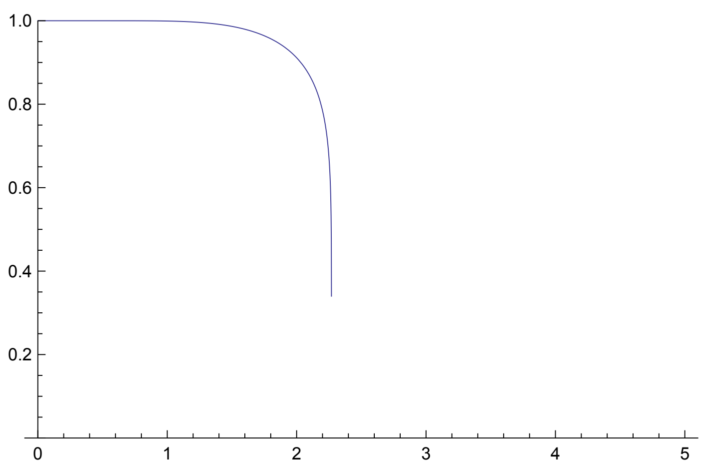
```
(*in this notebook we explore the exact solution of the Ising model in 2d*)
(*-beta f= lnZ*)

k[b_, J_] := Sinh[2 b J]^(-2)


lnZ[b_, J_] := Log[2] / 2 + 1 / (2 Pi) NIntegrate[Log[Cosh[2 b J]^2 +
      Sqrt[1 + k[b, J]^2 - 2 k[b, J] Cos[2 theta]] / k[b, J] ], {theta, 0, Pi}]

energy[b_, J_] := - J Coth[2 b J] (1 + 2 / Pi (2 Tanh[2 b J]^2 - 1) NIntegrate[
      1 / Sqrt[1 - 4 k[b, J] (1 + k[b, J])^(-2) Sin[theta]^2], {theta, 0, Pi / 2}])


Needs["NumericalCalculus`"]
```

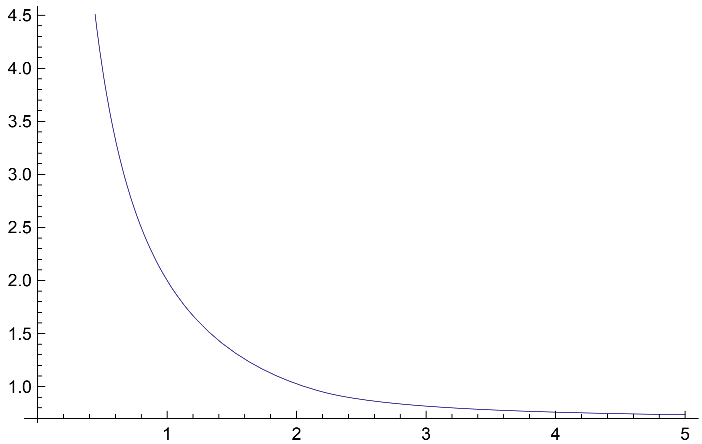
ND::shdw : Symbol ND appears in multiple contexts {NumericalCalculus` , Global` };
definitions in context NumericalCalculus` may shadow or be shadowed by other definitions . >>

```
specificheat[b_, J_] :=
 -b^2 ND[- J Coth[2 bp J] (1 + 2 / Pi (2 Tanh[2 bp J]^2 - 1) NIntegrate[1 / Sqrt[
         1 - 4 k[bp, J] (1 + k[bp, J])^(-2) Sin[theta]^2], {theta, 0, Pi / 2}]), bp, b]


m[b_, J_] := (1 - Sinh[2 b J]^-4)^(1 / 8)

Plot[m[1 / T, 1], {T, 0, 5}, PlotRange → {0, 1}]
```

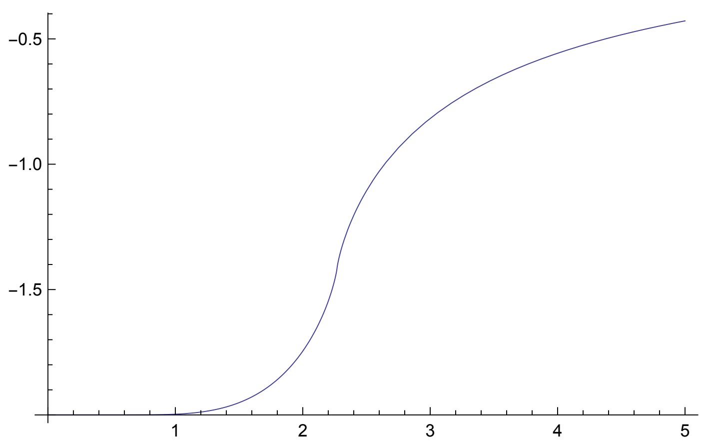



```
(*the spontaneous magnetisation at h=0 is nonzero below T_c=2.2...*)
```

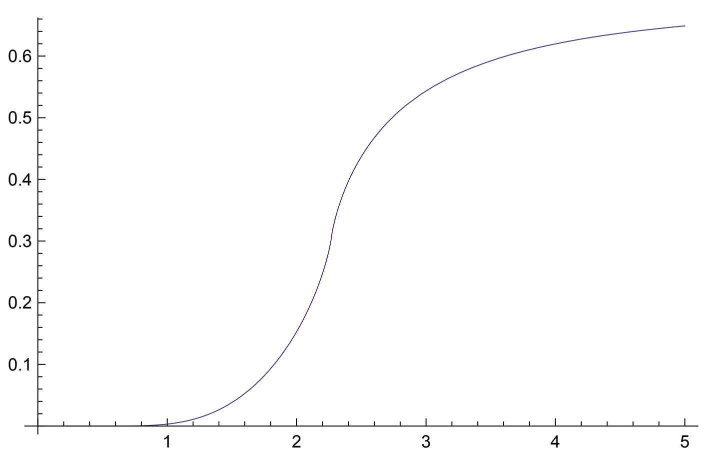
```
Plot[lnZ[1 / T, 1], {T, 0, 5}]
```




```
(*note the free energy does not have any
 peculiar feature at the phase transition at T_c*)

Plot[energy[1 / T, 1], {T, 0, 5}]
```




```
(*the energy has a large slope near T_c, as does the entropy below*)

Plot[lnZ[1 / T, 1] + energy[1 / T, 1] / T, {T, 0, 5}]
```



```
Plot[specificheat[1 / T, 1], {T, .1, 5}]
```

NIntegrate::inumr : The integrand $\frac{1}{\sqrt{1-\frac{4\,\text{Csch}[2\,\text{bp}]^2\,\text{Sin}[\text{theta}]^2}{(1+\text{Csch}[\ll 1 \gg]^2)^2}}}$ has evaluated to non-numerical values for all sampling points in the region with boundaries {{0, 1.5708}}. ≫

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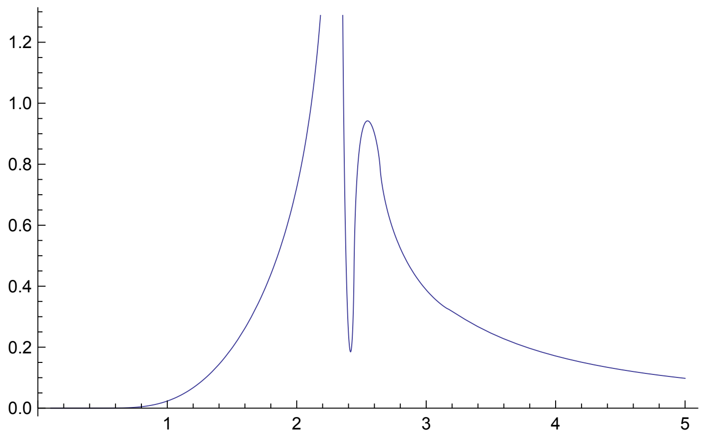
NIntegrate::inumr : The integrand $\frac{1}{\sqrt{1-\frac{4\,\text{Csch}[2\,\text{bp}]^2\,\text{Sin}[\text{theta}]^2}{(1+\text{Csch}[\ll 1 \gg]^2)^2}}}$ has evaluated to non-numerical values for all sampling points in the region with boundaries {{0, 1.5708}}. ≫

General::stop : Further output of NIntegrate::inumr will be suppressed during this calculation. ≫




```
(* you can find a nice collection of results at http://
 quantumtheory.physik.unibas.ch/people/bruder/Semesterprojekte2007/p1/
  index.html *)
```