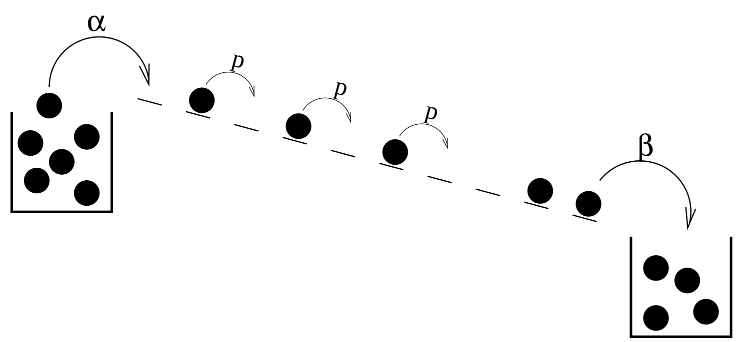
Nonequilibrium Physics: Problem Sheet 7

www.thp.uni-koeln.de/∼as/noneq13.html

13. Totally Asymmetric Exclusion Process (TASEP)

In Problem 11 we have derived the stochastic Hamiltonian for the TASEP with so-called "free boundary conditions". In the standard form of the model "open boundary conditions" are applied by coupling the system to reservoirs (see figure). If site 1 is empty, with rate α a particle will be inserted. If site L is occupied, the particle will be removed at rate β.



In the following we will study a system with just 2 sites ($L = 2$).

a) Determine the boundary Hamiltonians $\hat{h}_\alpha$ (acting only on site 1) and $\hat{h}_\beta$ (acting only on site 2) describing these processes. Determine the full boundary Hamiltonians $h_\alpha = \hat{h}_\alpha \otimes \mathbf{1}$ and $h_\beta = \mathbf{1} \otimes \hat{h}_\beta$ where $\mathbf{1}$ is the 2×2 unit matrix.

b) Determine the full Hamiltonian $\mathcal{H} = h_\alpha + h_{12} + h_\beta$ for the open system with 2 sites. Hint: h_{12} has been determined in Problem 11.

c) Show that one can choose $p = 1$ by rescaling $t \to \gamma t$ of the time variable. Determine γ. How are the boundary rates α and β rescaled?

d) Determine the eigenvalues of $\mathcal{H}$ for the special case $p = 1$ and $\alpha = \beta$.

e) Bonus: Determine the stationary state for the special case of c).

14. Cluster approximation for the TASEP

a) Use the Kolmogorov consistency equations to show that only one of the four cluster probabilities $P(n_j, n_{j+1})$ is independent, e.g. $P(1,0)$. How can the other three be determined once $P(1,0)$ is known?
Hint: We assume periodic boundary conditions so that in the stationary state the probabilities become independent of j due to translational invariance. Use the mean-field results for $P(0)$ and $P(1)$ to relate the cluster probabilities to the density ρ.

b) By applying the cluster approximation to the exact master equation for $P(1,0)$ one finds that it is determined by the equation

$$P(1,0) = \frac{P(1,0)P(0,0)}{1-\rho} + \frac{p^2 P^3(1,0)}{\rho(1-\rho)} + \frac{(1-p)P^2(1,0)}{1-\rho} + \frac{pP(1,0)P(1,1)}{\rho}$$

where $\rho = P(1) = N/L$ is the particle density.
Show that

$$P(1,0) = \frac{1}{2p}\left[1 - \sqrt{1-4p(1-\rho)\rho}\right] .$$

c) Compare the result for $P(1,0)$ with the mean-field result $P(1) \cdot P(0)$. Which one is larger? Interpret the result!

15. Matrix-product Ansatz for the TASEP

The MPA for the TASEP (with $p = 1$) leads to the following algebra for the matrices E and D and the vectors $\langle W|$ and $|V\rangle$:

$$\begin{aligned} DE &= D + E\,, \\ \alpha\langle W|E &= \langle W|\,, \\ \beta D|V\rangle &= |V\rangle\,. \end{aligned} \tag{1}$$

a) Show that this algebra has a one-dimensional representation (where E, D and $\langle W|$, $|V\rangle$ are real numbers) if $\alpha + \beta = 1$. Determine D and E in this case.

b) The density profile is defined by $\rho_j = \langle n_j \rangle$, i.e. the probability that site j is occupied. In the matrix-product formalism it is given by $\rho_j = \frac{1}{Z_L}\langle w|C^{j-1}DC^{L-j}|v\rangle$ since for the expectation value the occupations of sites $i \neq j$ are arbitrary (which gives a factor C), but for a non-zero contribution of a configuration site j has to be occupied (factor D). $Z_L = \langle w|C^L|v\rangle$ is a normalization.
Calculate the density profile for the 1d representation.

c) The current (between sites j and $j+1$) is defined by $J = \langle n_j(1-n_{j+1})\rangle$, or, in matrix-product form, $J = \frac{1}{Z_L}\langle w|C^{j-1}DEC^{L-j-1}|v\rangle$. Show that for arbitrary representations it is given by $J = \frac{Z_{L-1}}{Z_L}$.

d) Calculate the current for the 1d representation.

e) Show that

$$D = \begin{pmatrix} 1 & 1 & 0 & 0 & \cdots \\ 0 & 1 & 1 & 0 & \\ 0 & 0 & 1 & 1 & \\ 0 & 0 & 0 & 1 & \\ \vdots & & & & \ddots \end{pmatrix} \qquad E = \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots \\ 1 & 1 & 0 & 0 & \\ 0 & 1 & 1 & 0 & \\ 0 & 0 & 1 & 1 & \\ \vdots & & & & \ddots \end{pmatrix}$$

and

$$\langle W| = \kappa \left(1, a, a^2, \ldots\right) \,, \qquad |V\rangle = \kappa \begin{pmatrix} 1 \\ b \\ b^2 \\ \vdots \\ . \end{pmatrix} \,,$$

with $a = \frac{1-\alpha}{\alpha}$ and $b = \frac{1-\beta}{\beta}$ is a representation of the algebra (1) in the generic case. Which value of κ leads to $\langle W|V\rangle = 1$?

16. Doubly stochastic matrix

For a stochastic matrix $\mathbf{M}$ all sums of the elements in a column are zero: $\sum_j M_{ij} = 0$. A matrix is called *doubly stochastic* if also the row sums vanish: $\sum_j M_{ij} = 0$ for all i.

a) Show that the stationary state of a stochastic process which leads to a doubly stochastic Markov matrix is *uniform*, i.e. $P(n_1, \ldots, n_L) = const.$ independent of the state $(n_1, \ldots, n_L)$.

b) Show that in the case of particle conservation (where $P(n_1, \ldots, n_L) \neq 0$ only if $\sum_{j=1}^{L} n_j = N$) the stationary state factorizes, i.e.

$$P(n_1, \ldots, n_L) = P(n_1)P(n_2) \cdots P(n_L)\,.$$

This implies that mean-field theory is exact in this case.