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# Entangled Phases of Matter, WS 2020/21

**Exercise sheet 1**

This exercise will be discussed on 19.11.2020

## 1. Landau states of 2D electron gas

Consider two-dimensional gas of free fermions in the homogeneous magnetic field $B\hat{\mathbf{z}}$. The single particle states of this problem are described by the Hamiltonian

$$H = -\frac{1}{2m}\sum_{j=x,y}\left(\partial_j - \tfrac{ie}{c}A_j\right)^2, \qquad (A_x, A_y) = \frac{B}{2}(-y, x)$$

where the so-called radially symmetric gauge for the vector potential is chosen. To find the eigenstates and energy levels of this problem it is advantageous to introduce complex coordinates $z = x + iy$ and $\bar{z} = x - iy$, so that $\partial_z = \frac{1}{2}(\partial_x - i\partial_y)$ and $\partial_{\bar{z}} = \frac{1}{2}(\partial_x + i\partial_y)$ (check it!).

**a)** Show that in terms of 'complex' coordinates $(z, \bar{z})$ the Hamiltonian can be equivalently rewritten as

$$H = -\frac{1}{m}\left(D_z D_{\bar{z}} + D_{\bar{z}} D_z\right),$$

where 'complex' velocity operators become

$$D_z = \partial_z - \frac{ie}{c}A_z, \qquad D_{\bar{z}} = \partial_z - \frac{ie}{c}A_{\bar{z}},$$

and 'complex' vector potentials are $A_z = \frac{1}{2}(A_x - iA_y) = \frac{1}{4i}B\bar{z}$ and $A_{\bar{z}} = A_z^*$.

The Hamiltonian $H$ can be simplified via non-unitary transformation $H = e^{-|z|^2/4l_B^2}\tilde{H}e^{|z|^2/4l_B^2}$, where $l_B = (c/eB)^{1/2}$ is a magnetic length and $\hbar = 1$ for simplicity.

**b)** To find $\tilde{H}$ show that rotated velocity operators are transformed into

$$\tilde{D}_z = \partial_z - \frac{\bar{z}}{2l_B^2}, \qquad \tilde{D}_{\bar{z}} = \partial_{\bar{z}}.$$

**c)** With the above result verify that

$$\tilde{H} = -\frac{2}{m}\left(\partial_z - \frac{\bar{z}}{2l_B^2}\right)\partial_{\bar{z}} + \frac{1}{2}\hbar\omega_c,$$

where $\omega_c = eB/mc$ is the cyclotron frequency.

The latter relation demonstrates that the multiply degenerate Landau states with the ground state energy $\epsilon_0 = \frac{1}{2}\hbar\omega_c$ are of the form $\psi_0(x,y) = e^{-|z|^2/4l_B^2}f(z)$, where $f(z)$ is any polynomial in $z$, for instance $f(z) = z^m$.

**d)** Check that other excited eigenstates and energies have the form

$$\psi_n(x,y) = e^{-|z|^2/4l_B^2}(\tilde{D}_z)^n f(z), \qquad \epsilon_n = (n + \tfrac{1}{2})\hbar\omega_c, \qquad n = 1, 2, 3, ...$$

Here the operator $\tilde{D}_z$ acts on the function $f(z)$.

To begin with you may show the validity of the above relation in the case $n = 1$ and then prove it by induction for any other $n \geq 2$.

## 2. The non-Abelian Berry connection

For the $N$-dimensional orthogonal basis of degenerate ground states $|n_a(\lambda)\rangle$ with $a = 1, \ldots, N$ and the related Hamiltonian $H(\lambda)$, which parametrically depends on the set of parameters $\lambda = (\lambda_1, \lambda_2, ...)$, the non-Abelian Berry connection $A^i$ is defined as

$$(A^i)_{ab} = i\langle n_a(\lambda)|\partial_{\lambda_i}|n_b(\lambda)\rangle$$

At given $i$ it is the $\lambda$-dependent matrix $N \times N$ which lives in the Lie algebra $u(N)$ of the unitary group $U(N)$. Suppose that the basis transformation is performed

$$|n_a(\lambda)\rangle = \sum_b |n'_b(\lambda)\rangle G_{ba}(\lambda),$$

with $G(\lambda) \in U(N)$ being a unitary $\lambda$-dependent matrix, i.e. $\sum_c G_{ac}(\lambda)G^\dagger_{cb}(\lambda) = \delta_{ab}$. Verify that the Berry connection $A'^i$ evaluated in the new basis is related to $A^i$ via the non-Abelian gauge transformation

$$A'^i = GA^iG^\dagger - i(\partial_{\lambda_i}G)G^\dagger.$$

This law of transformation also emerges in all QFTs (quantum field theories) which incorporate the non-Abelian gauge fields: QCD in d=3+1 and CS (Chern-Simons) actions in d=2+1 are one of the most studied examples.