# Nonequilibrium Physics: Problem Sheet 6

www.thp.uni-koeln.de/∼as/noneq13.html

## 10. From the master equation to Fokker-Planck

The Fokker-Planck equation can be derived as an approximation to the master equation. We start from the master equation for a continuous state variable $x$ in continuous time:

$$\frac{\partial P(x,t)}{\partial t} = \int_{-\infty}^{\infty} W(x' \to x)P(x',t)dx' - \int_{-\infty}^{\infty} W(x \to x')P(x,t)dx' \,. \tag{1}$$

In the following we consider the transition probabilities as function of the "jump length" $r = x - x'$, i.e. $W(x' \to x) = \widetilde{W}(x', r)$.

**a)** Rewrite the master equation in terms of $\widetilde{W}(x', r)$ with integrals over $r$ instead of $x'$.

**b)** We now assume that $\widetilde{W}(x'-r, r)P(x'-r, t)$ is a sharply peaked function of $r$. Therefore it can be approximated by a Taylor expansion up to second order in the first argument of $\widetilde{W} \cdot P$. Use this to simplify the master equation derived in a).

**c)** Rewrite the result in b) in terms of the jump moments

$$a_n(x) = \int_{-\infty}^{\infty} r^n \widetilde{W}(x, r)dr$$

to obtain the (generalized) Fokker-Planck equation

$$\frac{\partial P(x,t)}{\partial t} = -\frac{\partial}{\partial x}\big(a_1(x)P(x,t)\big) + \frac{1}{2}\frac{\partial^2}{\partial x^2}\big(a_2(x)P(x,t)\big)\,.$$

**d)** Show that

$$\frac{\partial}{\partial t}\langle x\rangle = \langle a_1(x)\rangle \qquad \text{and} \qquad \frac{\partial}{\partial t}\langle x^2\rangle = 2\langle x a_1(x)\rangle + \langle a_2(x)\rangle\,.$$

*Hint:* You can assume that boundary terms (at $x = \pm\infty$) vanish.

**e)** Derive the result in d) directly from the master equation (1).

**f)** Determine the jump moments for the asymmetric random walk with jump lengths $r = \pm 1$ and hopping rates $p$ and $q$.

**g)** Generalize the result in c) to arbitrary order. This is called **Kramers-Moyal expansion**.

## 11. Stochastic Hamiltonian

The totally asymmetric exclusion process (TASEP) is a model of interacting random walks. A particle at site $j$ is allowed to move to site $j+1$ with rate $p$ only if the target site is empty. All other processes are not allowed.

**a)** Based on the definition of the tensor product, show that that the canonical basis vectors in the product space of two sites are indeed the tensor products of the corresponding vectors in each single-site space, i.e.

$$\begin{pmatrix}1\\0\end{pmatrix} \otimes \begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}1\\0\\0\\0\end{pmatrix}, \quad \begin{pmatrix}1\\0\end{pmatrix} \otimes \begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}0\\1\\0\\0\end{pmatrix}, \qquad \text{etc.}$$

**b)** Determine the local stochastic Hamiltonian $\hat{h}_{j,j+1}$ for this process.
Hint: Use the basis $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$.

**c)** Determine the Hamiltonian $\mathcal{H} = h_{12} + h_{23}$ corresponding to a system of 3 sites. For simplicity we assume free boundary conditions, i.e. sites 1 and 3 only interact with site 2. Convince yourself that $\mathcal{H}_{\text{free}}$ is a stochastic matrix.

## 12. BONUS: Quantum formalism for Fokker-Planck equation

**a)** Transform the Fokker-Planck equation

$$\frac{\partial P(x,t)}{\partial t} = -\frac{\partial}{\partial x}\left(K(x)P(x,t)\right) + D\frac{\partial^2}{\partial x^2}P(x,t) =: \mathcal{L}P(x,t)$$

(see problem 10c) with $a_1(x) = K(x)$ and $a_2(x) = 2D$ =const.) into the form

$$\frac{\partial P}{\partial t} = \mathcal{L}_{FP}P(x,t) \qquad \text{with the FP operator } \mathcal{L}_{FP} = D\frac{\partial^2}{\partial x^2} - V(x)\,.$$

Use the transformation $\mathcal{L} \to e^{U(x)/2D}\mathcal{L}e^{-U(x)/2D}$ with $K(x) = -\frac{\partial U}{\partial x}$ and find the relation between $U(x)$ and $V(x)$.

**b)** Transform the result of a) into a Schrödinger equation by introducing the imaginary time $t_S = -i\hbar t$. Find the relation between the diffusion coefficient $D$ and the particle mass $m$.

**c)** To which quantum mechanical problem does the choice $K(x) = \gamma x$ for the first jump moment correspond?