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Exercise sheet 3

Solution

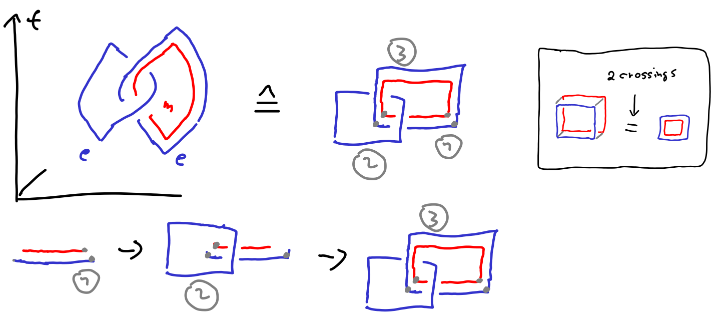
1. Braiding of anyons

In the lecture we considered the braiding statistics of e and m particles. In this exercise, we calculate the braiding statistics of e and $\epsilon = (e, m)$.

1. Write down the space-time diagram that corresponds to the definition of the entry $S_{e\epsilon}$ of the S-matrix.
2. Write down the corresponding expectation value.
3. Parameterize the braid by string and ribbon operators along suitable contours.
4. Evaluate the expectation value step-by-step by using the properties of the string operators.

Solution

Consider the following space time plot and the corresponding contours on the lattice.



They evaluate to the following expectation value when expressed in terms of string operators

$$S_{e,\epsilon} = \langle GS|S_m(\bar{C}_3)S_e(C_3)\, S_e(C_2)S_m(\bar{C}_1)S_e(C_1)|GS\rangle \ . \tag{1}$$

We apply the following properties of string operators to simplify the expression. First, string operators of the same anyon type commute

$$S_{e,\epsilon} = \langle GS|S_m(\bar{C}_3)S_e(C_2)S_e(C_3)\, S_m(\bar{C}_1)S_e(C_1)|GS\rangle \ . \tag{2}$$

String operators e and m anticommute if their contours cross an odd number of times, i.e.,

$$S_{e,\epsilon} = -\,\langle GS|S_e(C_2)S_m(\bar{C}_3)S_e(C_3)\, S_m(\bar{C}_1)S_e(C_1)|GS\rangle \ . \tag{3}$$

String operators with non-overlapping contours commute

$$S_{e,\epsilon} = -\,\langle GS|S_e(C_2)S_m(\bar{C}_3)\, S_m(\bar{C}_1)S_e(C_3)S_e(C_1)|GS\rangle \ , \tag{4}$$

contours can be joined

$$S_{e,\epsilon} = -\langle GS|S_e(C_2)S_m(\bar{C}_3 \cup \bar{C}_1)S_e(C_3 \cup C_1)|GS\rangle \ , \tag{5}$$

and string-operators along closed contours commute with the Hamiltonian and thus preserve the ground state space. In particular string-operators along topologically trivial closed contours act trivially on the ground state space, hence

$$S_{e,\epsilon} = -\langle GS|GS\rangle \ . \tag{6}$$

The ground state is normalized and we obtain

$$S_{e,\epsilon} = -1 \ . \tag{7}$$

2. Commuting projector Hamiltonians

In the lecture we discussed how the ground state space $\mathcal{G}$ of a commuting projector Hamiltonian

$$H = \sum_i P_i \ , \qquad [P_i, P_j] = 0 \ , \qquad P_i^2 = P_i \ , \tag{8}$$

can be obtained from sequentially projecting to the orthogonal complements

$$P_i^{\perp} = (1 - P_i) \tag{9}$$

of all individual projectors

$$P = \prod_i P_i^{\perp} \ , \qquad \mathrm{Im}[P] = \mathcal{G} \ , \qquad H\mathcal{G} = 0 \ . \tag{10}$$

As a warm up, convince yourself, that P_i and $P_i^{\perp}$ are indeed orthogonal and that $P_iP = 0$. Next, consider a tiny ferromagnet, i.e., a system of three spins on the vertices of a triangle with the following Hamiltonian

$$H_{\mathrm{F}} = -\sum_i z_i z_{i+1} \ . \tag{11}$$

1. Write this Hamiltonian as a sum of commuting projectors as in Eq. (8) by shifting the zero-energy and multiplication by a scalar and convince yourself that this does not change the eigenstates of the Hamiltonian.

2. Define the orthogonal complements of the individual projectors and 'calculate' their images $\mathcal{G}_i = \mathrm{Im}[P_i^{\perp}]$.

3. Convince yourself that the image of $P = \prod_i P_i^{\perp}$ is the intersection of the individual images, i.e., $\mathrm{Im}[P] = \mathcal{G} = \cap_i \mathcal{G}_i$ and calculate it.

In systems with frustration, the recipe to construct the ground state space as $\mathrm{Im}[P]$ fails. As an illustration consider the example for anti-ferromagnetic interactions, i.e.,

$$H_{\mathrm{AF}} = \sum_i z_i z_{i+1} \tag{12}$$

and repeat the three steps above. What do you find for $\mathcal{G}$ and why is this result expected? How does the actual ground state space look like? What it the ground state energy of the commuting projector Hamiltonian?

Solution

Ferromagnet
1. The Hamiltonian in Eq. (11) can be written as a commuting projector Hamiltonian by adding a constant term $3\mathbb{1}$ and dividing the whole Hamiltonian by 2

$$H'_F = \frac{H_F + 3\mathbb{1}}{2} = \sum_i \frac{1 - z_i z_{i+1}}{2} = \sum_i P_i^F \ . \tag{13}$$

This shifts the spectrum by an offset of 3/2 and rescales the level spacings by 1/2. The eigenstates are unchanged. If $|\Psi\rangle$ is an eigenstate to H_F with eigenvalue λ, the state $|\Psi\rangle$ is also an eigenstate of H'_F with a shifted ans scaled eigenvalue

$$H_F |\Psi\rangle = \lambda |\Psi\rangle \Rightarrow H'_F = \frac{H_F + 3\mathbb{1}}{2} |\Psi\rangle = \frac{\lambda + 3}{2} |\Psi\rangle \ . \tag{14}$$

2. The images $\mathcal{G}_i^F = \mathrm{Im}[P_i^\perp]$ of the orthogonal complements $P_i^\perp = 1 - P_i = \frac{1+z_i z_{i+1}}{2}$ are

$$\begin{aligned}
\mathcal{G}_1^F &= \mathrm{span}\{|000\rangle, |001\rangle, |110\rangle, |111\rangle\} \\
\mathcal{G}_2^F &= \mathrm{span}\{|000\rangle, |100\rangle, |011\rangle, |111\rangle\} \\
\mathcal{G}_3^F &= \mathrm{span}\{|000\rangle, |010\rangle, |101\rangle, |111\rangle\}
\end{aligned}$$

3. The ground state projector P is a product of all individual $P_i^\perp$, hence its image contains exactly the states that survive all the $P_i^\perp$ projections. This is the intersection of the images $\mathcal{G}_i$ which is

$$\mathcal{G}^F = \mathrm{span}\{|000\rangle, |111\rangle\} \ . \tag{15}$$

Anti-ferromagnet For the antiferrmagnet, the projector is $¶_i^{\mathrm{AF}} = \frac{1+z_i z_{i+1}}{2}$ and the images of the orthogonal complements are

$$\begin{aligned}
\mathcal{G}_1^{\mathrm{Af}} &= \mathrm{span}\{|010\rangle, |011\rangle, |100\rangle, |101\rangle\} \\
\mathcal{G}_2^{\mathrm{Af}} &= \mathrm{span}\{|001\rangle, |101\rangle, |010\rangle, |110\rangle\} \\
\mathcal{G}_3^{\mathrm{Af}} &= \mathrm{span}\{|100\rangle, |110\rangle, |001\rangle, |011\rangle\}
\end{aligned}$$

We note that the intersection is empty. This is expected because on a triangle not all pairs of spins can point in opposite directions. The ground state space can be calculated explicitly. It is six-fold degenerate since all states except for $|000\rangle$ and $|111\rangle$ are equally bad. We find that the ground state energy of H^{AF} is -1 and the ground state energy of the commuting projector Hamiltonian is $(3-1)/2 = 1$. This means, the minimal possible energy values (zero) of a commuting projector Hamiltonian is not reached.

3. Topological ground state degeneracy and Euler characteristic*

In the lecture we claimed that the ground state degeneracy of the toric code depends on the topology of the manifold on which it is defined as

$$\dim \mathcal{G} = 2^{2g} \ , \tag{16}$$

where g is the genus of the manifold, i.e, the number of (independent) holes. In this exercise we will sketch the proof of this formula. It is divided into two parts. First, we will repeat the counting of independent projectors to express $\dim \mathcal{G}$ as a function of the number of qubits N and the number of independent projectors $\#P_{\mathrm{ind}}$ as

$$\dim \mathcal{G} = 2^{N - \#P_{\mathrm{ind}}} \ . \tag{17}$$

Second, we will find that the number $N - \#P_{ind}$ does only depend on the genus of the manifold.
a) Independent projectors

1. Convince yourself that the total Hilbert space dimension is 2^N and that a single vertex or a single plaquette projector projects out half the space, i.e. $\dim \mathrm{Im}[P_v] = 2^N/2 = 2^{N-1}$.

2. When we count independent vertex projectors, we find that on any closed manifold, we have the following two constraints

$$\prod_v P_v = 1\ , \qquad \prod_p B_p = 1\ , \tag{18}$$

and thus the number of independent vertex and plaquette operators is reduced. Check this constraint for a small closed manifold of your choice and understand where it comes from[1].

3. Conclude that the ground state space dimension is

$$\dim \mathcal{G} = 2^{N_e - N_v - N_f + 2}\ , \tag{19}$$

where N_e is the number of edges of the lattice, N_v is the number of vertices and N_f is the number of faces (plaquettes).

b) Euler characteristic

The expression

$$\chi = N_v - N_e + N_f\ , \tag{20}$$

is well known and called the Euler characteristic of a manifold. It is independent of the discretization (drawing a mesh on the manifold) of the manifold and only depends on the genus g (the number of holes) as

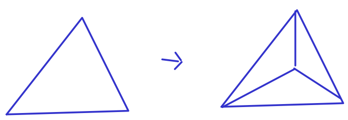
$$\chi = 2 - 2g\ . \tag{21}$$

Inserting Eq. (20) into Eq. (19) immediately yields the desired result in Eq. (16). However, we are a little more ambitious here and try to convince ourselves that Eq. (20) is indeed correct.

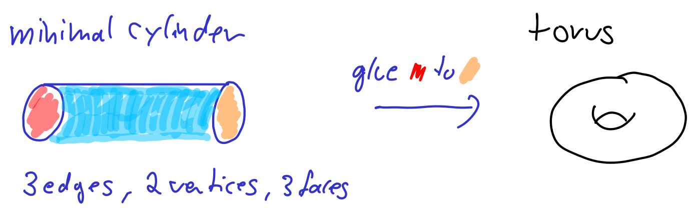
1. Count the number of vertices, edges and faces for a minimal sphere, i.e., a tetrahedron.

2. Convince yourself that the Euler characteristic does not change when you remove a single edge from a grid to form larger plaquettes.

3. Convince yourself that the following 'fine-graining' does also not change the Euler characteristic[2].



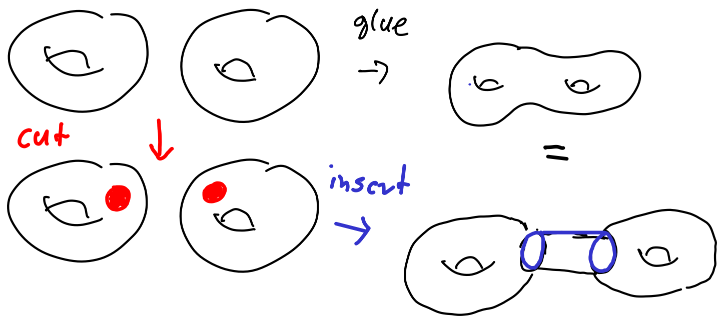
4. Understand the 'lattice' and the Euler characteristic of a minimal cylinder and calculate the Euler characteristic of a minimal torus.



[1] Hint: recall how $S_e[\partial D] = \prod_{p\in D} B_p$ was 'derived'.

[2] This fine-graining is called a 1-3 Pachner move. Sequences of this move and another Pachner move (the 2-2 move) can transform any triangulation of a manifold into another triangulation.

5. To increase the genus, we can glue another torus (handle) to our manifold. What happens to the Euler characteristic in this process? Hint: Consider to remove two discs from the two manifolds that we want to glue and to connect them with a minimal cylinder.



Solution

a) Counting independent projectors

1. The total Hilbert space dimension of a system with N qubits is 2^N. The vertex projector projects onto all states with even parity of the four spins around a vertex. Half of all the states in the Hilbert space fulfill this condition, while the other half does not. For the plaquette operator we can argue the same way. We can think about the space as being written in the basis of $\hat{x}$. Then, the plaquette operator again checks for even parity (now of x-basis states) and we find that it also divides the space in two halves.

2. We know that a $x(z)$-string operator along a closed contour can be expressed as the product of all plaquette (vertex) operators inside the contour. Consider the case of x-strings and plaquette. We choose a topologically trivial loop C on a closed manifold, such that the contour divides the manifold into an inside and outside part, i.e., $C = \partial R_{\text{in}} = \partial_{\text{out}}$ and $\mathcal{M} = R_{\text{in}} \cup R_{\text{out}}$. Its string operator $S_m(C)$ can be expressed as

$$S_m(C) = S_m(\partial R_{\text{in}}) = \prod_{p \in R_{\text{in}}} B_p \tag{22}$$

and likewise

$$S_m(C) = S_m(\partial R_{\text{out}}) = \prod_{p \in R_{\text{out}}} B_p \,. \tag{23}$$

We know that a string operator squares to one, i.e.,

$$S_m(C)^2 = S_m(C) S_m(C) = 1 \,. \tag{24}$$

Inserting Eq. (22) and Eq. (23) we find

$$S_m(C) S_m(C) = \prod_{p \in R_{\text{in}}} B_p \prod_{p \in R_{\text{out}}} B_p = \prod_{p \in R_{\text{in}} \cup R_{\text{out}}} B_p = \prod_{p \in \mathcal{M}} B_p = 1 \,. \tag{25}$$

The considerations for the vertex operators is the same (consider a z-string).

3. The ground state degeneracy on a manifold is given by the number of qubits and the number of independent vertex and plaquette projectors. There are $N_v - 1$ independent vertex projectors and $N_p - 1$ independent plaquette projectors, each reducing the ground state degeneracy by a factor of $\frac{1}{2}$. Thus, we obtain $\dim \mathcal{G} = 2^N / (2^{N_v - 1} 2^{N_p - 1})$ which gives the desired result.

b) Euler characteristic

Remarks. The questions of this exercise sheet serve the purpose to get familiar with combinatorial representations of manifolds and the Euler characteristic and introduce some concepts

used to calculate topological invariants on an informal level. Their answers do not result in a full proof of Eq. (21) (which is beyond the scope of this lecture), but give some intuition for why Eq. (21) is correct. As a first warm up, we consider a simple type of manifold discretization called simplicial complex. It is a tiling of the manifold with triangles. Every 2D manifold can be triangulated in that way. The minimal simplicial complex of a sphere is a tetrahedron. We can explicitly calculate the Euler characteristic (ex. 1).
To show that a quantity is independent of the particular discretization and only depends on the topology of the manifold, one often invokes the fact that two topological equivalent triangulations (i.e., the manifold has the same genus) can be transformed into each other by applying a sequence of so-called Pachner moves. One such move (a particular fine graining) is considered in ex. 3, where we see that the Euler characteristic is invariant under such a deformation.
To get an idea about more general cellulations, e.g., a mesh that contains also squares, or general n-gons, we convince ourselves that the Euler characteristic does not change when e.g. two triangles are 'merged' to a square in ex. 2.
General 'meshes' are more formally known as CW complexes or sometimes just 'cell complexes'. A simple class of 2D cell-complexes are the ones that consist of n-gons[3]. Here n can be as small as one. The edges of the n-gons have to be bounded by vertices, i.e., each edge ends in a vertex, but it can be the same vertex for both ends. As an example, consider a small cell complex for a sphere. It consists of two 1-gons (northern and southern hemisphere) connected along a common edge (equator), i.e, the cell complex consists of two faces, one edge and one vertex and its Euler characteristic is $2 = 2 - 2 \cdot 0$. In ex. 4 and 5 we consider manifolds discretized with cell complexes of this kind and investigate how the Euler characteristic changes when we glue these manifolds together changing their genus.
Answers. 1. A minimal sphere can be represented as a tetrahedron with four faces, six edges and four vertices. Its genus is zero. And we find

$$\chi_{\text{min sphere}} = 4 - 6 + 4 = 2 = 2 - 2 \cdot 0 \,, \tag{26}$$

which is consistent with Eq. (21).
2. If we remove an edge in a grid, the number of edges changes from $N_e \to N_e - 1$. The number of faces changes from $N_f \to N_f - 1$ and the number of vertices does not change, thus the EUler characteristic remains the same.
3. The 1-3 Pachner move has the following effect

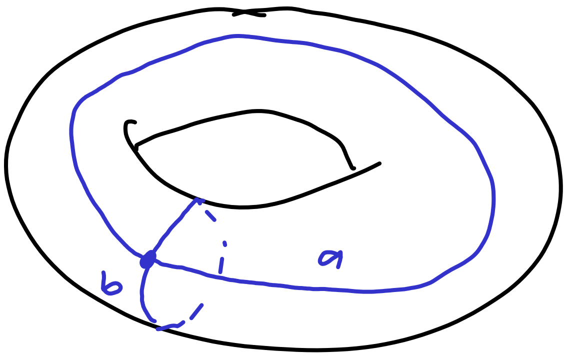
$$N_v - N_e + N_f \to (N_v + 1) - (N_e + 3) + (N_f + 2) \tag{27}$$

and thus does not change χ.
4. We now want to compare the Euler characteristic of a cylinder that is topologically equivalent to a sphere (genus zero) to that of a torus. First, we find a cellulation for the cylinder. It can be cellulated as depicted above, i.e., by three faces, two 1-gons for the left and right side (red and orange) and one square (blue) with two opposing edges identified for the middle part. It has three edges (one left, one right and one in the middle) and two vertices (one left and one right) and its Euler characteristic is two as expected ($\chi = 2 - 2 \cdot 0$). We now compare it to a torus. To this end, we imagine 'gluing' the left and right side together. In doing so orange and red faces are removed. The two vertices merge into one vertex and the two closed edges melt into one closed edge. Thus, we are left with one vertex, two edges and one face and

$$\chi = 1 - 2 + 1 = 0 = 2 - 2 \cdot 1 \,, \tag{28}$$

which is consistent with Eq. (21), because the genus of a torus is $g_{\text{torus}} = 1$.

[3] General 2D CW complexes are also allowed to have faces that are not bounded by edges, but by a single vertex. We will not consider these here.



5. When gluing another tours (handle) to a manifold, we increase its genus by one and hence we expect that the Euler characteristic changes by

$$\chi = 2 - 2g \to \ \chi' = 2 - 2(g+1) = 2 - 2g - 2 = \chi - 2\ . \tag{29}$$

In general, when we glue two manifolds with genus g_1 and g_2 in the way described above, we expect to obtain a manifold with genus $g' = g_1 + g_2$. We consider now how the number of edges, faces and vertices changes in such a process. Before we glue, we have all vertices, edges and faces of the two manifolds. The gluing itself has the following effect. We cut two holes in the manifold, i.e., we remove two discs (red). Then we insert a minimal open cylinder (i.e., the cylinder without the red and orange left and right face). Thus, we subtract two faces and then add one face and the total number of faces decreases by one. The attached cylinder adds one additional edge to the cell complex, thus the number of edges increases by one. The number of vertices stays the same. Thus, we have

$$\begin{aligned}
\chi' &= N'_v - N'_e + N'_f = N_v^{(1)} + N_v^{(2)} - (N_e^{(1)} + N_e^{(2)} + 1) + (N_f^{(1)} + N_f^{(2)} - 1) \\
&= (N_v^{(1)} - N_e^{(1)} + N_f^{(1)}) + (N_v^{(2)} - N_e^{(2)} + N_f^{(2)}) - 2 \\
&= \chi^{(1)} + \chi^{(2)} - 2 = (2 - 2g_1) + (2 - 2g_2) - 2 = 2 - 2(g_1 + g_2) = 2 - 2g'\ .
\end{aligned}$$

We have shown that the Euler characteristic changes as $\chi' = \chi^{(1)} + \chi^{(2)} - 2$ and verified the intuition, that the genus of two manifolds glued together in this way is the sum of the two genera of the individual manifolds. Thus, by induction we can verify that Eq. (21) holds for all manifolds glued from minimal tori. The fact that every 2D surface can be glued from tori is known as 'handle decomposition'.

Proof sketch. For a complete proof, we would also need to show that every cellulation can be obtained by fine graining a cellulation of glued minimal tori and that the Euler characteristic does not change under such fine grainings. Similiar to the considerations for the Pachner move, we can show, that adding a vertex to split one edge into two edges does not change the Euler characteristic and that adding an additional edge connecting two vertices and splitting a face into two does also not change the Euler characteristic. With these moves we can obtain arbitrary fine grainings and deform cellulations as we wish.

Note to tutorial question about minimal cellulation of a torus. It was suggested to cellulate a torus with a single edge, a vertex and one face. This is not a proper CW complex. A general 2D CW complex is obtained by gluing discs to a collection of points, lines and circles (cf wikipedia). We can not obtain the cellulation above in this way. This can be seen by cutting the torus open along the one given edge. We obtain an annulus and not a disk. The minimal CW complex of a torus is the one obtained in ex. 4. It can be constructed by taking two circles that meet at one vertex and glue a square-shaped disc to them.

4. Electric-magnetic duality

In the lecture, we mentioned several times, that vertex and plaquette operators are 'equivalent'. In this exercise, we convince ourselves that vertex and plaquette operators are indeed related by a local change of basis and going to the dual lattice. We start with the toric code Hamiltonian

$$H_{TC} = -\sum_v A_v - \sum_p B_p \ , \tag{30}$$

of vertex and plaquette operators, where A_v acts with z on all four spins around a vertex and B_p with x on all spins around a plaquette.

1. Local change of basis. Find a unitary 2×2 matrix H[4], for which $H^\dagger x H = z$ and $H z H^\dagger = x$. Show that one can choose H to have the additional properties $H^2 = \mathbb{1}$ and $H^T = H$.
2. Define the basis change $U = \otimes_i H_i$, i.e., applying H to every site, and write $H' = UHU^\dagger$ as a sum of new vertex and plaquette operators $H' = -\sum_v A'_v - \sum_p B'_p$.
3. Consider the dual lattice and write H' as a sum of vertex and plaquette operators on the dual lattice. Convince yourself that under both local basis change and going to the dual lattice, A_v is mapped to B_p. Conclude that likewise B_p is mapped to A_v and as a consequence the role of e and m-particles is interchanged.

This mapping is known as 'electric-magnetic'-duality and we note that the toric code is self-dual under this mapping.

Solution

1. With the ansatz $H = H^T$ and $H^\dagger = H^{-1} = H$ we have that H is real and symmetric. We write

$$H = \begin{pmatrix} a & b \\ b & c \end{pmatrix} \tag{31}$$

and evaluate $HzH = x$, i.e.,

$$\begin{pmatrix} a^2 - b^2 & ab - bc \\ ab - bc & b^2 - c^2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \ , \tag{32}$$

from which we fint that $a^2 = b^2 = c^2$ and $a = -c$. We guess $a = b = 1$ and $c = -1$ and normalize the matrix such that also $H^2 = \mathbb{1}$ is fulfilled. We find

$$H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \ . \tag{33}$$

2. Applying U, we know that x changes to z and vice-versa. We have $A'_v = \prod_{i \in v} x_i$ and $B'_p = \prod_{j \in p} z_i$.
3. On the dual lattice, vertices turn into plaquettes and plaquettes turn into vertices.

[4]Unfortunately, this particular matrix is called the Hadamard matrix/gate and is always denoted by the letter H, which is also the letter used for the Hamiltonian.