Institute of Theoretical Physics
University of Cologne

Dr. Carolin Wille
Dr. Dmitry Bagrets

Entangled Phases of Matter, WS 2020/21

Exercise sheet 3

This exercise will be discussed on 17.12.2020

1. Braiding of anyons

In the lecture we've considered the braiding statistics of e and m particles in Kitaev's toric code. In this exercise, we calculate the braiding statistics of e and $\epsilon = (e, m)$.

1. Write down the space-time diagram that corresponds to the definition of the entry $S_{e\epsilon}$ of the S-matrix.
2. Write down the corresponding expectation value.
3. Parameterize the braid by string and ribbon operators along suitable contours.
4. Evaluate the expectation value step-by-step by using the properties of the string operators.

2. Commuting projector Hamiltonians

In the lecture we discussed how the ground state space $\mathcal{G}$ of a commuting projector Hamiltonian

$$H = \sum_i P_i \ , \qquad [P_i, P_j] = 0 \ , \qquad P_i^2 = P_i \ , \tag{1}$$

can be obtained from sequentially projecting to the orthogonal complements

$$P_i^{\perp} = (1 - P_i) \tag{2}$$

of all individual projectors

$$P = \prod_i P_i^{\perp} \ , \qquad \mathrm{Im}[P] = \mathcal{G} \ , \qquad H\mathcal{G} = 0 \ . \tag{3}$$

As a warm up, convince yourself, that P_i and $P_i^{\perp}$ are indeed orthogonal and that $P_i P = 0$.

Next, consider a tiny ferromagnet, i.e., a system of three spins on the vertices of a triangle with the following Hamiltonian

$$H_{\mathrm{F}} = -\sum_i z_i z_{i+1} \ . \tag{4}$$

where z_i is the Pauli matrix σ_3 on a site i.

1. Write this Hamiltonian as a sum of commuting projectors as in Eq. (1) by shifting the zero-energy and multiplication by a scalar and convince yourself that this does not change the eigenstates of the Hamiltonian.
2. Define the orthogonal complements of the individual projectors and 'calculate' their images $\mathcal{G}_i = \mathrm{Im}[P_i^{\perp}]$.
3. Convince yourself that the image of $P = \prod_i P_i^{\perp}$ is the intersection of the individual images, i.e., $\mathrm{Im}[P] = \mathcal{G} = \cap_i \mathcal{G}_i$ and calculate it.

In systems with frustration, the recipe to construct the ground state space as Im$[P]$ fails. As an illustration consider the example for anti-ferromagnetic interactions, i.e.,

$$H_{\text{AF}} = \sum_i z_i z_{i+1} \tag{5}$$

and repeat the three steps above. What do you find for $\mathcal{G}$ and why is this result expected? How does the actual ground state space look like? What it the ground state energy of the commuting projector Hamiltonian?

3. Topological ground state degeneracy and Euler characteristic*

In the lecture we claimed that the ground state degeneracy of the toric code depends on the topology of the manifold on which it is defined as

$$\dim \mathcal{G} = 2^{2g} \ , \tag{6}$$

where g is the genus of the manifold, i.e, the number of (independent) holes. In this exercise we will sketch the proof of this formula. It is divided into two parts. First, we will repeat the counting of independent projectors to express $\dim \mathcal{G}$ as a function of the number of qubits N and the number of independent projectors $\#P_{\text{ind}}$ as

$$\dim \mathcal{G} = 2^{N-\#P_{\text{ind}}} \ . \tag{7}$$

Second, we will find that the number $N - \#P_{ind}$ does only depend on the genus of the manifold.
a) Independent projectors

1. Convince yourself that the total Hilbert space dimension is 2^N and that a single vertex or a single plaquette projector projects out half the space, i.e. $\dim \text{Im}[P_v] = 2^N/2 = 2^{N-1}$.
2. When we count independent vertex projectors, we find that on any closed manifold, we have the following two constraints

$$\prod_v P_v = 1 \ , \qquad \prod_p B_p = 1 \ , \tag{8}$$

   and thus the number of independent vertex and plaquette operators is reduced. Check this constraint for a small closed manifold of your choice and understand where it comes from[1].
3. Conclude that the ground state space dimension is

$$\dim \mathcal{G} = 2^{N_e - N_v - N_f - 2} \ , \tag{9}$$

   where N_e is the number of edges of the lattice, N_v is the number of vertices and N_f is the number of faces (plaquettes).

b) Euler characteristic
The expression

$$\chi = N_v - N_e + N_f \ , \tag{10}$$

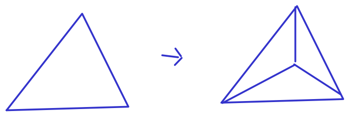
is well known and called the Euler characteristic of a manifold. It is independent of the discretization (drawing a grid on the manifold) of the manifold and only depends on the genus g as

$$\chi = 2 - 2g \ . \tag{11}$$

Inserting Eq. (10) into Eq. (9) immediately yields the desired result in Eq. (6). However, we are a little more ambitious here and try to convince ourselves that Eq. (10) is indeed correct.

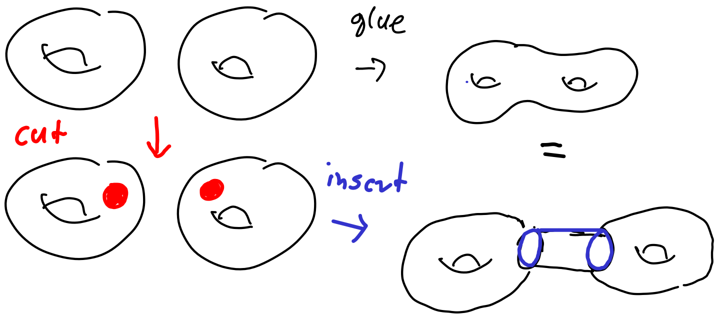
[1] Hint: recall how $S_e[\partial D] = \prod_{p \in D} B_p$ was 'derived'.

1. Count the number of vertices, edges and faces for a minimal sphere, i.e., a tetrahedron.
2. Convince yourself that the Euler characteristic does not change when you remove a single edge from a grid to form larger plaquettes.
3. Convince yourself that the following 'fine-graining' does also not change the Euler characteristic[2].



4. Understand the 'lattice' and the Euler characteristic of a minimal cylinder and calculate the Euler characteristic of a minimal torus.

5. To increase the genus, we can glue another torus to our manifold. What happens to the Euler characteristic in this process? Hint: Consider to remove two discs from the two manifolds that we want to glue and to connect them with a minimal cylinder.



4. Electric-magnetic duality

In the lecture, we mentioned several times, that vertex and plaquette operators are 'equivalent'. In this exercise, we convince ourselves that vertex and plaquette operators are indeed related by a local change of basis and going to the dual lattice. We start with the toric code Hamiltonian

$$H_{TC} = -\sum_v A_v - \sum_p B_p \, , \tag{12}$$

of vertex and plaquette operators, where A_v acts with z on all four spins around a vertex and B_p with x on all spins around a plaquette.

1. Local change of basis. Find a unitary 2×2 matrix H[3], for which $H^\dagger x H = z$ and $H z H^\dagger = x$. Show that one can choose H to have the additional properties $H^2 = \mathbb{1}$ and $H^T = H$.

[2]This fine-graining is called a 1-3 Pachner move. Sequences of this move and another Pachner move (the 2-2 move) can transform any triangulation of a manifold into another triangulation.

[3]Unfortunately, this particular matrix is called the Hadamard matrix/gate and is always denoted by the letter H, which is also the letter used for the Hamiltonian.

2. Define the basis change $U = \otimes_i H_i$, i.e., applying H to every site, and write $H' = UHU^\dagger$ as a sum of new vertex and plaquette operators $H' = -\sum_v A'_v - \sum_p B'_p$.

3. Consider the dual lattice and write H' as a sum of vertex and plaquette operators on the dual lattice. Convince yourself that under both local basis change and going to the dual lattice, A_v is mapped to B_p. Conclude that likewise B_p is mapped to A_v and as a consequence the role of e and m-particles is interchanged.

This mapping is known as 'electric-magnetic'-duality and we note that the toric code is self-dual under this mapping.