Mathematische Methoden WS 2011/12

Blatt 13

Aufgabe 1: Fourier-Reihe: allgemeine Eigenschaften

a) $\frac{1}{\pi}\int_0^{2\pi} \sin(mx)\sin(nx)\,dx =$ $\sin\alpha = \frac{1}{2i}\left(e^{i\alpha} - e^{-i\alpha}\right)$

$$= \frac{1}{\pi}\frac{1}{(2i)^2}\int_0^{2\pi}\left(e^{imx} - e^{-imx}\right)\left(e^{inx} - e^{-inx}\right)dx =$$

$$= -\frac{1}{4\pi}\int_0^{2\pi}\left(e^{i(m+n)x} - e^{i(m-n)x} - e^{i(-m+n)x} + e^{i(-m-n)x}\right)dx = \dots$$

es gilt: $\int_0^{2\pi} e^{i\tilde{m}x}\,dx = 2\pi\,\delta_{\tilde{m},0}$

$n = 1,2,3,\dots$, $m = 1,2,3,\dots$ $\Rightarrow$ $(m+n)$ auf jeden Fall $\neq 0$

$$\dots = -\frac{1}{4\pi}\Big(-2\pi\underbrace{\delta_{m-n,0}}_{=\delta_{m,n}} - 2\pi\underbrace{\delta_{-m+n,0}}_{=\delta_{mn}}\Big) = \delta_{mn} \quad \checkmark$$

b) i) $a_n = 0$ für $n = 0,1,2,\dots$

$\Rightarrow$ Fourier-Reihe: $f(x) = \sum_{n=1}^{\infty} b_n \sin(nx)$

$$\rightarrow f(-x) = \sum_{n=1}^{\infty} b_n \underbrace{\sin(-nx)}_{=-\sin(nx)} = -\sum_{n=1}^{\infty} b_n \sin(nx) = -f(x)$$

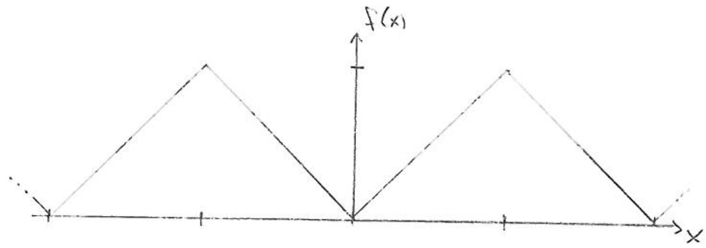
ii) $b_n = 0$ für $n = 1,2,3,\dots$

$\Rightarrow$ Fourier-Reihe: $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx)$

$$\rightarrow f(-x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \underbrace{\cos(-nx)}_{=\cos(nx)} = f(x)$$

Aufgabe 2: Darstellung einer gegebenen Funktion als Fourier-Reihe

$$f(x) = \begin{cases} x & : \quad 0 < x < \pi \\ -x & : \quad -\pi < x < 0 \end{cases}$$



und $f(x) = f(x+2\pi)$

$$\longrightarrow \quad a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos(nx)\,dx = \qquad n = 0,1,2,\dots$$

$$= \frac{1}{\pi}\int_0^{\pi} x\cos(nx)\,dx + \underbrace{\frac{1}{\pi}\int_{\pi}^{2\pi}(2\pi - x)\cos(nx)\,dx}_{} =$$

Substitution: $y = 2\pi - x$

$$\longrightarrow \int_{\pi}^{0} y \underbrace{\cos(n2\pi - ny)}_{=\cos(ny)}(-dy) = \int_0^{\pi} y\cos(ny)\,dy$$

$$= \frac{2}{\pi}\int_0^{\pi} x\cos(nx)\,dx$$

Fallunterscheidung: $\cdot\ n = 0 \longrightarrow a_0 = \frac{2}{\pi}\int_0^{\pi} x\,dx = \frac{1}{\pi}\left[x^2\right]_0^{\pi} = \pi$

$\cdot\ n \neq 0 \longrightarrow$ partielle Integration

$$\int_0^{\pi} x\cos(nx)\,dx = \underbrace{\frac{1}{n}\left[x\sin(nx)\right]_0^{\pi}}_{=0} - \underbrace{\frac{1}{n}\int_0^{\pi}\sin(nx)\,dx}_{= -\frac{1}{n}\left(\cos(n\pi) - 1\right)} =$$

$$= \frac{1}{n^2}\left((-1)^n - 1\right)$$

$$\Longrightarrow \quad a_n = \begin{cases} 0 & : \quad n \text{ gerade} \\ -\frac{4}{\pi n^2} & : \quad n \text{ ungerade } (n \neq 0) \end{cases}$$

$$\longrightarrow \quad b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin(nx)\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi} \underbrace{f(x)\sin(nx)}_{}\,dx$$

es gilt: $f(x) = f(-x)$, $\sin(nx) = -\sin(n(-x))$

$\rightarrow$ Integrand $g(x) = -g(-x)$ und damit alle $b_n = 0$

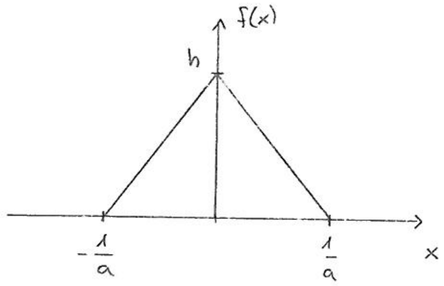
die Fourier-Reihe hat damit die Form:

$$f(x) = \frac{\pi}{2} - \frac{4}{\pi} \sum_{\substack{n=1 \\ n \text{ ungerade}}}^{\infty} \frac{1}{n^2} \cos(nx) =$$

$$= \frac{\pi}{2} - \frac{4}{\pi} \left(\cos x + \frac{1}{9} \cos(3x) + \frac{1}{25} \cos(5x) + \dots \right)$$

Aufgabe 3: Fourier-Transformation

$$f(x) = \begin{cases} h(1 - a|x|) & : |x| < \frac{1}{a} \\ 0 & : |x| > \frac{1}{a} \end{cases}$$



Fourier-Transformierte: $g(\alpha) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{i\alpha x} dx =$

$$= \underbrace{\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{0} f(x) e^{i\alpha x} dx}_{} + \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} f(x) e^{i\alpha x} dx = \dots$$

Subtitution: $y = -x \rightarrow \int_{\infty}^{0} f(-y) e^{-i\alpha y} (-dy) = \int_{0}^{\infty} f(y) e^{-i\alpha y} dy$

$$\dots = \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} f(x) \underbrace{\left(e^{i\alpha x} + e^{-i\alpha x} \right)}_{= 2\cos(\alpha x)} dx = \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} f(x) \cos(\alpha x)\, dx =$$

$$= \sqrt{\frac{2}{\pi}}\, h \underbrace{\int_{0}^{1/a} (1 - ax) \cos(\alpha x)\, dx}_{} = -\sqrt{\frac{2}{\pi}} \frac{ha}{\alpha^2} \left(\cos\left(\frac{\alpha}{a}\right) - 1 \right)$$

$$= \underbrace{\int_{0}^{1/a} \cos(\alpha x)\, dx}_{= \frac{1}{\alpha} \sin\left(\frac{\alpha}{a}\right)} - a \underbrace{\int_{0}^{1/a} x \cos(\alpha x)\, dx}_{= \frac{1}{\alpha a} \sin\left(\frac{\alpha}{a}\right) + \frac{1}{\alpha^2} \left(\cos\left(\frac{\alpha}{a}\right) - 1 \right)}$$