excitations, for all t.

Def. A Hamiltonian H is said to be of symmetry class AIII if

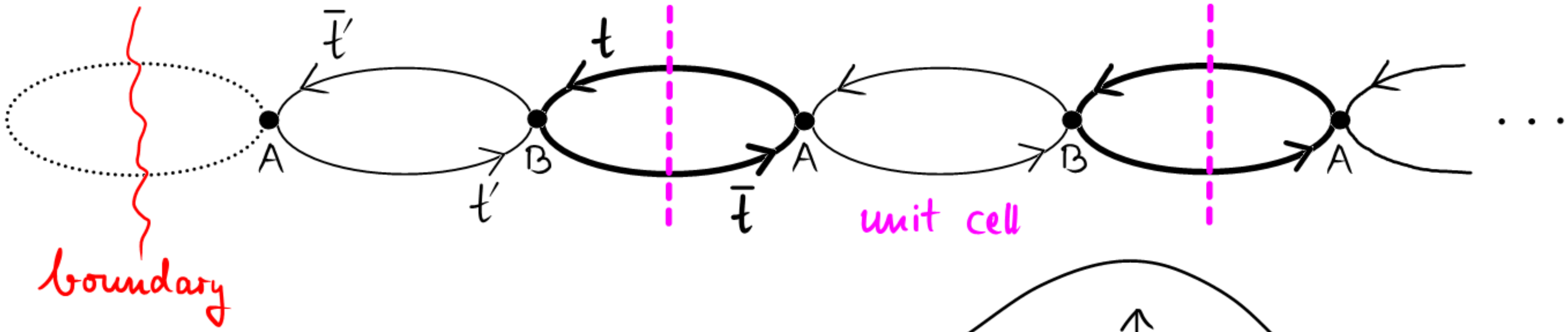
$$H = e^{i\theta Q} H e^{-i\theta Q} = C H C^{-1}.$$

(charge operator Q, particle-hole transformation C)

$\rightsquigarrow$ symmetry-protected topological phase of class AIII

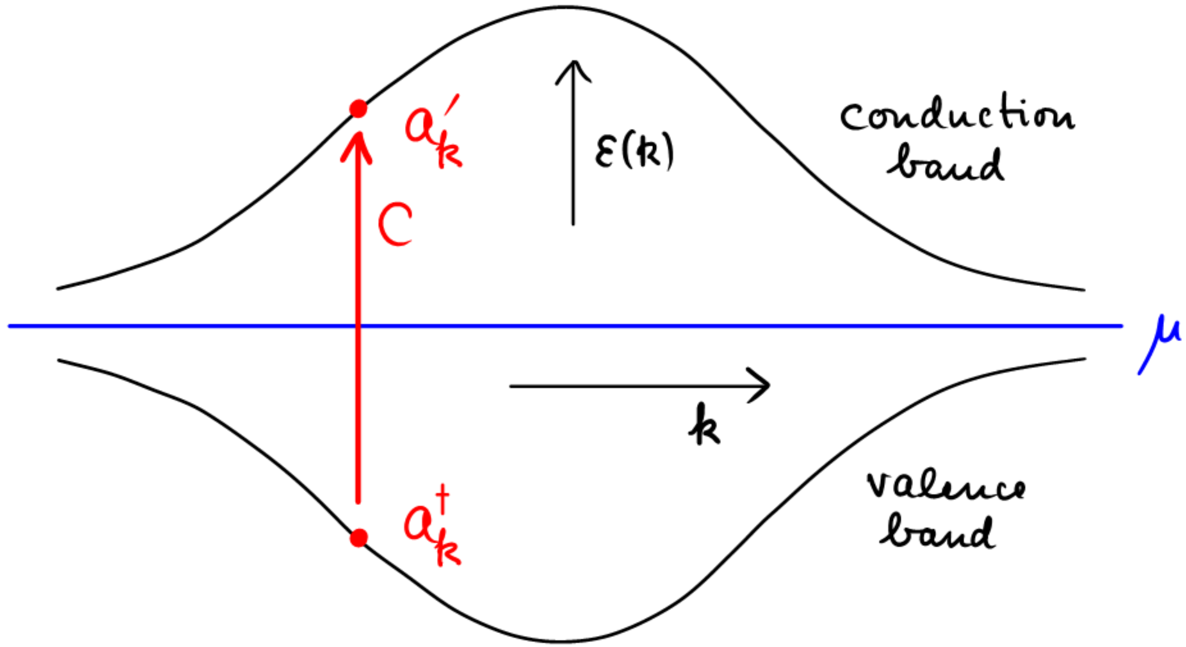
A3 (here): free fermion SPT phase in 1D with symmetry group $U(1)_Q \times \mathbb{Z}_2^C \overset{\text{AIII}}{\sim}$ Haldane phase

Su-Schrieffer-Heeger model ("polyacetylene"; class AIII)



$|t| > |t'|$ (Peierls)

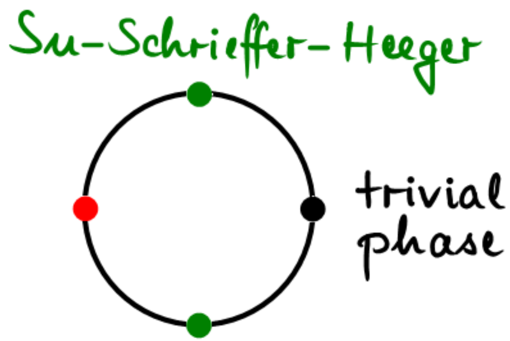
⇝ double the unit cell:



Valence band carries non-trivial topological invariant (e.g. from bulk-boundary correspondence).

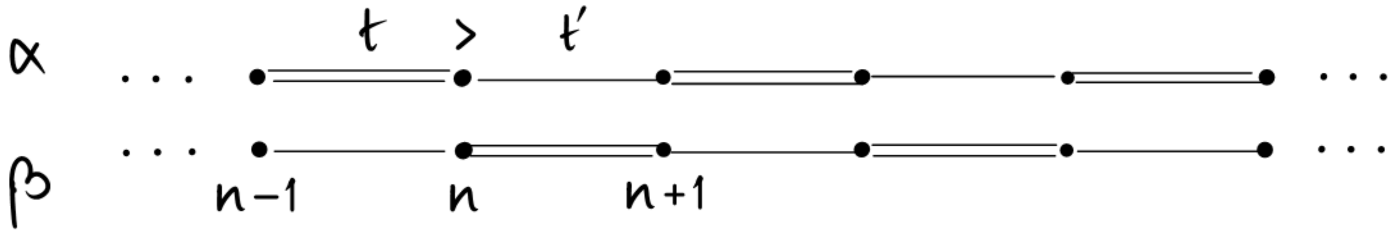
Zero mode localized at boundary (cut strong bond):

$$\psi(A_{n+1}) = -\frac{t'}{t}\,\psi(A_n)\,, \quad \psi(B_n) = 0\,.$$



From SSH to Haldane-AKLT

starting point: two chains of SSH



spin $S = 1/2$ (each)

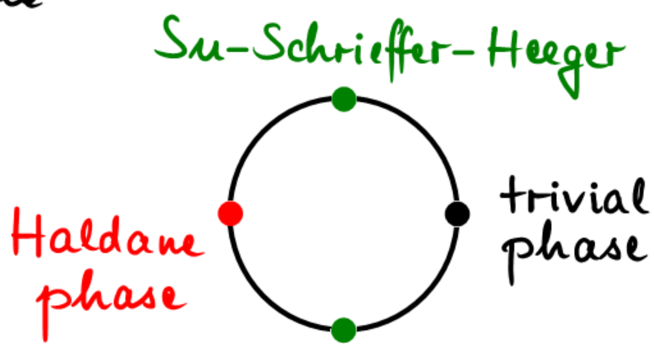
Symmetry group $G = U(1)_Q \times \mathbb{Z}_2^C$ (class AIII)

Recall $a^\dagger(n) \overset{C}{\longmapsto} (-1)^n a(n)$, $Q(n) \overset{C}{\longmapsto} -Q(n)$, $S(n) \overset{C}{\longmapsto} -S(n)$.

Hamiltonian (path in class AIII):

$$H(t, t', U, J) = H_{\text{free}} + U \sum_n Q^2(n) - J \sum_n S_\alpha(n) \cdot S_\beta(n)$$

"Hubbard" "Hund's rule"

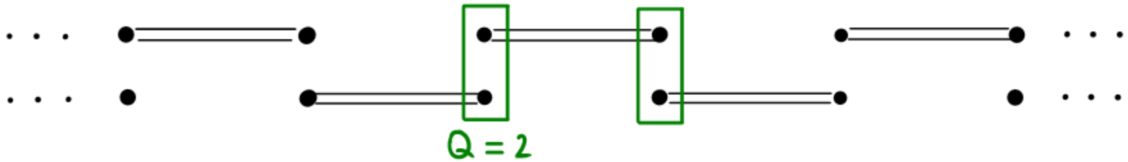


Deformation

Step 1. Turn off the hopping t' $\rightsquigarrow$ flat bands

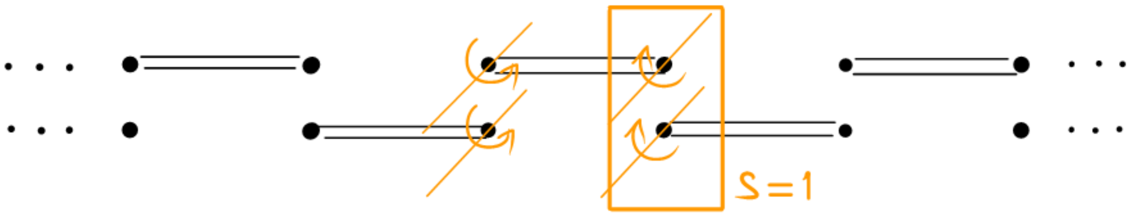
(energy gap stays open; topological invariant remains unchanged)

Step 2. Turn on the Hubbard coupling $(U \gg |t|)$ $\rightsquigarrow$ 2 charges/site



(singlet bonds form; energy gap stays open)

Step 3. Turn on the Hund's rule coupling $(J \sim U)$ $\rightsquigarrow$ $S=1$ on each site



(antiferromagnetic exchange coupling; energy gap stays open)

The resulting ground state is the AKLT state $\hookrightarrow$ Haldane phase

III. Half-filled lowest Landau level

Halperin, Lee, Read (1994)

Composite fermions (i.e. electrons with 2 fictitious magnetic flux quanta attached) experience zero net B-field and populate a Fermi sea (in free-fermion approximation).

$\rightsquigarrow$ Effective field theory with Lagrangian:

$$\mathcal{L} = i\psi^\dagger(\partial_t - iA_0 + ia_0)\psi - \frac{1}{2m}|(\partial_i - iA_i + ia_i)\psi|^2 + \frac{1}{2}\frac{1}{4\pi}\epsilon^{\mu\nu\lambda}a_\mu\partial_\nu a_\lambda + \ldots$$

Phenomenologically quite successful!

Problems: – mass renormalization ?

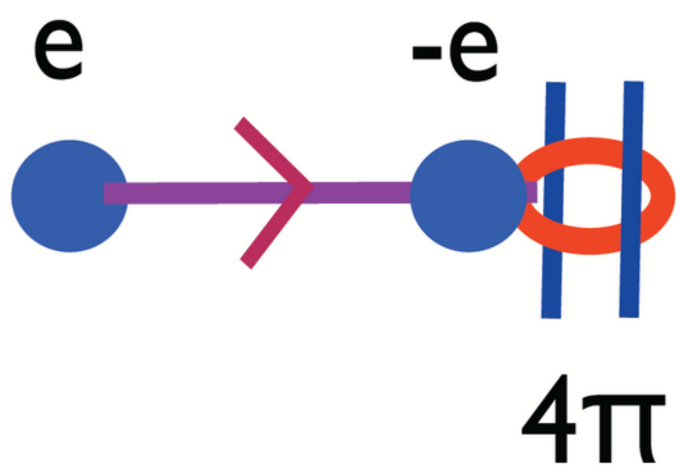
– particle-hole symmetry ?

<u>Note</u>. $H_{eff} = P_{LLL} V_2 P_{LLL}$, V_2 quadratic in charge density, is exactly particle-hole symmetric … …

Remark. Son's new theory resolves both of these issues!

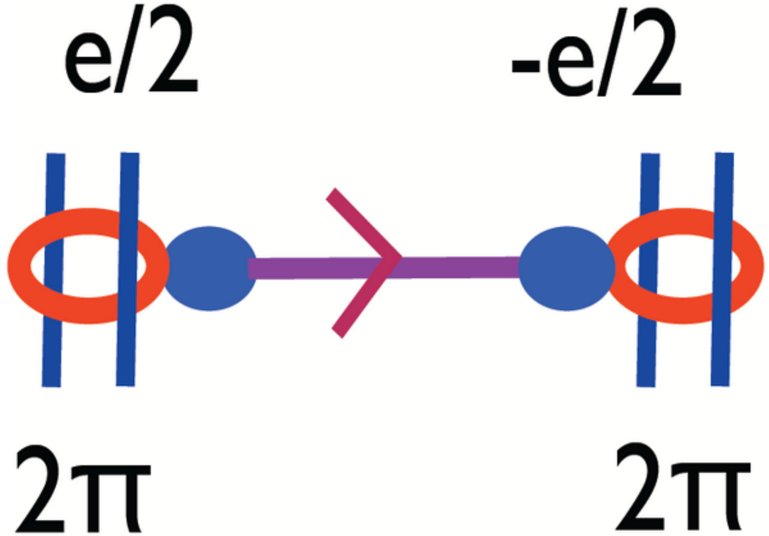
Pictures of composite fermion as a dipole

Wang & Senthil (2016)



traditional picture due to Halperin-Lee-Read

new picture of CF with particle-hole symmetry



Son's Logic

- Realize lowest Landau level as zero-energy sector of Dirac fermion (in a homogeneous magnetic field):

$$S = \int d^3x i\bar{\Psi}\gamma^\mu(\partial_\mu - iA_\mu)\Psi + S_{E.M.}$$

- Switch to dual description (QED_3 ; fermionic particle-vortex duality) by $\bar{\Psi}\gamma^\mu\Psi = J^\mu = \frac{1}{2\pi}\epsilon^{\mu\nu\lambda}\partial_\nu a_\lambda$

  (a_0 = magnetization, $\epsilon^{ij}a_j$ = electric polarization field):

$$S_{\text{eff}} = \int d^3x \left(i\bar{\psi}\gamma^\mu(\partial_\mu + 2ia_\mu)\psi + \frac{1}{2\pi}\epsilon^{\mu\nu\lambda}A_\mu\partial_\nu a_\lambda \right) + S_{E.M.}$$

Remarks. — No mass term, no Chern–Simons term !

— Particle-hole symmetry (antilinear) is implemented as CT

(charge conjugation & time reversal)

Symmetry Considerations

- Electromagnetic gauge field $A = A_\mu dx^\mu$ is time-twisted:
  under time reversal $A \longmapsto -T^*A$ or $A_0 \longmapsto +A_0$, $A_j \longmapsto -A_j$
  and under parity $A \longmapsto +P^*A$ or $A_0 \longmapsto +A_0$, ...

- Charge 3-current $J = da$ is space-twisted. Hence
  time reversal : $a \longmapsto +T^*a$
  parity : $a \longmapsto -P^*a$
  $\rightsquigarrow \int A \wedge J$ invariant

Consequence: First-quantized Hamiltonian for the fermionic vortex field

$$H = v\sigma_1(p_1 + 2a_1) + v\sigma_2(p_2 + 2a_2) + a_0$$

is <u>odd</u> under each of time reversal, parity, and charge conjugation.

<u>Time reversal</u>: $\psi \mapsto \overset{\mathbb{C}\text{-linear}}{\sigma_3\psi}$, $a_0 \mapsto -a_0$, $a_1 \mapsto +a_1$, $a_2 \mapsto +a_2$.

<u>Parity</u> $(x_2 \mapsto -x_2)$: $\psi \mapsto \overset{\mathbb{C}\text{-antilinear}}{\overline{\psi}}$, $a_0 \mapsto -a_0$, $a_1 \mapsto -a_1$, $a_2 \mapsto +a_2$.

<u>Charge conjugation</u>: $\psi \mapsto \overset{\mathbb{C}\text{-antilinear}}{\sigma_1\overline{\psi}}$, $a_\mu \mapsto -a_\mu$.

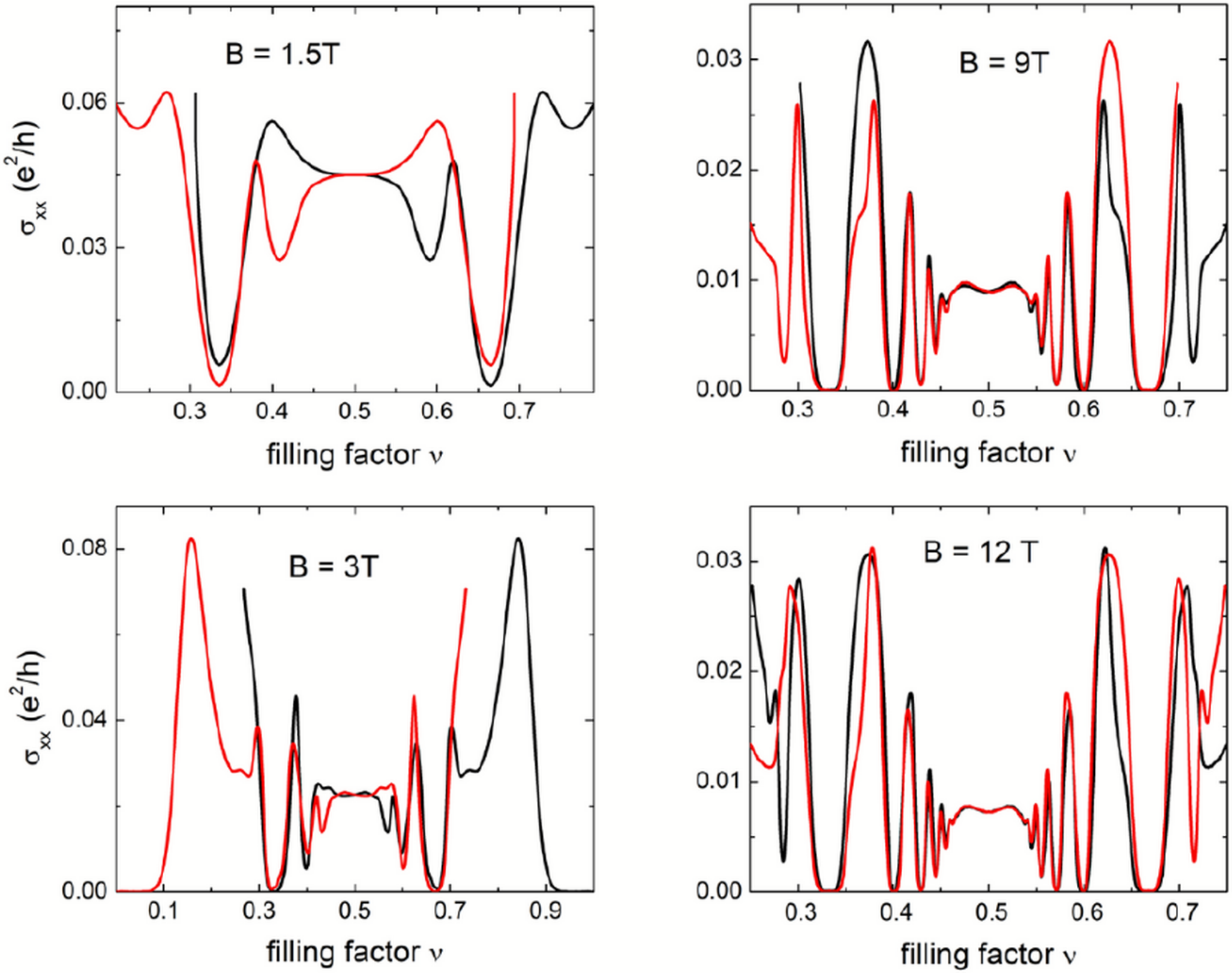


Figure 4. Examination of reflection symmetry in σ_{xx} over a large range of magnetic fields, from 1.5T to 12T.

Thank you!

(The End)