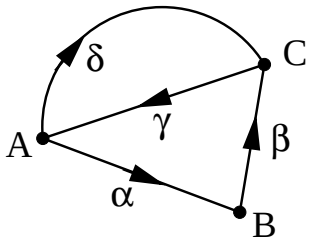
4 Networks

In this chapter we develop some parts of discrete exterior calculus, introducing the laws of electrostatics, Kirchhoff's rules and certain types of networks along the way.

4.1 Chains on 1-complexes

The basic objects in discrete exterior calculus are *(differential) k-complexes.* For the moment we look only at the simple case of 1-complexes (k-complexes with $k > 1$ will be introduced later). These are made up of vertices (or 0-cells) connected by oriented edges (or 1-cells, or links).

Example. In the following example of a 1-complex K, there are 3 vertices A, B, C and 4 oriented edges $\alpha, \beta, \gamma, \delta$:



Orientation of an edge means a sense of direction (determined pictorially by an arrow).

Chains. In discrete exterior calculus, the vertices of a complex K are regarded as the basis vectors of a real vector space $C_0(K)$. Thus the elements c of $C_0(K)$ are linear combinations of vertices A_i with real coefficients c_i: $c = \sum_i c_i A_i$. Such formal sums are called *0-chains.* Similarly, the oriented edges of K are regarded as the basis vectors of a real vector space $C_1(K)$. Its elements are formal sums $\sum_j r_j \alpha_j$ of oriented edges α_j with real coefficients r_j, and they are called *1-chains.*

Boundary operator. Every 1-complex K comes with a linear operator

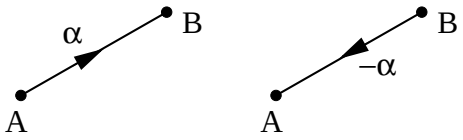
$$\partial : \ C_1(K) \to C_0(K), \tag{4.1}$$

called the *boundary operator* of K. The boundary of an oriented edge is defined to be the end point of the edge minus the starting point. By linearity, this defines ∂ on all of $C_1(K)$.

Example. Recall the complex K shown above. There we have

$$\begin{aligned}
&\partial\alpha = B - A, \\
&\partial(\alpha + \beta) = \partial\alpha + \partial\beta = (B - A) + (C - B) = C - A, \\
&\partial(\alpha + \beta + \gamma) = 0, \\
&\partial(\alpha + \beta - \gamma) = -2\partial\gamma = 2C - 2A.
\end{aligned}$$

Notice that, if $\partial\alpha = B - A$ (as in the example), then $\partial(-\alpha) = A - B$ by the linearity of ∂. Thus, in going from α to $-\alpha$ the starting point A and end point B switch roles, and we should think of $-\alpha$ as being the same edge as α but with the orientation (i.e. the sense of direction) reversed.



Kirchhoff's first rule. Let I be the electrical current flow in an electrical circuit K, and let us consider I as a 1-chain on K, viewing K as a 1-complex (by assigning orientations to the wires of the circuit). In formulas:

$$I = \sum_j I_j \, \gamma_j \,, \tag{4.2}$$

where γ_j are the (oriented) wires of the circuit K, and the real coefficient I_j is the current flowing along γ_j. ($I_j > 0$ means that the current is flowing in the direction of the chosen orientation of γ_j, whereas $I_j < 0$ means that the current flows the opposite way.) If the current flow is stationary (i.e. both the current density and the charge density are constant in time), then the law of electrical current conservation applies and is succinctly expressed by *Kirchhoff's first rule*:

$$\partial I = 0 \,. \tag{4.3}$$

Example. Consider again the complex K of the previous example. The current in that case is a sum of 4 terms:

$$I = I_\alpha \alpha + I_\beta \beta + I_\gamma \gamma + I_\delta \delta.$$

Computing its boundary we obtain

$$\begin{aligned} \partial I &= I_\alpha \partial\alpha + I_\beta \partial\beta + I_\gamma \partial\gamma + I_\delta \partial\delta \\ &= I_\alpha (B - A) + I_\beta (C - B) + I_\gamma (A - C) + I_\delta (C - A) \\ &= (-I_\alpha + I_\gamma - I_\delta) A + (I_\alpha - I_\beta) B + (I_\beta - I_\gamma + I_\delta) C \,. \end{aligned}$$

Because of the linear independence of the "vectors" A, B, C – recall that they furnish a basis of the vector space $C_0(K)$ by postulate – setting $\partial I = 0$ is equivalent to setting the coefficients of each of A, B, C to zero:

$$\partial I = 0 \quad \Longleftrightarrow \quad -I_\alpha + I_\gamma - I_\delta = 0 \,, \quad I_\alpha - I_\beta = 0 \,, \quad I_\beta - I_\gamma + I_\delta = 0 \,.$$

The equation $I_\alpha = I_\beta$ obviously means that there is as much current flowing into the node B as there is current flowing out of it. The other two equations mean the same thing for the nodes A, C. This is just the statement of current conservation: for every node N in a circuit with stationary current flow (and no charges piling up at the nodes) the current flowing into N exactly balances the current flowing out of N.

4.2 Cochains and coboundary operator

Recall from linear algebra that every vector space V comes with a dual vector space V^*. The elements of V^* are linear functions $f : V \to \mathbb{R}$. By this token, given the vector space $C_0(K)$ we have the dual vector space of linear functions $f : C_0(K) \to \mathbb{R}$. The latter is conventionally

denoted by $C^0(K) \equiv C_0(K)^*$, and its elements are called *0-cochains.* If c is a 0-chain of K and f is a 0-cochain of K, we write

$$\langle f, c \rangle \tag{4.4}$$

for the value of f on c. By linearity, a 0-cochain f is completely determined by its values on the 0-cells (the vertices) A_i of K. Such values are alternatively written as

$$\langle f, A_i \rangle \equiv f(A_i). \tag{4.5}$$

We also refer to the (non-degenerate) linear mapping $C^0(K) \times C_0(K) \to \mathbb{R}$ by $(f, c) \mapsto \langle f, c \rangle$ as the *pairing* between 0-cochains and 0-chains.

Electric energy. The electric scalar potential (also called the electrostatic potential in the present context) Φ assigns to every node A_i of an electric circuit K a real number $\Phi(A_i)$, namely the value of the potential at that point. The function Φ can be viewed (and this is a view we will often take) as a 0-cochain on K regarded as a 1-complex. The value of Φ on an arbitrary 0-chain is

$$\left\langle \Phi, \sum\nolimits_i c_i A_i \right\rangle = \sum_i c_i \, \Phi(A_i). \tag{4.6}$$

While it may seem odd to the novice that we view the electrostatic potential as a function not just on points but on arbitrary 0-chains, this point of view is often quite useful. One application of this viewpoint is the following. Let $\rho = \sum_i \rho_i A_i$ (a 0-chain) be the electric charge density of the circuit K – here we assume that all charges sit on nodes and the amount of charge on the circuit node A_i is ρ_i . Then the pairing between Φ and ρ is (twice) the electrostatic energy of the circuit:

$$\langle \Phi, \rho \rangle = \sum_i \rho_i \, \Phi(A_i). \tag{4.7}$$

1-cochains. Just as $C_0(K)$ is associated with its dual, $C^0(K)$, the vector space of 1-chains $C_1(K)$ is associated with its dual vector space, $C^1(K)$. The elements of $C^1(K)$ are called 1-cochains. The pairing between a 1-cochain ω and a 1-chain c is still denoted by $\langle \omega, c \rangle$, and for the value of ω on a 1-cell γ_j we still use the alternative notation

$$\langle \omega, \gamma_j \rangle \equiv \omega(\gamma_j). \tag{4.8}$$

Electrical power. To every edge (or link) γ_j of an electric circuit K we associate the voltage $V(\gamma_j)$ along γ_j . We can view this association as a function $V : \gamma_j \mapsto V(\gamma_j) \in \mathbb{R}$, which linearly extends to a function $V : C_1(K) \to \mathbb{R}$. Thus V is a 1-cochain, $V \in C^1(K) = C_1(K)^*$. In a resistive network (see Section 4.4 below) with current $I = \sum_j I_j \, \gamma_j$ the pairing of V with I,

$$\langle V, I \rangle = \sum_j I_j \, V(\gamma_j), \tag{4.9}$$

has a clear physical meaning: it is the *power* dissipated by the network.

Canonical adjoint. We now recall another general principle from linear algebra: given a linear mapping L from a vector space V into a vector space W, there exists a canonically defined linear mapping $L^t : W^* \to V^*$ called the *canonical adjoint* of L. It is defined by

$$\langle L^t\omega, c\rangle = \langle \omega, Lc\rangle \quad (\omega \in W^*,\ c \in V). \tag{4.10}$$

The canonical adjoint is sometimes called the "transpose": if one introduces bases $v_1, v_2, \ldots$ of V and $w_1, w_2, \ldots$ of W and expands $Lv_i = \sum_j w_j L_{j\,i}$, then the matrix of L^t w.r.t. the dual bases of W^* and V^* is the transpose of the matrix of L: $(L^t)_{ij} = L_{j\,i}$.

Coboundary operator. Application of this general principle to the boundary operator

$$\partial :\ C_1(K) \to C_0(K) \tag{4.11}$$

yields a linear operator (the canonical adjoint or transpose of ∂)

$$\mathrm{d} :\ C^0(K) \to C^1(K), \tag{4.12}$$

which is called the *coboundary operator.* The boundary operator ∂ is a very intuitive object, but what is the intuition for the coboundary operator d? We will see that d is the discrete variant of a differential operator (hence the notation d). First we give a concrete example illustrating d.

Example. Consider once again the complex K of the example given in Section 4.1. The matrix of ∂ w.r.t. the (ordered) bases A, B, C of $C_0(K)$ and $\alpha, \beta, \gamma, \delta$ of $C_1(K)$ is

$$\begin{pmatrix} -1 & 0 & 1 & -1 \\ 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 1 \end{pmatrix}.$$

Since d is the canonical adjoint of ∂, its matrix w.r.t. the corresponding dual bases is obtained by transposition:

$$\begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 0 & 1 \end{pmatrix}.$$

If A^*, B^*, C^* and $\alpha^*, \beta^*, \gamma^*, \delta^*$ are the dual bases of $C^0(K)$ and $C^1(K)$, this means that

$$\begin{aligned} \mathrm{d}A^* &= -\alpha^* + \gamma^* - \delta^*, \\ \mathrm{d}B^* &= \alpha^* - \beta^*, \\ \mathrm{d}C^* &= \beta^* - \gamma^* + \delta^*, \end{aligned}$$

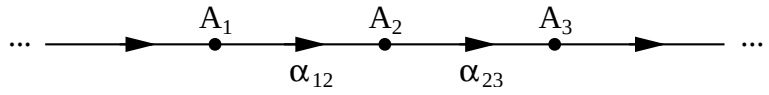
since $\mathrm{d}A_i^* = \sum_j \alpha_j^*\, \mathrm{d}_{j\,i}$.

Rule for d. Here comes the general rule for the linear operator d. (Of course, by linearity it suffices to specify d on some basis of $C^0(K)$.) We adopt the convention of denoting dual bases with an asterisk. Let then p be any vertex of K. To compute $\mathrm{d}p^*$ we look for the oriented edges

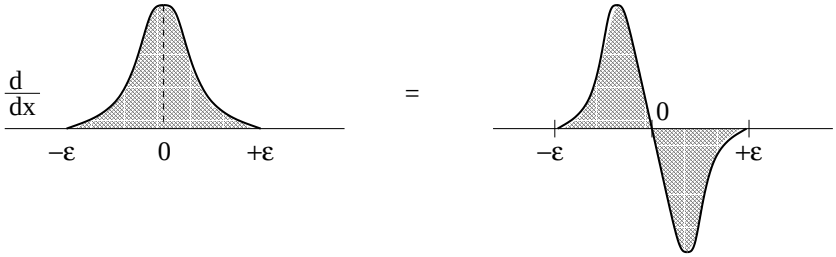
of K that have p as a boundary point. Let $\alpha_{i_1}, \ldots, \alpha_{i_m}$ $(\beta_{j_1}, \ldots, \beta_{j_n})$ be the oriented edges that end (resp. begin) on p. Then

$$\mathrm{d}p^* = \alpha^*_{i_1} + \alpha^*_{i_2} + \ldots + \alpha^*_{i_m} - \beta^*_{j_1} - \beta^*_{j_2} - \ldots - \beta^*_{j_n}\,. \tag{4.13}$$

d as a differential. Why did we say that the coboundary operator d resembles a differential operator? Consider d on a network of linear structure:



and compare $\mathrm{d}A^*_2 = \alpha^*_{12} - \alpha^*_{23}$ with the derivative of a smeared δ-function:



The similarity should be clear.

Voltage as a coboundary. We will put the coboundary operator to many good uses in the sequel. A first application is as follows. Consider, once again, the complex of the example in Section 4.1. Expressing the potential Φ in the basis of characteristic functions A^*, B^*, C^* we have

$$\Phi = \Phi(A)\, A^* + \Phi(B)\, B^* + \Phi(C)\, C^*. \tag{4.14}$$

We then compute the coboundary of Φ:

$$\begin{aligned}
\mathrm{d}\Phi &= \Phi(A)\,\mathrm{d}A^* + \Phi(B)\,\mathrm{d}B^* + \Phi(C)\,\mathrm{d}C^* \\
&= \Phi(A)(\gamma^* - \alpha^* - \delta^*) + \Phi(B)(\alpha^* - \beta^*) + \Phi(C)(\delta^* + \beta^* - \gamma^*) \\
&= \big(\Phi(B) - \Phi(A)\big)\,\alpha^* + \big(\Phi(C) - \Phi(B)\big)\,\beta^* + \big(\Phi(A) - \Phi(C)\big)\,\gamma^* + \big(\Phi(C) - \Phi(A)\big)\,\delta^*.
\end{aligned}$$

By the definition of Φ, the potential difference $\Phi(B) - \Phi(A)$ is the negative of the voltage along α: $V(\alpha) = \Phi(A) - \Phi(B)$. Similarly, $V(\beta) = \Phi(B) - \Phi(C)$, and so on. Thus we see that

$$V = -\,\mathrm{d}\Phi. \tag{4.15}$$

This corresponds to the equation $\vec{E} = -\mathrm{grad}\,\Phi$ of vector calculus.

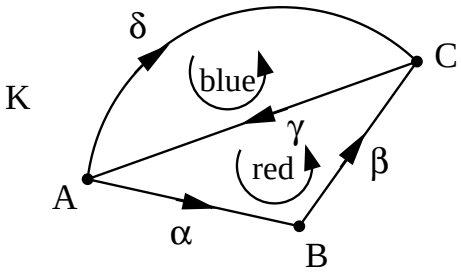
4.3 2-complexes: d∘d = 0

The complexes K under consideration so far were built from vertices and edges only. Now we add another element to the setup: oriented faces (or 2-cells). Beside the vector spaces $C_0(K)$ and

$C_1(K)$, we then have the vector space of 2-chains, $C_2(K)$. The boundary operator $\partial : C_1(K) \to C_0(K)$ naturally extends to a sequence of linear operators

$$C_2(K) \xrightarrow{\partial} C_1(K) \xrightarrow{\partial} C_0(K). \tag{4.16}$$

Example. The following 2-complex is obtained by still taking the 1-complex of the example in Section 4.1 and sewing in two oriented faces, denoted by "red" and "blue":



The boundaries of the two oriented faces of K are

$$\partial(\text{red}) = \alpha + \beta + \gamma, \quad \partial(\text{blue}) = -\gamma - \delta.$$

Notice that the boundary of each of the two boundaries is zero:

$$(\partial \circ \partial)(\text{red}) = \partial(\alpha + \beta + \gamma) = B - A + C - B + A - C = 0,$$
$$(\partial \circ \partial)(\text{blue}) = \partial(-\gamma - \delta) = C - A + A - C = 0.$$

This property is not particular to the special 2-complex under consideration, but will always be true for any "reasonable" complex (think about it, and you'll see that it's true!). Hence the following definition is rather natural.

Definition. A *k-complex* (for $k \in \mathbb{N}$) is a sequence of vector spaces $C_0(K), \ldots, C_k(K)$ connected by linear operators (the *boundary operator*),

$$0 \xleftarrow{\partial} C_0(K) \xleftarrow{\partial} C_1(K) \xleftarrow{\partial} \cdots \xleftarrow{\partial} C_{k-1}(K) \xleftarrow{\partial} C_k(K), \tag{4.17}$$

such that $\partial \circ \partial = 0$. □

Now, given a k-complex (of chains) we immediately get another k-complex (of cochains) by dualization:

$$C^0(K) \xrightarrow{\mathrm{d}} C^1(K) \xrightarrow{\mathrm{d}} \cdots \xrightarrow{\mathrm{d}} C^{k-1}(K) \xrightarrow{\mathrm{d}} C^k(K) \xrightarrow{\mathrm{d}} 0. \tag{4.18}$$

Indeed, from $\partial \circ \partial = 0$ we deduce that

$$\langle \mathrm{d}\mathrm{d}\omega, c \rangle = \langle \mathrm{d}\omega, \partial c \rangle = \langle \omega, \partial\partial c \rangle = \langle \omega, 0 \rangle = 0 \tag{4.19}$$

for any $\omega \in C^k(K)$ and $c \in C_{k+2}(K)$, so the coboundary operator d satisfies

$$\mathrm{d} \circ \mathrm{d} = 0. \tag{4.20}$$

Kirchhoff's second rule. Consider some 2-complex and recall the electrostatic law $V = -\mathrm{d}\Phi$; in words: the voltage is (minus) the coboundary of the potential. Applying d to both sides of this equation and using $\mathrm{d} \circ \mathrm{d} = 0$ we get

$$\mathrm{d}V = 0 \qquad \text{(Kirchhoff's second rule).} \tag{4.21}$$

What does this mean? The equation $\mathrm{d}V = 0$ is equivalent to $\langle \mathrm{d}V, c \rangle$ being zero for every 2-chain c. But $\langle \mathrm{d}V, c \rangle = \langle V, \partial c \rangle$, so

$$\mathrm{d}V = 0 \quad \Longleftrightarrow \quad \forall c \in C_2(K) : \ \langle V, \partial c \rangle = 0. \tag{4.22}$$

Any 1-chain ∂c consists of boundary loops, so the statement $\mathrm{d}V = 0$ is equivalent to saying that the voltage V vanishes along any such loop. For example, the boundary of the red face in the example on the previous page is $\alpha + \beta + \gamma$, and Kirchhoff's second rule states that

$$\langle V, \partial(\mathrm{red}) \rangle = V(\alpha) + V(\beta) + V(\gamma) = 0.$$

Exercise. Show from the definition of the coboundary operator by $\langle \mathrm{d}\omega, c \rangle = \langle \omega, \partial c \rangle$ that $\mathrm{d} : C^1(K) \to C^2(K)$ is computed by the following rule. Let $\gamma_1, \gamma_2, \ldots$ be the oriented edges of K and denote the element dual to γ_j by γ_j^*, as before. To compute the coboundary $\mathrm{d}\gamma_j^*$ we look for the (oriented) faces that contain γ_j in their boundary. Then $\mathrm{d}\gamma_j^*$ is the sum (with signs) of the duals of these faces, where the plus/minus sign is used if the orientations agree/disagree.

Example. For the 2-complex K from the beginning of this subsection we have

$$\mathrm{d}\alpha^* = \mathrm{red}^*, \quad \mathrm{d}\beta^* = \mathrm{red}^*, \quad \mathrm{d}\gamma^* = \mathrm{red}^* - \mathrm{blue}^*, \quad \mathrm{d}\delta^* = -\mathrm{blue}^*.$$

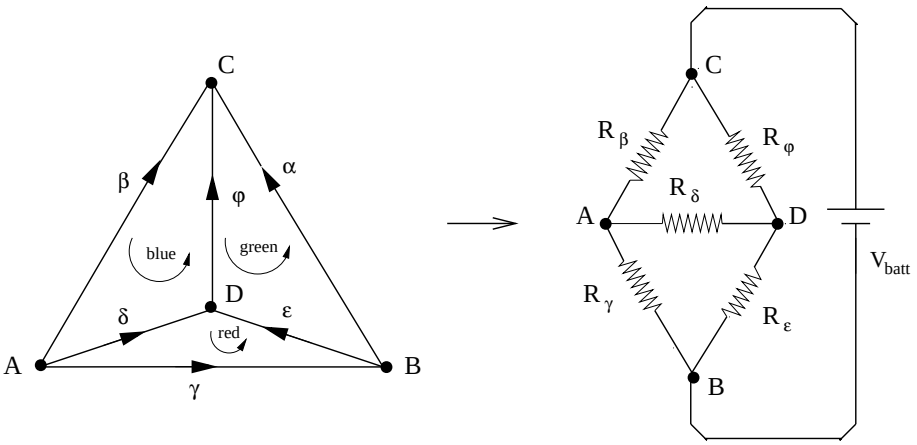
4.4 Resistive networks

Let K be some 2-complex with a number N_1 of 1-cells $\gamma_1, \gamma_2, \ldots, \gamma_{N_1}$. Place resistors with resistances R_i on $N_1 - 1$ of these, and on the residual 1-cell place a current source (i.e. a battery, say with voltage V_{batt}). What you get in this way is called a *resistive network*. We assume that the resistors and the battery are connected by perfectly conducting wires via the 0-cells of K (the *nodes* of the resistive network). The situation is illustrated in the next diagram, showing a 2-complex on the left-hand side and a corresponding resistive network on the right-hand side.

Now let the resistive network be in a stationary state. This means that everything is constant in time (except for the state of the battery, which is gradually getting discharged), and there is a certain time-independent electrical current flowing in each 1-cell γ_j of K, i.e. in each resistor of the network. These (local) currents I_j determine a 1-chain

$$I = \sum_{j=1}^{N_1} I_j \, \gamma_j \tag{4.23}$$

called the (global) current. Assuming the resistances R_j and the battery voltage V_{batt} to be given, our goal will be to compute the I_j and hence I.



There exists 3 sets of linear relations, which altogether determine the global current I. The first set is given by Kirchhoff's first rule: $\partial I = 0$. As was explained earlier, it expresses the law of current conservation at the nodes. The second set relates the currents I_j to the voltages V_j by Ohm's law: $V_j = R_j I_j$. Viewing the resistances as a mapping $R : \ C_1(K) \to C^1(K)$, we write Ohm's law in the succinct global form $V = R\, I$. The third set of relations is given by Kirchhoff's second rule, $\mathrm{d}V = 0$. Its meaning, as we saw, is that the voltage about any boundary loop of the network vanishes. In summary, the equations of a stationary resistive network are

$$\mathrm{d}V = 0, \quad V = R\, I, \quad \partial I = 0. \tag{4.24}$$

In the absence of a driving term, the solution of these equations is the trivial one, $I = 0 = V$. The solution becomes nontrivial if we take one of the voltages, say $V_j = V(\gamma_j)$, to be externally prescribed, by forcing $V_j = V_{\mathrm{batt}}$. Of course, if we then insisted on prescribing the ratio $V_{\mathrm{batt}}/I_{\mathrm{batt}} = V_j/I_j = R_j$, the system would be overdetermined. To allow a solution to exist, we have to treat the current I_{batt} drawn from the battery as one of the unknowns of the problem.

Solution strategies. We distinguish between two basic strategies for solving the equations of a resistive network:

1. Node potential method:

   (i) Solve $\mathrm{d}V = 0$ by making an ansatz for the potentials at the nodes: $V = -\mathrm{d}\Phi$.

   (ii) Express the current by the potential: $I = -R^{-1}\mathrm{d}\Phi$.

   (iii) Determine the unknowns in Φ from the equation $0 = \partial I = -\partial R^{-1}\mathrm{d}\Phi$.

2. Mesh current method:

   (i) Solve $\partial I = 0$ by setting $I = \partial H$, with H an unknown 2-chain.

   (ii) Express V by H via Ohm's law: $V = R\,\partial H$.

   (iii) Determine H from the equation $0 = \mathrm{d}V = \mathrm{d}R\,\partial H$.

We are not going to enter into an explanation as to whether/why these two methods always work (this is done at length in the textbook by Bamberg & Sternberg: *A Course in Mathematics for Students of Physics*, volume 2, Cambridge University Press, 1990). Instead, we content ourselves by illustrating them at the above example.

Node potential method. The first remark to make here is that the electrostatic potential Φ never is uniquely determined: if Φ solves $V = -\,\mathrm{d}\Phi$ for a given V, then so does $\Phi + \text{const}$. We fix this arbitrariness by setting (any) one of the node potentials to zero. For the network at hand (which is the example network given at the beginning of this subsection) we choose $\Phi(B) = 0$. Since $V_{\text{batt}} = V(\alpha) = \Phi(B) - \Phi(C)$, we then have

$$\Phi = \Phi(A)\, A^* - V_{\text{batt}} C^* + \Phi(D)\, D^*$$

with unknowns $\Phi(A)$ and $\Phi(D)$. Application of the coboundary operator gives the voltage:

$$-V = \mathrm{d}\Phi = \Phi(A)(-\beta^* - \gamma^* - \delta^*) - V_{\text{batt}}(\alpha^* + \beta^* + \varphi^*) + \Phi(D)(\delta^* + \varepsilon^* - \varphi^*).$$

The 1-cochain V so obtained automatically satisfies Kirchhoff's second rule.

Next we use Ohm's law to express the current by the unknowns:

$$\begin{aligned} I = R^{-1}V = I_{\text{batt}}\alpha + R_\beta^{-1}\big(V_{\text{batt}} + \Phi(A)\big)\,\beta + R_\gamma^{-1}\Phi(A)\,\gamma \\ + R_\delta^{-1}\big(\Phi(A) - \Phi(D)\big)\,\delta - R_\varepsilon^{-1}\Phi(D)\,\varepsilon + R_\varphi^{-1}\big(V_{\text{batt}} + \Phi(D)\big)\,\varphi. \end{aligned}$$

Finally, we take the boundary of the current I, and organize the answer by nodes:

$$\begin{aligned} \partial I \;=\; & \Big(- R_\beta^{-1}\big(V_{\text{batt}} + \Phi(A)\big) - R_\gamma^{-1}\Phi(A) - R_\delta^{-1}\big(\Phi(A) - \Phi(D)\big)\Big) A \\ + \; & \Big(- I_{\text{batt}} + R_\gamma^{-1}\Phi(A) + R_\varepsilon^{-1}\Phi(D)\Big) B \\ + \; & \Big(I_{\text{batt}} + R_\beta^{-1}\big(V_{\text{batt}} + \Phi(A)\big) + R_\varphi^{-1}\big(V_{\text{batt}} + \Phi(D)\big)\Big) C \\ + \; & \Big(R_\delta^{-1}\big(\Phi(A) - \Phi(D)\big) - R_\varepsilon^{-1}\Phi(D) - R_\varphi^{-1}(V_{\text{batt}} + \Phi(D))\Big) D. \end{aligned}$$

It would now appear that the system is overdetermined, as there are 4 equations to satisfy with 3 unknowns, namely $\Phi(A)$, $\Phi(D)$ and I_{batt}. Fortunately all of the equations are not independent: if Kirchhoff's first rule is satisfied at all nodes but one, then it is automatically satisfied also at the last node. To see why this always holds true, let $f \in C^0(K)$ be some constant function, $f(A_1) = f(A_2) = \ldots = f(A_{N_0})$. Evaluating such a function on the 0-chain ∂I we are guaranteed to get zero: $\langle f, \partial I\rangle = \langle \mathrm{d}f, I\rangle = 0$, as the coboundary of any constant function vanishes (recall that d "differentiates"). [Another characterization of the situation is to say that the boundary operator $\partial : C_1(K) \to C_0(K)$ of a connected network fails to be surjective by $\dim \operatorname{coker} \partial = 1$.] Thus $\langle f, \partial I\rangle = 0$ or, equivalently, $\sum_i (\partial I)_i = 0$. And, indeed, if we add up the coefficients of the nodes A, B, C, D in our example, we do get identically zero. Therefore, we may delete (any) one of these coefficients, and solve the remaining 3 linear equations from $\partial I = 0$ for the 3 unknowns $\Phi(A)$, $\Phi(D)$, and I_{batt}.

Mesh current method. The resistive network under consideration has three 2-cells, which we chose to denote by red, blue, and green. Thus, the vector space $C_2(K)$ is 3-dimensional and the most general ansatz for H reads

$$H = H_{\text{red}} \cdot \text{red} + H_{\text{blue}} \cdot \text{blue} + H_{\text{green}} \cdot \text{green}$$

with unknowns H_{red}, H_{blue}, and H_{green}. An expression for the current I results on forming the boundary of H,

$$I = \partial H = H_{\text{red}}(\gamma - \delta + \varepsilon) + H_{\text{blue}}(-\beta + \delta + \varphi) + H_{\text{green}}(\alpha - \varepsilon - \varphi).$$

From this we get the voltage 1-cochain V by using Ohm's law:

$$\begin{aligned} V = R\,\partial H = V_{\text{batt}}\,\alpha^* - R_\beta\,H_{\text{blue}}\,\beta^* + R_\gamma\,H_{\text{red}}\,\gamma^* + R_\delta\,(-H_{\text{red}} + H_{\text{blue}})\,\delta^* \\ + R_\varepsilon\,(H_{\text{red}} - H_{\text{green}})\,\varepsilon^* + R_\varphi\,(H_{\text{blue}} - H_{\text{green}})\,\varphi^*. \end{aligned}$$

Here we isolated the battery 1-cell α and set the coefficient of the dual element α^* to V_{batt}, in keeping with the setup specified at the outset.

In the next step we compute the coboundary of V:

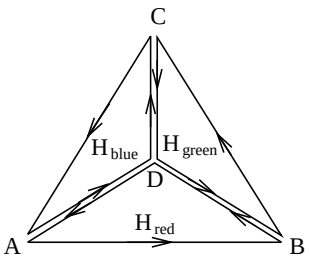
$$\begin{aligned} \mathrm{d}V &= \big(R_\gamma\,H_{\text{red}} - R_\delta\,(-H_{\text{red}} - H_{\text{blue}}) + R_\varepsilon\,(H_{\text{red}} - H_{\text{green}})\big)\,\text{red}^* \\ &+ \big(R_\beta\,H_{\text{blue}} + R_\delta\,(-H_{\text{red}} + H_{\text{blue}}) + R_\varphi\,(H_{\text{blue}} - H_{\text{green}})\big)\,\text{blue}^* \\ &+ \big(V_{\text{batt}} - R_\varepsilon\,(H_{\text{red}} - H_{\text{green}}) - R_\varphi\,(H_{\text{blue}} - H_{\text{green}})\big)\,\text{green}^*. \end{aligned}$$

Imposing Kirchhoff's second rule, $\mathrm{d}V = 0$, we then obtain 3 equations for the 3 unknowns H_{red}, H_{blue}, and H_{green}. These equations can be arranged in matrix form as follows:

$$\begin{pmatrix} R_\gamma + R_\delta + R_\varepsilon & -R_\delta & -R_\varepsilon \\ -R_\delta & R_\beta + R_\delta + R_\varphi & -R_\varphi \\ -R_\varepsilon & -R_\varphi & R_\varepsilon + R_\varphi \end{pmatrix} \begin{pmatrix} H_{\text{red}} \\ H_{\text{blue}} \\ H_{\text{green}} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ -V_{\text{batt}} \end{pmatrix},$$

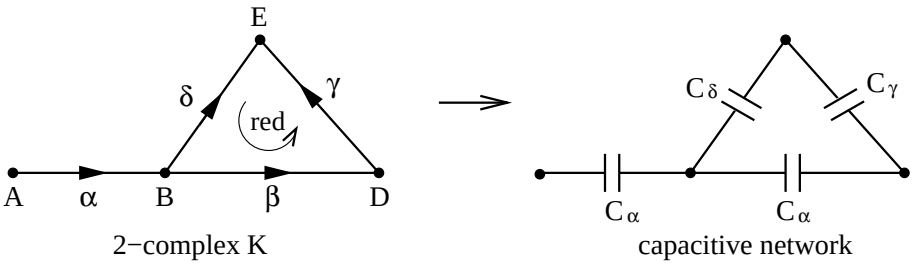
which is a linear system of the type $\mathcal{A}x = b$. The solution $x = \mathcal{A}^{-1}b$ always exists here, since the symmetric matrix $\mathcal{A}$ formed from the resistances is positive and hence invertible.

Let us end with a word on the physical meaning of the 2-chain $H = \sum_k H_k \Sigma_k$. Ultimately, we will recognize H (from Ampere's law) as the magnetic excitation (or, rather, as the 2-chain model thereof), but for now we will simply say this: H_k can be viewed as the partial current (or mesh current) circulating around the k^{th} face. In our example, this is clear from the relations $I_\alpha = H_{\text{green}}$, $I_\beta = -H_{\text{blue}}$, $I_\gamma = H_{\text{red}}$, $I_\delta = H_{\text{blue}} - H_{\text{red}}$, $I_\varphi = H_{\text{blue}} - H_{\text{green}}$, $I_\varepsilon = H_{\text{red}} - H_{\text{green}}$.



4.5 Capacitive networks

In the previous section we took a 2-complex K and placed resistors on its 1-cells to form a resistive network. Now we do the same thing using capacitors instead of resistors. Thus, let there be some 2-complex with 1-cells γ_j $(j = 1, \ldots, N_1)$. On each of these we put a capacitor, denoting the capacitance on the j^{th} 1-cell by C_j. The capacitors are connected to each other via the nodes of K. An example of such a capacitive network is shown in the next figure. We will use this example for illustrative purposes below.



Initial state. To get started, let the network be in a neutral state. This means that we took care to discharge all the capacitor plates (in case they had been charged earlier) and there are no superfluous charges on the nodes either. Starting from this neutral situation we then prepare the network in a specific charged state by transferring charges (usually, electrons) between the nodes of the network by using, say, a battery. After that process there is a nonzero amount of charge on each node, although the total charge (summed over all nodes) still vanishes since all we did was to instigate internal charge transfer. To do the book keeping, we assemble all the information about the node charges into a 0-chain $\rho = \sum_{i=1}^{N_0} \rho_i A_i$, with ρ_i being the charge transferred onto node A_i. The vanishing of the total charge – referred to as global charge neutrality – implies $\langle f, \rho \rangle = 0$ for any $f \in C^0(K)$ with constant values on the nodes.

Stationary state. The extra charges placed on the nodes will not stay there, as such a configuration would not be energetically favorable. Rather, the positive and negative charges forced upon the nodes will seek to recombine so as to restore the original state of *local* charge neutrality (the state of minimal electric energy). They are prevented from doing so by the insulating layer between the two plates of each capacitor. Thus the node charges move to the capacitors and accumulate on the capacitor plates. After a brief transient period of charge redistribution the capacitive network will have settled down in a stationary state, which is the energetically lowest configuration accessible in the presence of the constraints posed by the experimental setup.

Capacitor charges and voltages. The capacitors are now charged – each of them by some pair of charges $(Q_j, -Q_j)$ on opposite plates. Our goal is to compute these capacitor charges Q_j, given the capacitances C_j and the data about the charge transfer between nodes, ρ. For that purpose it is convenient (and also natural) to assemble the information about the charge state of the capacitors into a 1-chain: $Q = \sum_{j=1}^{N_1} Q_j \gamma_j$. Now by the definition of what is meant by a capacitor with capacitance C_j, the charge Q_j on the j^{th} capacitor is accompanied by a voltage $V_j = Q_j/C_j$ between the plates of that capacitor. All these voltages determine a voltage 1-cochain

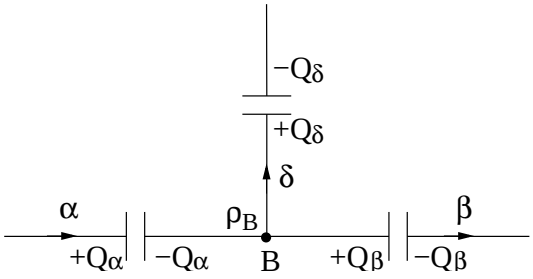
$V = \sum_j V_j \gamma_j^*$. This concludes our discussion of basic definitions and setup, and we can now turn to formulating the algebraic problem to be solved.

Network equations. The equilibrium state of the capacitive network is determined by three sets of equations:

$$-\partial Q = \rho, \quad Q = \mathcal{C}V, \quad \mathrm{d}V = 0. \tag{4.25}$$

The last set is familiar: it is Kirchhoff's second rule, stating that the sum of the voltages around any boundary loop of the capacitive network vanishes. The middle set encodes the information about the capacitances of the network. We define $\mathcal{C} : C^1(K) \to C_1(K)$ to be the linear operator with the property $\mathcal{C}\gamma_j^* = C_j\gamma_j$ $(j = 1, \ldots, N_1)$. The "global" equation $Q = \mathcal{C}V$ then succinctly summarizes all the local relations $Q_j = C_j V_j$.

The first set, $-\partial Q = \rho$, expresses the fact that all capacitor charges stem from node charges. The minus sign in the equation invites some explanation. What enters here is a sign convention for making the assignment of a pair $(Q_j, -Q_j)$ to a pair of capacitor plates. The convention is that as we follow along the directed 1-cell γ_j , the first capacitor plate we encounter carries charge Q_j , and the second one charge $-Q_j$. To illustrate, consider the node B of our example:



Charge conservation means that $\rho_B = -Q_\alpha + Q_\beta + Q_\delta$, and this equals $-(\partial Q)_B$ because

$$\begin{aligned} -Q_\alpha + Q_\beta + Q_\delta &= -Q_\alpha(\partial\alpha)_B - Q_\beta(\partial\beta)_B - Q_\delta(\partial\delta)_B \\ &= -\big(\partial(Q_\alpha\alpha + Q_\beta\beta + Q_\gamma\gamma + Q_\delta\delta)\big)_B = -(\partial Q)_B. \end{aligned}$$

Given a set of node charges ρ, how do we solve the equations of a capacitive network for Q and V? As with resistive networks, there exist (at least) two major strategies. The first of these proceeds by reduction to Poisson's equation and will now be described in some detail.

Poisson's equation. Just as in the node potential method of Section 4.4, we start by solving $\mathrm{d}V = 0$ through the introduction of a potential: $V = -\mathrm{d}\Phi$, thereby obtaining an expression for the capacitor charges as $Q = \mathcal{C}V = -\mathcal{C}\mathrm{d}\Phi$. Inserting this expression into $-\partial Q = \rho$, we get an equation solely for the unknown potential:

$$\partial\,\mathcal{C}\mathrm{d}\Phi = \rho\,. \tag{4.26}$$

The operator on the left-hand side,

$$\triangle := -\partial\,\mathcal{C}\mathrm{d}, \tag{4.27}$$

is called the *Laplace operator* of the capacitive network. Using it, we can write the equation for Φ in the concise form of *Poisson's equation*:

$$-\triangle\Phi = \rho. \tag{4.28}$$

What can we say about the existence and uniqueness of solutions to Poisson's equation? To give some sort of answer, we need a few basic facts about the Laplace operator $\triangle$. First of all, note that $\triangle = -\partial\,\mathcal{C}\mathrm{d}$ is a linear mapping from the vector space $W = C^0(K)$ into the dual vector space $W^* = C_0(K)$. It therefore makes sense to ask whether $\triangle$ is *symmetric.* Indeed, the canonical adjoint of any linear operator $L:\ W \to W^*$ is still a linear operator $L^t:\ W \to W^*$, and one may compare L with L^t. The operator L is called symmetric if $L = L^t$.

Symmetry of $\triangle$. The Laplace operator

$$\triangle = -\partial\,\mathcal{C}\mathrm{d}\Phi:\ C^0(K) \to C_0(K) \tag{4.29}$$

of a capacitive network is symmetric: $\triangle = \triangle^t$.

Proof. Consider first the capacitance operator $\mathcal{C}:\ C^1(K) \to C_1(K)$. If $U = \sum_j U_j\,\gamma_j^*$ and $V = \sum_j V_j\,\gamma_j^*$ are any two 1-cochains, we have

$$\big\langle U, \mathcal{C}V\big\rangle = \sum_{j=1}^{N_1} C_j\,U_j\,V_j = \big\langle V, \mathcal{C}U\big\rangle. \tag{4.30}$$

Thus $\mathcal{C}$ is symmetric. Now let f and Φ be any two functions (or 0-cochains) on K. The proposed statement then follows from a one-line computation using the fact that the coboundary operator d is canonically adjoint to the boundary operator ∂:

$$\big\langle f, \triangle\Phi\big\rangle = -\big\langle \mathrm{d}f, \mathcal{C}\mathrm{d}\Phi\big\rangle = -\big\langle \mathrm{d}\Phi, \mathcal{C}\mathrm{d}f\big\rangle = \big\langle \Phi, \triangle f\big\rangle. \quad \square \tag{4.31}$$

Kernel and cokernel of $\triangle$. In order for the solution of Poisson's equation $-\triangle\Phi = \rho$ to exist and be unique, the Laplace operator would have to be bijective. Unfortunately, $\triangle:\ C^0(K) \to C_0(K)$ is neither injective nor surjective. Indeed, the coboundary operator d annihilates the constants, so $\triangle = -\partial\,\mathcal{C}\mathrm{d}$ has a nonvanishing kernel; and any boundary in $C_0(K)$ is annihilated by the constant functions ($\langle f, \partial Q\rangle = 0$ if $\mathrm{d}f = 0$), so $\triangle$ also has a nonvanishing cokernel.

If the network K is *connected,* as we shall suppose henceforth, the constant functions $f \in C^0(K)$ are the only functions annihilated by d. If all capacitances are nonzero and positive, the same holds true for $\triangle$. The kernel of $\triangle$ is then one-dimensional:

$$\dim\,\ker\triangle = 1, \tag{4.32}$$

and to improve the situation concerning the existence and uniqueness of solutions of Poisson's equation, we may proceed as follows. Let $Z^0(K)$ denote the one-dimensional vector space of constant functions $f \in C^0(K)$, and $B_0(K)$ the vector space of boundaries in $C_0(K)$

$$Z^0(K) = \{f \in C^0(K) \mid \mathrm{d}f = 0\}, \quad B_0(K) = \{c \in C_0(K) \mid \exists\gamma \in C_1(K):\ c = \partial\gamma\}. \tag{4.33}$$

Given $Z^0(K)$ we may pass to the quotient $C^0(K)/Z^0(K)$, the vector space of 0-cochains modulo the constants. [More formally, $C^0(K)/Z^0(K)$ is the vector space of equivalence classes $[f]$ given by the equivalence relation $f_1 \sim f_2 \Leftrightarrow f_1 - f_2 = \text{const.}$] The space $C^0(K)/Z^0(K)$ can be viewed as being dual to $B_0(K)$. Indeed, if $\partial Q \in B_0(K)$, then $\langle f, \partial Q\rangle = \langle f + \text{const}, \partial Q\rangle$, so the value of an equivalence class $[f] \in C^0(K)/Z^0(K)$ on ∂Q is well-defined, and $[f]$ lies in the dual of $B_0(K)$. Conversely, any linear function on $B_0(K)$ is easily seen to correspond to a unique element of $C^0(K)/Z^0(K)$. Moreover, the pairing between $C^0(K)/Z^0(K)$ and $B_0(K)$ by $([f], \partial Q) \mapsto \langle \mathrm{d}f, Q\rangle$ is non-degenerate. A particular consequence of all this is that the two spaces have the same dimension:

$$\dim B_0(K) = \dim C^0(K)/Z^0(K). \tag{4.34}$$

Consider now the Laplace operator $[\triangle]$ obtained from $\triangle : C^0(K) \to C_0(K)$ by restriction:

$$[\triangle] : C^0(K)/Z^0(K) \to B_0(K), \tag{4.35}$$

which is well-defined since $\triangle$ annihilates $Z^0(K)$. By construction, the restricted operator $[\triangle]$ has zero-dimensional kernel and hence is injective. The operator $[\triangle]$ is still symmetric – this follows by essentially the same argument as for $\triangle$. As a result, injectivity of $[\triangle]$ also implies surjectivity of this operator, so $[\triangle]$ is a bijection and possesses an inverse $[\triangle]^{-1}$. Thus the restricted Poisson equation has a unique solution:

$$-[\triangle][\Phi] = \rho \quad \Longleftrightarrow \quad [\Phi] = -[\triangle]^{-1}\rho\,. \tag{4.36}$$

Physically speaking, two aspects are of importance here. For one, Poisson's equation $-\triangle\Phi = \rho$ has a solution if and only if ρ is a boundary, i.e. if the total charge on the capacitive network vanishes. [The condition $\rho \in B_0(K)$ is equivalent to $\sum_i \rho_i = 0$.] Thus it was by no means accidental that ρ was constructed by internal charge transfer starting from a neutral network: this setup was to ensure global charge neutrality, $\rho \in B_0(K)$, which in turn guarantees the existence of a solution of the network equations.

The second aspect of importance is that the solution spaces of $-\triangle\Phi = \rho$ are always of the affine form $\Phi_0 + Z^0(K)$. Thus solutions come as one-parameter families (parameterized by $Z^0(K)$), and to make the solution unique we should mod out the constants $Z^0(K)$. An alternative recipe is to pick some node, say N, and remove the indeterminacy by setting $\Phi(N) \equiv 0$.

Weyl's method of orthogonal projection. After this discussion of the solution strategy via Poisson's equation, we briefly touch on a second strategy, which will be seen to have some similarity with the mesh current method for resistive networks:

(i) Pick any set of capacitor charges $Q_0 \in C_1(K)$ that satisfy $-\partial Q_0 = \rho$. (Such a set exists if $\rho \in B_0(K)$, and is usually easy to find.)

(ii) Make the most general ansatz for Q:

$$Q = Q_0 + Q_1\,,$$

where the 1-chain Q_1 is subject to the condition $\partial Q_1 = 0$ (to preserve $-\partial Q = -\partial Q_0 = \rho$).

(iii) To satisfy Kirchhoff's second rule $\mathrm{d}V = 0$, choose Q_1 in such a way that $V = \mathcal{C}^{-1}Q = \mathcal{C}^{-1}(Q_0 + Q_1)$ becomes a coboundary: $V = -\mathrm{d}\Phi$.

The last step – known as Weyl's method of orthogonal projection – deserves some explanation.

Let the vector space of closed chains (or cycles) in $C_1(K)$ be denoted by $Z_1(K)$, and the vector space of coboundaries in $C^1(K)$ by $B^1(K)$:

$$Z_1(K) = \{c \in C_1(K) \mid \partial c = 0\}, \quad B^1(K) = \{\omega \in C^1(K) \mid \exists f \in C^0(K) : \ \omega = \mathrm{d}f\}. \tag{4.37}$$

$B^1(K)$ is the *annihilator space* of $Z_1(K)$: a coboundary $\mathrm{d}f \in B^1(K)$ gives zero on every 1-cycle $c \in Z_1(K)$ and (by general principles of linear algebra) every annihilator of $Z_1(K)$ must be of that form. To establish the mathematical basis of Weyl's method, consider the inner product $(\cdot \,|\, \cdot)_{\mathcal{C}}$ on $C^1(K)$ defined by

$$(U \,|\, V)_{\mathcal{C}} := \big\langle U, \mathcal{C}V \big\rangle. \tag{4.38}$$

This is a positive symmetric bilinear form, by the symmetry and positivity of $\mathcal{C}$.

Given $B^1(K) \subset C^1(K)$, we can ask: what is the orthogonal complement of $B^1(K)$ with respect to this inner product? The orthogonal complement certainly contains the vector space $\mathcal{C}^{-1}Z_1(K)$, for the inner product of $\mathrm{d}f \in B^1(K)$ with $\mathcal{C}^{-1}\gamma \in \mathcal{C}^{-1}Z_1(K)$ always vanishes:

$$(\mathrm{d}f \,|\, \mathcal{C}^{-1}\gamma)_{\mathcal{C}} = \langle \mathrm{d}f, \gamma \rangle = \langle f, \partial\gamma \rangle = \langle f, 0 \rangle = 0. \tag{4.39}$$

Because $B^1(K)$ is the annihilator space of $Z_1(K)$, the dimensions of this pair must add up to that of $C^1(K)$, and since $\mathcal{C} : \ C^1(K) \to C_1(K)$ is an isomorphism, we infer that

$$\dim B^1(K) + \dim \mathcal{C}^{-1}Z_1(K) = \dim C^1(K). \tag{4.40}$$

Thus the dimensions match and we conclude that $\mathcal{C}^{-1}Z_1(K)$ is precisely the orthogonal complement of $B^1(K)$ with respect to the inner product $(\cdot \,|\, \cdot)_{\mathcal{C}}$:

$$C^1(K) = B^1(K) \oplus \mathcal{C}^{-1}Z_1(K). \tag{4.41}$$

Having established this orthogonal decomposition, we take π to be the orthogonal projector $\pi : C^1(K) \to B^1(K)$. Then, if $V_0 = \mathcal{C}^{-1}Q_0$ (and $-\partial Q_0 = \rho$), Weyl's method of solution is to put

$$V := \pi(V_0). \tag{4.42}$$

Indeed, $V \in B^1(K)$ by construction, so Kirchhoff's second rule ($\mathrm{d}V = 0$) is satisfied. Moreover, since

$$V - V_0 = -(1 - \pi)(V_0) \in \mathcal{C}^{-1}Z_1(K), \tag{4.43}$$

we have $V = V_0 + \mathcal{C}^{-1}Q_1 = \mathcal{C}^{-1}(Q_0 + Q_1)$ with $Q_1 \in Z_1(K)$. Thus $-\partial Q = -\partial(Q_0 + Q_1) = \rho$ still holds, and Weyl's answer $V = \pi(V_0)$ solves the equations of the capacitive network. Because the solution for the voltage (as opposed to the potential) is unique, it is the solution of the problem.

4.5.1 Math tutorial: quotient of vector spaces

Given a vector space V (over $\mathbb{K} = \mathbb{R}$, or $\mathbb{K} = \mathbb{C}$) let there be a subvector space $U \subset V$. By definition, the *quotient space* V/U is the set of equivalence classes $[v] = [v+U]$ (for $v \in V$) defined by the equivalence relation

$$v_1 \sim v_2 \iff v_1 - v_2 \in U. \tag{4.44}$$

Any element $v_0 \in V$ such that $[v_0] = [v]$ is called a *representative* of the equivalence class $[v] \in V/U$. The quotient space V/U is another vector space (still over $\mathbb{K}$). The operations of addition and scalar multiplication on V/U are defined by

$$[v] + [v'] = [v + v'], \qquad \lambda \cdot [v] = [\lambda \cdot v]. \tag{4.45}$$

One easily verifies that these definitions are independent of the choices of representative. The vector space dimension of V/U is

$$\dim V/U = \dim V - \dim U. \tag{4.46}$$

Example & exercise. Let $V = \mathbb{R}^3$ (with $\mathbb{K} = \mathbb{R}$), and let $U = \mathbb{R} \cdot u \subset V$ where $u \in \mathbb{R}^3$, $u \neq 0$. Then the quotient vector space V/U is two-dimensional. It can be visualized as the space of straight lines in $\mathbb{R}^3$ which are parallel to the line $U = \mathbb{R} \cdot u$. Such a space of straight lines has the structure of a vector space because straight lines can be added and multiplied by scalars. How?

4.5.2 Math tutorial: annihilator space

Let V be a vector space (still over $\mathbb{K} = \mathbb{R}$, or $\mathbb{K} = \mathbb{C}$), and let V^* be its dual vector space. By the definition of V^* we have the canonical pairing $V^* \otimes V \to \mathbb{K}$ by evaluation $f \otimes v \mapsto f(v)$. Now let $U \subset V$ be a subvector space. Then the *annihilator space* $U^\perp \subset V^*$ is defined to be

$$U^\perp = \{f \in V^* \mid \forall u \in U : \ f(u) = 0\}. \tag{4.47}$$

Thus $U^\perp$ consists of the linear functions $f \in V^*$ that annihilate every vector $u \in U$. The dimensions of U and $U^\perp$ are related by

$$\dim U^\perp = \dim V - \dim U. \tag{4.48}$$

Indeed, $U^\perp$ is canonically paired with V/U.

Remark & exercise. Given the ambient vector space V (and its dual V^*), the correspondence $U \leftrightarrow U^\perp$ is a one-to-one correspondence between subvector spaces. Thus when dealing with, say a linear form $f \in V^*$, we have the option of visualizing it via $(\mathbb{K} \cdot f)^\perp$ (actually, as an element in $V/(\mathbb{K} \cdot f)^\perp$ dual to f). This is the fundamental idea behind our scheme of "visualization" of forms which was introduced in the Lecture Course on Mathematical Methods (winter term 14/15). Can you fill in the details of how this goes?

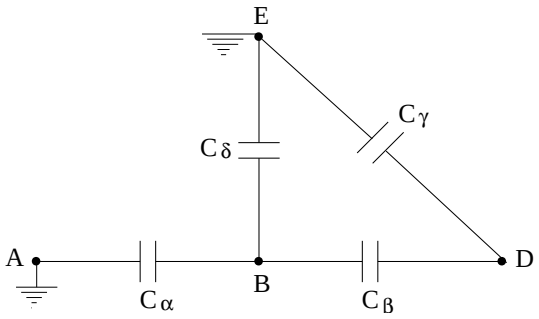
4.6 Boundary-value problems

In this section we continue our study of capacitive networks. What we have investigated so far were *isolated* networks. We assumed the distribution of node charges $\rho \in B_0(K)$ to be known and held fixed, and we viewed the electric potential Φ (and the derived capacitor voltages $V = -\mathrm{d}\Phi$ and capacitor charges $Q = \mathcal{C}V$) as the unknowns of the problem.

A different though related type of question arises if we connect some of the nodes to the external world, thereby enabling charge flow into or out of the capacitive network. Such a change in the physical setup results in a reshuffle between the knowns and unknowns of the mathematical formulation of the problem. There will still be nodes where the charge is given and the potential is to be determined. However, there will now be some other nodes – those connected to the external world – where what we prescribe is the potential, and what we seek is the charge.

Basic setting. As before, we are given some complex K with capacitances assigned to its 1-cells, and what we are looking for are solutions of Poisson's equation $-\triangle\Phi = \rho$ with network Laplacian $\triangle = -\partial\mathcal{C}\mathrm{d}$. The new feature is a split of the set of nodes into two subsets: *interior* (i) nodes, and *boundary* (b) nodes. Mathematically, this means that we are given vector space decompositions $C_0(K) = C_0^{\mathrm{i}}(K) \oplus C_0^{\mathrm{b}}(K)$ and $C^0(K) = C^{0,\mathrm{i}}(K) \oplus C^{0,\mathrm{b}}(K)$ (direct sums).

For example, in the network used for illustration in the previous section, we could take the interior nodes to be B and D, spanning the vector space $C_0^{\mathrm{i}}(K)$. The remaining two nodes, A and E, would then be boundary nodes, spanning the vector space $C_0^{\mathrm{b}}(K)$.



Problem posed. The general question we can ask in such a setting is this: given $\rho^{(\mathrm{i})} \in C_0^{\mathrm{i}}(K)$ (charges in the interior) and $\Phi^{(\mathrm{b})} \in C^{0,\mathrm{b}}(K)$ (prescribed potential on the boundary), what is the potential in the interior, $\Phi^{(\mathrm{i})} \in C^{0,\mathrm{i}}(K)$, and the charge distribution on the boundary, $\rho^{(\mathrm{b})} \in C_0^{\mathrm{b}}(K)$? In other words, given $\rho^{(\mathrm{i})}$ and $\Phi^{(\mathrm{b})}$, we are to find $\Phi^{(\mathrm{i})}$ and $\rho^{(\mathrm{b})}$ from the condition that $\Phi = \Phi^{(\mathrm{i})} + \Phi^{(\mathrm{b})}$ and $\rho = \rho^{(\mathrm{i})} + \rho^{(\mathrm{b})}$ satisfy Poisson's equation $-\triangle\Phi = \rho$. By linearity, this general problem decomposes into two complementary subproblems:

(i) In a *Poisson problem*, one specifies some charge distribution $\rho^{(\mathrm{i})}$ in the interior, while setting $\Phi^{(\mathrm{b})} \equiv 0$ on the boundary nodes.

(ii) In a *Dirichlet problem*, the prescribed data is the boundary potential $\Phi^{(\mathrm{b})}$, while the interior is free of charges: $\rho^{(\mathrm{i})} \equiv 0$.

In both cases the unknowns to be determined are the potential in the interior, $\Phi^{(\mathrm{i})}$, and the boundary charges, $\rho^{(\mathrm{b})}$.

A standard tool in the construction of solutions for both types of boundary-value problem is the *Green's function*, which we are now going to define. For that purpose, let $\triangle_{\mathrm{int}}$ be the restriction of the Laplacian $\triangle$ to the interior of the network:

$$-\triangle_{\mathrm{int}}: \quad C^{0,\mathrm{i}}(K) \xrightarrow{\mathrm{d}} C^1(K) \xrightarrow{\mathcal{C}} C_1(K) \xrightarrow{\partial} C_0^{\mathrm{i}}(K). \tag{4.49}$$

Note that this is not just the Laplacian of the network truncated to its interior part, as the intermediate spaces $C^1(K)$ and $C_1(K)$ are the full spaces of 1-cochains resp. 1-chains of K.

Green's function. If K is a complex with a decomposition $C_0(K) = C_0^{\mathrm{i}}(K) \oplus C_0^{\mathrm{b}}(K)$ into an interior and a boundary part, the Green's function G for K is a function on pairs of nodes

$$A, B \mapsto G(A,B) \in \mathbb{R}, \tag{4.50}$$

with the properties

(i) $(-\triangle_{\mathrm{int}} G)(\bullet, p) = 1 \cdot p$ for every interior node p (and $\triangle_{\mathrm{int}}$ acts on $G(\bullet, p)$ viewed as a function of its first argument with p being a parameter);

(ii) $G(A,B) = 0$ if at least one of A, B is a boundary node.

Existence and uniqueness. The Green's function G exists and is unique provided that the capacitance operator $\mathcal{C}: C^1(K) \to C_1(K)$ is positive and $\dim C_0^{\mathrm{b}}(K) \geq 1$ (i.e. there must be at least one boundary node). In the proof one first shows that $\triangle_{\mathrm{int}}: C^{0,\mathrm{i}}(K) \to C_0^{\mathrm{i}}(K)$ is an isomorphism under the specified conditions. The existence and uniqueness of the Green's function G then follows because G is essentially the inverse of $-\triangle_{\mathrm{int}}$. We add the trivial remark that G depends on the choice of boundary: if we change the decomposition $C_0(K) = C_0^{\mathrm{i}}(K) \oplus C_0^{\mathrm{b}}(K)$, then the Green's function changes.

Interpretation. From the definition of the Green's function we see that $\Phi \equiv G(\bullet, p)$ solves the Poisson problem with interior charge distribution $\rho^{(\mathrm{i})} = 1 \cdot p$. Thus, $G(A,p)$ is to be interpreted as *the value of the electric potential at the node A that results on placing a unit charge on the interior node p when all boundary nodes are grounded.*

Exercise. The Green's function is symmetric: $G(A,B) = G(B,A)$.

Example. Consider the capacitive network displayed at the beginning of the section. A quick computation shows its restricted Laplacian to be

$$-\triangle_{\mathrm{int}} = (C_\alpha + C_\beta + C_\delta) B \otimes B - C_\beta (B \otimes D + D \otimes B) + (C_\beta + C_\gamma) D \otimes D. \tag{4.51}$$

The tensor product notation simply means that $A \otimes B: C^0 \to C_0$ is the mapping $\Phi \mapsto A \cdot \Phi(B)$ (evaluation on the second factor). We see that $\triangle_{\mathrm{int}}$ is symmetric, as expected on general grounds.

By definition, the Green's function (restricted to the interior) is given by inverse of the matrix of $-\triangle_{\text{int}}$:

$$\begin{pmatrix} G(B,B) & G(B,D) \\ G(D,B) & G(D,D) \end{pmatrix} = \begin{pmatrix} C_\alpha + C_\beta + C_\delta & -C_\beta \\ -C_\beta & C_\beta + C_\gamma \end{pmatrix}^{-1} . \quad \square \tag{4.52}$$

Solution of Poisson problem. Since $G(\bullet, p)$ solves the Poisson problem with data $\rho^{(\mathrm{i})} = 1 \cdot p$, it is clear by linearity that the solution for the potential of the general Poisson problem with data $\rho^{(\mathrm{i})} = \sum \rho_p^{(\mathrm{i})} \cdot p$ is

$$\Phi = \sum_{\text{interior}} G(\bullet, p)\, \rho_p^{(\mathrm{i})} . \tag{4.53}$$

Poisson kernel. To find the corresponding boundary charge distribution $\rho^{(\mathrm{b})}$ we simply apply the (full) Laplacian to the potential and evaluate Poisson's equation on the boundary. Restricting the 0-chain $(\triangle G)(\bullet, p)$ to the boundary and expanding it as

$$(\triangle G)(\bullet, p)^{(\mathrm{b})} = \sum_{\text{boundary}} R \cdot K_R(p) \tag{4.54}$$

with real coefficients $K_R(p)$, we obtain

$$\rho_R^{(\mathrm{b})} = - \sum_{\text{interior}} K_R(p)\, \rho_p^{(\mathrm{i})} . \tag{4.55}$$

The coefficients $K_R(p)$ organize into what is called the *Poisson kernel* of the complex K. The Poisson kernel obviously has the following physical meaning: $K_R(p)$ is the *influence charge that accumulates on the boundary node R if a negative unit charge is placed at the interior node p and the boundary is grounded.*

Green reciprocity. We turn to describing the solution of the Dirichlet problem in terms of the Green's function and the Poisson kernel. To that end, we need to develop some further theoretical background. Recall that the network Laplacian is symmetric: $\triangle = \triangle^t$. Thus if Φ, Φ' (not a derivative) is any pair of potentials, we have

$$\big\langle \Phi', \triangle \Phi \big\rangle = \big\langle \Phi, \triangle \Phi' \big\rangle . \tag{4.56}$$

Now let $\rho = -\triangle\Phi$ and $\rho' = -\triangle\Phi'$ be any two solutions of Poisson's equation. Then the symmetry of $\triangle$ entails

$$\big\langle \Phi', \rho \big\rangle = \big\langle \Phi, \rho' \big\rangle . \tag{4.57}$$

This identity is referred to as *Green's reciprocity theorem.* (Note that no reference is made here to a choice of boundary for K.) With $\rho = -\partial Q$ and $\rho' = -\partial Q'$ a related statement is

$$\big\langle V', Q \big\rangle = \big\langle V, Q' \big\rangle . \tag{4.58}$$

Solution of Dirichlet problem. Green's reciprocity theorem opens a quick path to the solution of Dirichlet-type problems as follows. We decompose both sides of Eq. (4.57) into an interior and a boundary contribution:

$$\sum_{\text{interior}} \Phi'(A)\, \rho_A + \sum_{\text{boundary}} \Phi'(B)\, \rho_B = \sum_{\text{interior}} \Phi(A)\, \rho'_A + \sum_{\text{boundary}} \Phi(B)\, \rho'_B . \tag{4.59}$$

Next, let Φ, ρ be the solution of a Dirichlet problem and Φ', ρ' the solution of a Poisson problem. Both terms on the left-hand side of Eq. (4.59) then vanish; the first one does so by $\rho^{(\mathrm{i})} = 0$ (Dirichlet problem), and the second one by $\Phi'^{(\mathrm{b})} = 0$ (Poisson problem). If we now take the Poisson data to be $\rho'_A = \delta_{A,p}$, then by inserting the known solution $\rho'_R = -K_R(p)$ for the boundary charges of the Poisson problem, we deduce

$$\Phi^{(\mathrm{i})}(p) = \sum_{\text{boundary}} \Phi^{(\mathrm{b})}(R)\, K_R(p)\,. \tag{4.60}$$

Summary. We see that the Poisson kernel plays a double role in the theory. On one hand, in a Poisson problem it determines the boundary charges in terms of the interior charges. In a Dirichlet problem, on the other hand, it expresses the potential in the interior by the potential on the boundary. As we have shown, this double role is a consequence of Green's reciprocity theorem. In conclusion, we have:

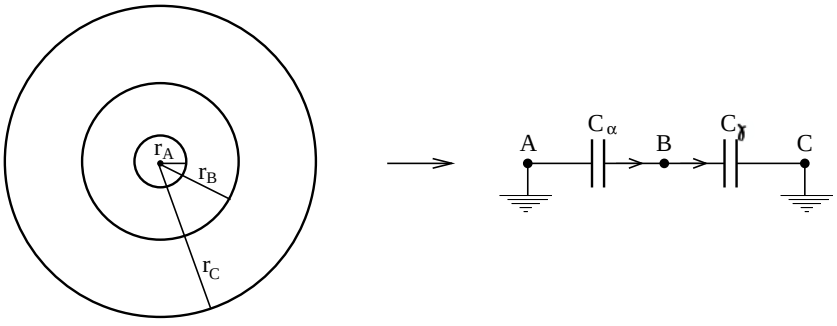
$$\text{Poisson problem:} \qquad \rho_R^{(\mathrm{b})} = -\sum_{\text{interior}} K_R(p)\, \rho_p^{(\mathrm{i})}\,.$$

$$\text{Dirichlet problem:} \qquad \Phi^{(\mathrm{i})}(p) = +\sum_{\text{boundary}} \Phi^{(\mathrm{b})}(R)\, K_R(p).$$

Of course, given the solution for the potential we can easily compute the boundary charges of a Dirichlet problem from $\rho = -\triangle\Phi$.

Example. To illustrate, consider an arrangement of three concentric conducting spheres (in relative isolation) with radii $r_A < r_B < r_C$. Place a charge ρ_B on the middle sphere and connect the inner and outer sphere to the ground (this is a Poisson-type problem). We then ask: what are the influence charges ρ_A and ρ_C on the two grounded spheres? This question has a quick answer using Green reciprocity. Before disclosing it, let us construct the answer from the full formalism.

We map the system of three concentric spheres to a capacitive network with three nodes A, B, C. The nodes of the network are arranged in sequence, with a 1-cell α connecting A and B, and a 1-cell γ connecting B and C. From elementary considerations, the capacitances to be assigned to these 1-cells are $C_\alpha = 4\pi\varepsilon_0(r_A^{-1} - r_B^{-1})^{-1}$ and $C_\gamma = 4\pi\varepsilon_0(r_B^{-1} - r_C^{-1})^{-1}$. A and C are boundary nodes, B is an interior node.



From rotational symmetry, it is clear that the capacitive network faithfully reflects the physics of the system of spheres. The answer to the question posed can then be found by applying the general machinery: we write down the interior Laplacian $\triangle_{\mathrm{int}}$, find the Green's function $G(B, B)$

by inversion of $-\triangle_{\rm int}$, and compute the Poisson kernel coefficients $K_A(B)$ and $K_C(B)$. The result for the influence charges is

$$\rho_A = -K_A(B)\,\rho_B = -\frac{C_\alpha}{C_\alpha + C_\gamma}\,\rho_B\,, \quad \rho_C = -K_C(B)\,\rho_B = -\frac{C_\gamma}{C_\alpha + C_\gamma}\,\rho_B\,. \tag{4.61}$$

Short cut. Here is how to get this answer more quickly. Instead of attacking the Poisson problem directly, we first solve the corresponding Dirichlet problem. Thus we prescribe potential values $\Phi(A)$ and $\Phi(C)$ and set $\rho_B = 0$. Since there is now zero charge between the inner and the outer sphere, the intermediate potential $\Phi(B)$ must divide the interval between $\Phi(A)$ and $\Phi(C)$ according to Coulomb's law:

$$\frac{\Phi(A) - \Phi(B)}{\Phi(B) - \Phi(C)} = \frac{r_A^{-1} - r_B^{-1}}{r_B^{-1} - r_C^{-1}} = \frac{C_\gamma}{C_\alpha}\,. \tag{4.62}$$

From the solution of this equation for $\Phi(B)$,

$$\Phi(B) = \Phi(A)\frac{C_\alpha}{C_\alpha + C_\gamma} + \Phi(C)\frac{C_\gamma}{C_\alpha + C_\gamma}\,, \tag{4.63}$$

we read off the Poisson kernel $K_A(B) = C_\alpha/(C_\alpha + C_\gamma)$ and $K_C(B) = C_\gamma/(C_\alpha + C_\gamma)$. Green reciprocity then immediately returns our answer (4.61) for the boundary charges ρ_A and ρ_C of the Poisson problem with interior charge ρ_B.