Mathematische Methoden WS 2011/12

Blatt 10

Aufgabe 1: Trägheitstensor

$$T_R = \frac{1}{2} \sum_{l,m=1}^{3} J_{lm}\, \omega_l\, \omega_m \qquad \text{im Koordinatensystem } k$$

Kinetische Energie im Koordinatensystem k':

$$T_R' = \frac{1}{2} \sum_{lm} J_{lm}'\, \omega_l'\, \omega_m' = \ldots$$

Komponenten des Trägheitstensors in k': $\quad J_{lm}' = \sum_{no} d_{ln}\, d_{mo}\, J_{no}$

—— " — Vektors $\vec{\omega}$ in k': $\quad \omega_l' = \sum_p d_{lp}\, \omega_p$

(und $\omega_m' = \sum_q d_{mq}\, \omega_q$)

$$\ldots = \frac{1}{2} \sum_{lm} \sum_{nopq} J_{no}\, \omega_p\, \omega_q\, d_{ln}\, d_{mo}\, d_{lp}\, d_{mq} =$$

$$= \frac{1}{2} \sum_{nopq} J_{no}\, \omega_p\, \omega_q \underbrace{\Big[\sum_l d_{ln}\, d_{lp} \Big]}_{=\delta_{np}} \underbrace{\Big[\sum_m d_{mo}\, d_{mq} \Big]}_{=\delta_{oq}} = \frac{1}{2} \sum_{no} J_{no}\, \omega_n\, \omega_o = T_R \quad \checkmark$$

Aufgabe 2: Gradient

a) $\varphi_1(\vec{r}) = \cos(x)\cos(y)\cos(z)$

$$\vec{\nabla} \varphi_1(\vec{r}) = \begin{pmatrix} \frac{\partial \varphi_1}{\partial x} \\ \frac{\partial \varphi_1}{\partial y} \\ \frac{\partial \varphi_1}{\partial z} \end{pmatrix} = \begin{pmatrix} -\sin(x)\cos(y)\cos(z) \\ -\cos(x)\sin(y)\cos(z) \\ -\cos(x)\cos(y)\sin(z) \end{pmatrix}$$

b) $\varphi_2(\vec{r}) = r^n = \left(x^2+y^2+z^2\right)^{\frac{n}{2}} \qquad n \in \mathbb{N}$

$$\frac{\partial \varphi_2}{\partial x} = \frac{n}{2} \underbrace{\left(x^2+y^2+z^2\right)^{\left(\frac{n}{2}-1\right)}}_{=\left(x^2+y^2+z^2\right)^{\frac{1}{2}(n-2)}} 2x = n\, r^{n-2}\, x$$

$$\Rightarrow \vec{\nabla} \varphi_2(\vec{r}) = n\, r^{n-2}\, \vec{r}$$

Aufgabe 3 : Divergenz

$$\vec{E}(\vec{r}) = q \frac{\vec{r}}{r^3} \quad , \quad \text{zu zeigen:} \quad \vec{\nabla} \cdot \vec{E}(\vec{r}) = 0 \quad \text{für } \vec{r} \neq \vec{0}$$

$$\frac{\partial}{\partial x} E_x = \frac{\partial}{\partial x} q \frac{x}{r^3}$$

$$= q \frac{\partial}{\partial x} \left(x \left(x^2 + y^2 + z^2\right)^{-\frac{3}{2}} \right) = q \left(\frac{1}{r^3} - \frac{3x^2}{r^5} \right)$$

$$\Rightarrow \frac{\partial}{\partial x} E_x + \frac{\partial}{\partial y} E_y + \frac{\partial}{\partial z} E_z = q \left(\frac{3}{r^3} - \frac{3}{r^5} \underbrace{\left(x^2 + y^2 + z^2\right)}_{= r^2} \right) = 0 \quad \checkmark$$

Aufgabe 4 : Laplace - Operator

a) $\varphi_1(\vec{r}) = \sin(\vec{k} \cdot \vec{r}) \quad , \quad \vec{k} = (k_1, k_2, k_3)$

$$= \sin(k_1 x + k_2 y + k_3 z)$$

$$\frac{\partial \varphi_1}{\partial x} = k_1 \cos(\vec{k} \cdot \vec{r}) \quad , \quad \frac{\partial^2 \varphi_1}{\partial x^2} = -k_1^2 \sin(\vec{k} \cdot \vec{r})$$

$$\Rightarrow \Delta \varphi_1(\vec{r}) = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) \sin(\vec{k} \cdot \vec{r})$$

$$= -\left(k_1^2 + k_2^2 + k_3^2\right) \sin(\vec{k} \cdot \vec{r}) = -k^2 \sin(\vec{k} \cdot \vec{r})$$

mit $k = |\vec{k}|$

b) $\varphi_2(\vec{r}) = e^{-\alpha r^2} \quad , \quad \alpha \in \mathbb{R}$

$$\frac{\partial \varphi_2}{\partial x} = \frac{\partial}{\partial x} \exp\left(-\alpha \left(x^2 + y^2 + z^2\right)\right) = -2\alpha x \, e^{-\alpha r^2}$$

$$\frac{\partial^2 \varphi_2}{\partial x^2} = -2\alpha \frac{\partial}{\partial x} \left(x e^{-\alpha r^2} \right) = -2\alpha \left(e^{-\alpha r^2} - 2\alpha x^2 e^{-\alpha r^2} \right)$$

$$\Rightarrow \Delta \varphi_2(\vec{r}) = -2\alpha \, e^{-\alpha r^2} \left(3 - 2\alpha r^2\right)$$

Aufgabe 5: Rotation – allgemeine Rechenregeln

zu zeigen: $\vec{\nabla} \times (\varphi \vec{A}) = \varphi \, \vec{\nabla} \times \vec{A} - \vec{A} \times \vec{\nabla} \varphi$

$$\vec{\nabla} \times (\varphi \vec{A}) = \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{pmatrix} \times \begin{pmatrix} \varphi A_1 \\ \varphi A_2 \\ \varphi A_3 \end{pmatrix} = \begin{pmatrix} \frac{\partial}{\partial y}(\varphi A_3) - \frac{\partial}{\partial z}(\varphi A_2) \\ \frac{\partial}{\partial z}(\varphi A_1) - \frac{\partial}{\partial x}(\varphi A_3) \\ \frac{\partial}{\partial x}(\varphi A_2) - \frac{\partial}{\partial y}(\varphi A_1) \end{pmatrix}$$

erste Komponente:

$$\left(\frac{\partial \varphi}{\partial y}\right) A_3 + \varphi \frac{\partial A_3}{\partial y} - \left(\frac{\partial \varphi}{\partial z}\right) A_2 - \varphi \left(\frac{\partial A_2}{\partial z}\right) =$$

$$= \underbrace{\varphi \left(\frac{\partial A_3}{\partial y} - \frac{\partial A_2}{\partial z} \right)}_{\text{1. Komponente von } \vec{\nabla} \times \vec{A}} + \underbrace{\left(\frac{\partial \varphi}{\partial y}\right) A_3 - \left(\frac{\partial \varphi}{\partial z}\right) A_2}_{\text{1. Komponente von } \vec{\nabla}\varphi \times \vec{A} = -\vec{A} \times \vec{\nabla}\varphi}$$

die anderen Komponenten analog ✓