Quantum Mechanics and the Gauge Principle

Two 90 min Lectures by MRZ

DPG Summer School Bad Honnef (09/2018)

Lecture 1:

Schrödinger wave functions are sections of a complex line bundle

1. Dirac monopole problem 2. Dirac quantization condition 3. Berry connection 4. Aharonov–Bohm effect

Reminder: QM for a charged particle (Schrödinger eqn)

- Gauge transformations:

$$A \longmapsto A + d\chi, \quad \psi \longmapsto e^{ie\chi/\hbar}\psi.$$

- Wave function ψ not gauge-invariant.
- Hamiltonian $H = \frac{1}{2m}\sum_j \left(\frac{\hbar}{i}\frac{\partial}{\partial x^j} - eA_j\right)^2$ depends on choice of gauge.

Q: Is gauge dependence inevitable?

A: No! A gauge-invariant notion of wave functions, Hamiltonians, etc., <u>does exist</u>.

> Gauge symmetry is a structure imposed to remove redundancy from an imperfect mathematical model of physical reality.

Dirac monopole problem: my favorite example.

Consider a charged particle moving freely in the magnetic field of a monopole with magnetic charge nh/e for $n=2$. For simplicity (and without much loss) restrict the motion to a sphere, S^2, around the monopole.

CLAIM. In this setting the wave function of the charged particle can be visualized as a vector field on S^2 (= section of the tangent bundle TS^2).

Sanity check.

Q: Shouldn't the values of a Schrödinger wave function be in $\mathbb{C}$?

A: $v(x) \in T_x S^2 \cong \mathbb{R}^2 \cong \mathbb{C}$.

Q: you mean <u>real</u> vector fields? (To write the Schrödinger equation, we need multiplication by $i = \sqrt{-1}$.)

A: Yes! Multiplication by i in our picture is rotation by $\pi/2$ in $T_x S^2$.

Q: What are the operators of momentum and energy?

A: Momentum $p = \frac{\hbar}{i} \nabla$ (Levi-Civita covariant derivative ∇)

Energy $= \frac{p^2}{2m}$.

Note: $[\nabla_u, \nabla_v] = -i \frac{e}{\hbar} B(u,v)$.

Q: How to retrieve the picture taught in class?

A: Fix a unit-vector field $s(x)$ as a reference/standard.

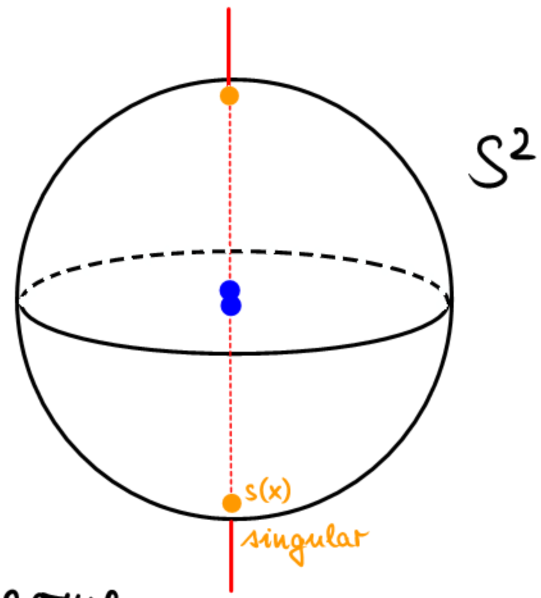
Use $T_x S^2 \longleftarrow \mathbb{C} \otimes T_x S^2$ to write $v(x) = \psi(x)\, s(x)$.

choice of gauge

$x \longmapsto \psi(x) \in \mathbb{C}$ gauge-dependent

Q: Mustn't the reference vector field $s(x)$ have some zeroes?

A: Yes, in fact $n=2$ zeroes. That's a problem for the naive approach. In the Dirac-string approach one assumes $s(x)$ with singularities. The ensuing singularities in $\psi(x)$ are attributed to fictitious magnetic flux lines entering at the singular points.



Q: This vector-field picture is great! Why isn't it used all the time?

A: In the general situation, our vector fields become sections of a complex line bundle, and working with these is not a piece of cake.

Q: What changes for monopole charge $n \neq 2$?

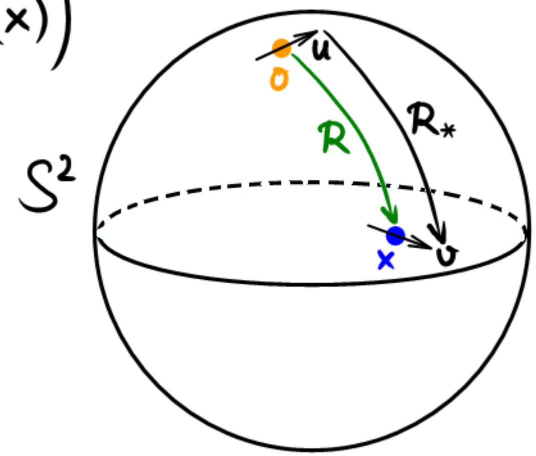
A: Write $T_x S^2 \ni v = R_* u$ where $u \in T_o S^2$ ("north pole" o)

and R_* differential of $R \in SO(3)$: $R \cdot o = x$.

Now $v(x) = R_*(x)\, u(x) = \big(R_*(x)\, g(x)\big)\big(g(x)^{-1} u(x)\big)$

with $u(x) \in T_o S^2 \cong \mathbb{C}$ gauge-dependent;

$g(x) \in SO(2) \cong U(1)$ gauge transfn.



For $n \neq 2$ form gauge equivalence classes:

$$\psi^{(n)}(x) = \big[R_*(x)\, ;\, u(x)\big] \equiv \big[R_*(x)\, g(x)\, ;\, g(x)^{-n/2} u(x)\big]$$

change the charge / or representation

Language / Notation. $S^2 = SO(3)/SO(2)$.

Associated vector bundle $E^{(n)} = SO(3) \times_{SO(2)} \mathbb{R}^2_{n/2}$

Dirac quantization condition.

$$\boxed{\text{electric charge} \times \text{magnetic charge}/\hbar \in 2\pi\mathbb{Z}.}$$

— (transversal) section has $e \cdot m/h$ zeroes.

— Gauss–Bonnet–Chern: integrated curvature $\frac{ie}{\hbar}\iint B = 2\pi\, c_1$

Chern number

Generalization.

Associated vector bundle $\quad E = P \times_G V \longrightarrow P/G$

principal bundle: P; standard fiber: V; structure group: G; base space: P/G

Our case: $\quad P = SO(3) \quad$ (actually, $Spin(3)$)

$G = SO(2) \quad$ (actually, $Spin(2)$)

$P/G = S^2; \quad V = \mathbb{C}$ (carries G-representation)

Connection ($\approx$ covariant derivative ∇).

Let s be a section of the vector bundle E, i.e. $s(x) \in E_x$.

A priori, there is no meaningful way to take derivatives of s!
(Indeed, for $x \neq y$ it makes no sense to subtract $s(x) \in E_x$ from $s(y) \in E_y$.)

Given a notion of parallel transport $\mathcal{T}_{\gamma(t)} : E_x \longrightarrow E_{\gamma(t)}$ $\quad x = \gamma(0)$

one defines
$$\underset{u = \dot{\gamma}(0)}{(\nabla_u s)}(x) = \lim_{t \to 0} \frac{1}{t}\left(\mathcal{T}^{-1}_{\gamma(t)} s(\gamma(t)) - s(x)\right).$$

Curvature: $\nabla_u \nabla_v - \nabla_v \nabla_u - \nabla_{[u,v]} = \mathcal{F}^\nabla(u,v)$
"field strength", "Faraday".

Our example (Dirac monopole $n = 2$): $i\mathcal{F}^\nabla = \frac{e}{\hbar} B = R$ Riemannian curvature of TS^2.

Exercise. Compute the Christoffel symbol of the connection for $n = 1$.

Berry connection.

Given a trivial vector bundle $M \times V \to M$ with trivial connection $\nabla = d$ and Hermitian structure $\langle \cdot , \cdot \rangle_V$, any subvector bundle $M \times V \supset E \to M$ inherits a connection $\nabla^E = d\big|_E$

(restriction and projection)

Application: quantum dynamics in the adiabatic limit.

M = space of adiabatically varying parameters; V = Hilbert space;

$E \to M$ line bundle of (say) ground states.

Example (Berry, 1985): spin $S = 1/2$ in magnetic field B with variable axis.

$$M = S^2 \ni \frac{B}{|B|}, \quad V = \mathbb{C}^2, \quad H \propto \begin{pmatrix} \cos\theta & \sin\theta\, e^{i\phi} \\ \sin\theta\, e^{-i\phi} & -\cos\theta \end{pmatrix},$$

$$E_{\theta,\phi} = \mathbb{C} \cdot \begin{pmatrix} \cos(\theta/2) \\ \sin(\theta/2)\, e^{-i\phi} \end{pmatrix} \quad \rightsquigarrow \quad \nabla^E = \ ? \ \text{(Exercise)}$$

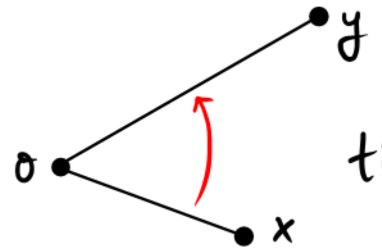
Aharonov–Bohm Effect.

Def.: a connection ∇ with zero curvature is called *flat*.

Remark. There exist flat connections which are non-trivial (i.e. with *holonomy*).

Example. For $M = E_2 \setminus \{o\}$ (punctured Euclidean plane) consider the tangent bundle TM with connection ∇ determined from parallel transport $\tau_{x \xrightarrow{\gamma} y} : T_xM \longrightarrow T_yM$

where $\tau_{x \xrightarrow{\gamma} y}$ is $SO(2)$-rotation by angle



times $r \in \mathbb{R}$

(and thus depends only on the homotopy class of the path γ).

Exercise. — This connection ∇ is flat.

— ∇ is trivial (non-trivial) for $r \in \mathbb{Z}$ ($r \notin \mathbb{Z}$).

— (TM, ∇) can serve as a model for the Aharonov–Bohm effect with magnetic flux rh/e through o.

Lesson Learned.

Schrödinger wave fcts are sections of a complex line bundle:

- Locality: every space point x comes with its own line $E_x \cong \mathbb{C}$.
- Sections s are differentiated using a connection ∇ ($\rightsquigarrow$ momentum $= \frac{\hbar}{i}\nabla$).
- Curvature of the connection: $\nabla^2 = \mathcal{F}^\nabla$ $\left(\iint_\Sigma \mathcal{F}^\nabla = \int_{\partial\Sigma} \alpha \right.$ magnetic flux, or "Berry phase" $\left.\right)$

Note: s, ∇ are manifestly gauge-invariant.

Gauge dependence of ψ, H, etc. enters through $\psi(x) \in \mathbb{C}$:

$$s(x) = \psi(x)\, s_0(x) \quad \rightsquigarrow \quad \nabla s = s_0 \left(d - i\tfrac{e}{\hbar}A\right)\psi .$$

Lesson from AB effect: there is more gauge-invariant physics in A than just its curl $B = dA$ (indeed: ∇ flat implies A closed but ∇ trivial requires A exact).

Remark. Gauge "symmetry" is <u>not</u> a symmetry!

— Associated vector bundle: $E = P \times_G V$

symmetries act here

there act the gauge transformations

— (Unitary) symmetries lead to conservation laws (Noether), but gauge "symmetries" lead to nothing of the sort.

— Symmetries can be broken (spontaneously or explicitly), but gauge "symmetries" cannot ever be broken.

Simple analogy: vector space V with basis $\{e_a\}$.

— Active transformation ($\curvearrowleft$ physical motion):

$$v \mapsto gv = g(e_a v^a) = (g e_b) v^b = e_a g^a{}_b v^b$$

— Passive transformation ($\curvearrowleft$ gauge transformation):

$$v = e_a v^a = e_a (g^{-1} g)^a{}_b v^b = \tilde{e}_a g^a{}_b v^b, \quad \tilde{e}_a = e_b (g^{-1})^b{}_a$$

References.

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Lecture 2:

Non-relativistic condensed matter has $U(1)_{em} \times SU(2)_{spin}$ local gauge invariance

1. Pauli equation completed 2. Dirac equation 3. Spin group & Spin structure 4. Aharonov–Casher effect

Pauli equation (completed) for the electron.

$$\boxed{i\hbar c\, D_0\psi = \frac{-\hbar^2}{2m}\sum_{k=1}^{3} D_k^2\psi} \qquad \text{(Fröhlich et al., 1993–)}$$

$$D_\mu = \frac{\partial}{\partial x^\mu} + \alpha_\mu + \beta_\mu\,, \quad \alpha_\mu(x) \in \mathrm{Lie}\,U(1), \quad \beta_\mu(x) \in \mathrm{Lie}\,SU(2).$$

$$\alpha_0 = \frac{ie}{\hbar c}\Phi\,, \qquad \alpha_k = -\frac{ie}{\hbar} A_k \quad (k = 1,2,3).$$

$$\beta_0 = -\frac{e}{8mc} B_{k\ell}\left[\sigma^k, \sigma^\ell\right], \qquad \beta_k = -\frac{e}{8mc^2}\left[\sigma^k, E_\ell\sigma^\ell\right].$$

Invariance under local gauge transformations.

$$U(1)_{em}: \; \psi \mapsto e^{i\chi}\psi\,, \qquad \alpha_\mu \mapsto \alpha_\mu - i\,\partial_\mu\chi\,.$$

$$SU(2)_{spin}: \; \psi \mapsto g\,\psi\,, \qquad \beta_\mu \mapsto g\,\beta_\mu g^{-1} + g\,\partial_\mu g^{-1}, \qquad g(x) \in SU(2).$$

Heuristic. The Dirac equation has $SU(2)_{spin}$ gauge invariance. Therefore, if nonrelativistic reduction is performed consistently, then $SU(2)_{spin}$ gauge invariance must be passed on to the Pauli equation.

Dirac equation (on Minkowski space-time $M \cong \mathbb{R}^{1,3}$).

In components:
$$(\gamma^\mu)^a{}_b \frac{\partial}{\partial x^\mu} \psi^b + i \frac{mc}{\hbar} \psi^a = 0 \qquad (a = 1, \dots, 4).$$

The gamma matrices satisfy the defining relations of a Clifford algebra $Cl_{1,3}(\mathbb{R})$:
$$(\gamma^\mu)^a{}_b (\gamma^\nu)^b{}_c + (\gamma^\nu)^a{}_b (\gamma^\mu)^b{}_c = 2\eta^{\mu\nu} \delta^a_c ,$$
$$\eta^{00} = 1, \quad \eta^{11} = \eta^{22} = \eta^{33} = -1 \qquad \text{Minkowski metric.}$$

Standard choice: $\gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $\gamma^k = \begin{pmatrix} 0 & \sigma^k \\ -\sigma^k & 0 \end{pmatrix}$ $(k = 1, 2, 3)$.

Invariant formulation.

— Dirac field is section of spinor bundle: $\psi \in \Gamma(M, S)$.

$\psi(x) = e_a(x)\,\psi^a(x) \in S_x$ (spinor space at world point $x \in M$).

— Clifford map $\gamma(x) : T_x^* M \longrightarrow \mathrm{End}(S_x)$ is expressed by

$e_a(x)\,(\gamma^\mu)^a{}_b\, e^b(x)\,(\partial_\mu)_x \equiv \gamma(x) \in T_x M \otimes \mathrm{End}(S_x)$.

— Spin connection $\nabla^S : \Gamma(M, S) \longrightarrow \Gamma(M, T^*M \otimes S)$ given by

$\nabla^S e_a = dx^\mu \otimes e_b\,(\beta_\mu)^b{}_a \;\curvearrowright\; \nabla^S(e_a \psi^a) = dx^\mu \otimes e_b\left(\delta^b_a \frac{\partial}{\partial x^\mu} + (\beta_\mu)^b{}_a\right)\psi^a$.

Note: ∇^S becomes nontrivial in gravitational backgrounds.

— Dirac operator $\not{D} = \gamma \circ \nabla^S = \gamma(dx^\mu)\, \nabla^S_{\partial_\mu}$.

$dx^\mu \partial_\mu$

Spin gauge invariance $\leadsto$ ψ, ∇^S invariantly defined.

$$\psi(x) = e_a(x)\psi^a(x) = \tilde{e}_b(x)\tilde{\psi}^b(x),$$

where $e_a(x) = \tilde{e}_b(x)\, g(x)^b{}_a$, $\tilde{\psi}^b(x) = g(x)^b{}_a\, \psi^a(x)$

and $\tilde{\beta}^a{}_c = g^a{}_b\, \beta^b{}_{b'} (g^{-1})^{b'}{}_c + g^a{}_b\, d\, (g^{-1})^b{}_c$.

Enter the electromagnetic field.

Take the tensor product $E_x \otimes_{\mathbb{C}} S_x \cong S_x$.

The Dirac field for electrically charged particles becomes a section $\psi \in \Gamma(M, E \otimes S)$.

Spin^c connection $\nabla^{E\otimes S} : \Gamma(M, E\otimes S) \longrightarrow \Gamma(M, T^*M \otimes E \otimes S)$,

$$\nabla^{E\otimes S}\psi = \nabla^{E\otimes S}((s_0 \otimes e_a)\psi^a) = (s_0 \otimes e_b)\left(\delta^b_a(d+\alpha) + \beta^b{}_a\right)\psi^a.$$

Spin group

Q: Can we allow <u>any</u> change of frame $e_a(x)\psi^a(x) = \tilde{e}_b(x)\tilde{\psi}^b(x)$?

A: In principle yes. However, in order to preserve the geometric structure of the theory, one wants to restrict to transformations from the spin group, a double cover of the Lorentz group:

$\gamma_\mu = \eta_{\mu\lambda}\gamma^\lambda$

$$\mathrm{Spin}(1,3) \xrightarrow{2:1} SO(1,3)$$

$$g = \exp\left(\tfrac{1}{8}\omega^\mu{}_\nu[\gamma_\mu,\gamma^\nu]\right) \longmapsto \exp\left(\omega^\mu{}_\nu e_\mu \otimes e^\nu\right) \equiv R(g).$$

Simple example: $\mathrm{Spin}(2) \xrightarrow{2:1} SO(2)$

$$\exp\left(\tfrac{\theta}{4}[\sigma_x,\sigma_y]\right) \longmapsto \exp\left(\theta\,(e_x\otimes e^y - e_y\otimes e^x)\right).$$

Q: Who ordered the double cover?

A: The Clifford map $\gamma: TM \to \mathrm{End}(S)$ must be equivariant:

$$g\,\gamma_\nu\, g^{-1} = \gamma_\mu\, R(g)^\mu{}_\nu .$$

Spin structure.

Q: Is the spinor bundle $S \to M$ an associated vector bundle?

A: yes. $S = \mathrm{Spin}(M) \times_{\mathrm{Spin}(1,3)} \mathbb{C}^4 \longrightarrow M$.

The principal bundle $\mathrm{Spin}(M) \longrightarrow M = \mathrm{Spin}(M)/\mathrm{Spin}(1,3)$ is called a *spin structure*.

Exercise. S^2 has a spin structure (we used it earlier, secretly).

Q: What is the associated vector bundle good for?

A: It helps us understand how to actively Lorentz-transform Dirac spinor fields (in fact, $\mathrm{Spin}(M)$ carries a left action by the Lorentz group).

Non-relativistic reduction

is applicable when mc^2 is the largest energy scale.

$\rightsquigarrow$ Do perturbation theory on the Dirac equation in Hamiltonian form:

$$i\hbar \frac{\partial}{\partial t} = \mathcal{H}\psi, \qquad \mathcal{H} = \gamma^0 \sum_{k=1}^{3} \gamma^k \left(\frac{\hbar}{i} \frac{\partial}{\partial x^k} - eA_k \right) + e\Phi + \gamma^0 mc^2 .$$

Foldy–Wouthuysen transformation: $\quad \gamma^0 W = -W\gamma^0$

$$\psi \mapsto \mathcal{U}\psi, \quad \mathcal{H} \mapsto \mathcal{U}\mathcal{H}\mathcal{U}^{-1}, \quad \mathcal{U} = \exp(iW)$$

and projection to the electron sector gives the Pauli equation (plus corrections).

Spin rotation group.

$$\mathrm{Spin}(3) = \left\{ g \in \mathrm{Spin}(1,3) \,\middle|\, \gamma^0 g = +g\gamma^0 \right\} = SU(2) \text{ generated by}$$

$$[\gamma^k, \gamma^l] = -\begin{pmatrix} [\sigma^k, \sigma^l] & 0 \\ 0 & [\sigma^k, \sigma^l] \end{pmatrix}$$ transfers to the non-relativistic limit as left action ($\rightsquigarrow$ physical symmetry) and right action ($\rightsquigarrow$ gauge transformations).

Aharonov–Casher effect.

Place charged wire (charge/length = Q) on the z-axis.

$\rightsquigarrow$ Electric field strength $E = \frac{Q}{2\pi\varepsilon_0}\frac{dr}{r}$, $\quad r = \sqrt{x^2+y^2}$.

Neutral particle (e.g. neutron) with spin-magnetic moment μ:

$$D_k = \frac{\partial}{\partial x^k} - \frac{\mu}{8\hbar c^2}\left[\sigma_k, E_\ell \sigma^\ell\right].$$

Restrict the motion to some plane $z = \text{const}$.

Curvature of the SU(2) connection:

$$\left[D_x, D_y\right] = \frac{\mu}{8\hbar c^2}\left[\sigma_x, \sigma_y\right]\left(\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y}\right) = i\sigma_z\, \delta(x,y)\, \frac{\mu Q}{4\varepsilon_0 \hbar c^2}.$$

The connection is flat but non-trivial!

Aharonov & Casher (1984) predict wave interference effect.

Exercise. Look up the experimental situation with the AC effect.

Is the charged-wire experiment feasible with neutrons?

References.

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