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Exercise sheet 1

Solution

1. Landau states of 2D electron gas

Consider two-dimensional gas of free fermions in the homogeneous magnetic field $B\hat{\mathbf{z}}$. The single particle states of this problem are described by the Hamiltonian

$$H = -\frac{1}{2m}\sum_{j=x,y}\left(\partial_j - \tfrac{ie}{c}A_j\right)^2, \qquad (A_x, A_y) = \frac{B}{2}(-y, x)$$

where the so-called radially symmetric gauge for the vector potential is chosen. To find the eigenstates and energy levels of this problem it is advantageous to introduce complex coordinates $z = x + iy$ and $\bar{z} = x - iy$, so that $\partial_z = \frac{1}{2}(\partial_x - i\partial_y)$ and $\partial_{\bar{z}} = \frac{1}{2}(\partial_x + i\partial_y)$ (check it!).

a) One needs to show that $H = -\frac{1}{m}\{D_z, D_{\bar{z}}\}$, where $\{A, B\} = AB + BA$ is anticommutator, while

$$D_z = \partial_z - \frac{ie}{c}A_z, \qquad D_{\bar{z}} = \partial_z - \frac{ie}{c}A_{\bar{z}},$$

with complex vector potentials

$$A_z = \tfrac{1}{2}(A_x - iA_y) = \tfrac{1}{4i}B\bar{z}, \qquad A_{\bar{z}} = \tfrac{1}{2}(A_x + iA_y) = -\tfrac{1}{4i}Bz.$$

The Hamiltonian in original variables reads

$$H = -\frac{1}{2m}\left(\partial_x^2 + \partial_y^2 - \frac{ie}{c}\{\partial_x, A_x\} - \frac{ie}{c}\{\partial_x, A_x\} - \frac{e^2}{c^2}(A_x^2 + A_y^2)\right).$$

Using $\partial_x = (\partial_z + \partial_{\bar{z}})$ and $\partial_x = i(\partial_z + \partial_{\bar{z}})$, different terms of H can be rewritten as

$$\partial_x^2 + \partial_y^2 = (\partial_z + \partial_{\bar{z}})^2 - (\partial_z - \partial_{\bar{z}})^2 = 4\partial_z\partial_{\bar{z}} = 2\{\partial_z, \partial_{\bar{z}}\} \qquad (1)$$

$$\{\partial_x, A_x\} = \{\partial_z + \partial_{\bar{z}}, A_z + A_{\bar{z}}\} = \{\partial_z, A_z\} + \{\partial_z, A_{\bar{z}}\} + \{\partial_{\bar{z}}, A_{\bar{z}}\} + \{\partial_{\bar{z}}, A_{\bar{z}}\}.$$

$$\{\partial_y, A_y\} = i^2\{\partial_z - \partial_{\bar{z}}, A_z - A_{\bar{z}}\} = -\{\partial_z, A_z\} + \{\partial_z, A_{\bar{z}}\} - \{\partial_{\bar{z}}, A_{\bar{z}}\} + \{\partial_{\bar{z}}, A_{\bar{z}}\}.$$

Hence

$$-\frac{ie}{c}\{\partial_x, A_x\} - \frac{ie}{c}\{\partial_y, A_y\} = -\frac{2ie}{c}\{\partial_z, A_{\bar{z}}\} - \frac{2ie}{c}\{\partial_{\bar{z}}, A_z\}. \qquad (2)$$

and

$$-\frac{e^2}{c^2}(A_x^2 + A_y^2) = \frac{e^2}{c^2}\left((A_z + A_{\bar{z}})^2 + i^2(A_z - A_{\bar{z}})^2\right) = \frac{4e^2}{c^2}A_zA_{\bar{z}} = \frac{2e^2}{c^2}\{A_z, A_{\bar{z}}\} \qquad (3)$$

Adding (1), (2) and (3) we obtain the required result

$$H = -\frac{1}{m}\{\partial_z - \frac{ie}{c}A_z, \partial_{\bar{z}} - \frac{ie}{c}A_{\bar{z}}\} = -\frac{1}{m}\{D_z, D_{\bar{z}}\}.$$

b) The transformed gradient operators read (using that $|z|^2 = z\bar{z}$ and implicitly acting on any function from the right)

$$\tilde{\partial}_z = e^{|z|^2/4l_B^2}\partial_z e^{-|z|^2/4l_B^2} = \partial_z - \frac{\bar{z}}{4l_B^2}, \qquad \tilde{\partial}_{\bar{z}} = e^{|z|^2/4l_B^2}\partial_{\bar{z}} e^{-|z|^2/4l_B^2} = \partial_{\bar{z}} + \frac{z}{4l_B^2}.$$

from where required expressions for velocity operators

$$\tilde{D}_z = \partial_z - \frac{\bar{z}}{2l_B^2}, \qquad \tilde{D}_{\bar{z}} = \partial_{\bar{z}}.$$

follow.

c) With the above result we get

$$\begin{aligned}\tilde{H} &= -\frac{1}{m}(\tilde{D}_z\partial_{\bar{z}} + \partial_{\bar{z}}\tilde{D}_z) = -\frac{2}{m}\tilde{D}_z\partial_{\bar{z}} - \frac{1}{m}[\partial_{\bar{z}}, \tilde{D}_z] = -\frac{2}{m}\tilde{D}_z\partial_{\bar{z}} + \frac{1}{2ml_B^2} \\ &= -\frac{2}{m}(\partial_z - \frac{\bar{z}}{2l_B^2})\partial_{\bar{z}} + \frac{1}{2}\omega_c, \qquad \hbar = 1.\end{aligned}$$

d) Let's check the $n = 1$ case. As it has been used just above, we notice that

$$[\partial_{\bar{z}}, \tilde{D}_z] = -\frac{1}{2l_B^2}[\partial_{\bar{z}}, \bar{z}] = -\frac{1}{2l_B^2}. \tag{4}$$

By denoting $\tilde{\psi}_1(x, y) = \tilde{D}f(z)$ and omitting a constant term $\hbar\omega_c/2$ we check that

$$(\tilde{H} - \tfrac{1}{2}\omega_c)\tilde{\psi}_1 = -\frac{2}{m}\tilde{D}_z\partial_{\bar{z}}\tilde{D}_z f(z) = -\frac{2}{m}\tilde{D}_z[\partial_{\bar{z}}, \tilde{D}_z]f(z) = \frac{1}{ml_B^2}\tilde{D}_z f(z) = \hbar\omega_c\tilde{\psi}_1.$$

It was important here that $f(z)$ is a holomorphic function of z, in other words it doesn't depend on $\bar{z}$. Hence $\psi_1(x, y)$ is the eigenfunction of H with the same eigenvalue $\frac{3}{2}\hbar\omega_c$.

Let's further study the case $n \geq 2$. We assume that by induction

$$[\partial_{\bar{z}}, \tilde{D}_z^{n-1}] = -\frac{n-1}{2l_B^2}\tilde{D}_z^{n-2} \tag{5}$$

holds, where the case $n = 2$ reduces to Eq. (4). Then by using a property of commutators $[A, BC] = B[A, C] + [A, B]C$, one derives

$$[\partial_{\bar{z}}, \tilde{D}_z^n] = [\partial_{\bar{z}}, \tilde{D}_z^{n-1}]\tilde{D}_z + \tilde{D}_z^{n-1}[\partial_{\bar{z}}, \tilde{D}_z] = -\frac{n-1}{2l_B^2}\tilde{D}_z^{n-1} - \frac{1}{2l_B^2}\tilde{D}_z^{n-1} = -\frac{n}{2l_B^2}\tilde{D}_z^{n-1},$$

and hence (5) is proved with $n \to n + 1$. Therefore if $\tilde{\psi}_n = \tilde{D}_z^n f(z)$, then

$$(\tilde{H} - \tfrac{1}{2}\omega_c)\tilde{\psi}_n = -\frac{2}{m}\tilde{D}_z^n\partial_{\bar{z}}\tilde{D}_z^n f(z) = -\frac{2}{m}\tilde{D}_z^n[\partial_{\bar{z}}, \tilde{D}_z^n]f(z) = \frac{n}{ml_B^2}\tilde{D}_z^n f(z) = n\hbar\omega_c\tilde{\psi}_n.$$

In this way one sees that $\psi_n(x, y) = e^{-|z|^2/(4l_B^2)}\tilde{\psi}_n(x, y)$ is the eigenfunction of H with the same eigenvalue $(n + \frac{1}{2})\hbar\omega_c$.

2. The non-Abelian Berry connection

Provided

$$|n_a(\lambda)\rangle = \sum_b |n'_b(\lambda)\rangle G_{ba}(\lambda),$$

then the new basis set $|n'_a(\lambda)\rangle$ is expressed via the original one with the help of an inverse unitary transformation,

$$|n'_a(\lambda)\rangle = \sum_b |n_b(\lambda)\rangle G^\dagger_{ba}(\lambda)$$

Therefore for the Berry connection in the new basis we have

$$\begin{aligned}
(A'^i)_{ab} &= i\langle n'_a(\lambda)|\partial_{\lambda_i}|n'_b(\lambda)\rangle = i\sum_{cd}(G^\dagger_{ca})^*(\lambda)\langle n_c(\lambda)|\partial_{\lambda_i}\left(|n_d(\lambda)\rangle G^\dagger_{db}(\lambda)\right) \\
&= i\sum_{cd} G_{ac}(\lambda)\left(\langle n_c(\lambda)|\partial_{\lambda_i}|n_d(\lambda)\rangle\right) G^\dagger_{db}(\lambda) + i\sum_c G_{ac}(\lambda)\partial_{\lambda_i}G^\dagger_{cb}(\lambda) \\
&= \sum_{cd} G_{ac}(\lambda)(A^i)_{cd}G^\dagger_{db}(\lambda) + i\sum_c G_{ac}(\lambda)\partial_{\lambda_i}G^\dagger_{cb}(\lambda).
\end{aligned}$$

In the matrix form the above expression reads

$$A'^i = GA^iG^\dagger - i(\partial_{\lambda_i}G)G^\dagger.$$

which is a required law of the non-Abelian gauge transformation for the Berry connection, and it was used that $(\partial_{\lambda_i}G)G^\dagger + G\partial_{\lambda_i}G^\dagger = 0$ since $GG^\dagger = 1$.