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Exercise sheet 4

This exercise will be discussed on 26.01.2020

1. Braiding of Majorana fermions

Consider four Majorana modes, $\gamma_1, \ldots, \gamma_4$, satisfying anti-commutation relations $\{\gamma_i, \gamma_j\} = 2\delta_{ij}$. They can be realized, e.g., as zero bound states of half-quantum vorticies in a 2D p-wave superconductor. Let's introduce operators

$$c_1 = \frac{1}{2}(\gamma_1 + i\gamma_2), \qquad c_2 = \frac{1}{2}(\gamma_3 + i\gamma_4) \tag{1}$$

which combine two pairs of Majoranas in two 'complex' fermions. Let further $|0\rangle \equiv |0\rangle_c$ be the vacuum state ($c_i|0\rangle_c = 0$) and $|1\rangle \equiv c_1^\dagger c_2^\dagger |0\rangle_c$ be the doubly occupied state.

a) Upto a global phase, three braiding operators B_{12}, B_{23} and B_{34} for Majorana fermions read $B_{i\,i+1} = \exp(\frac{\pi}{4}\gamma_{i+1}\gamma_i)$ with $i = 1, 2, 3$. Verify that $B_{i\,i+1}$ can be written in the equivalent form

$$B_{i\,i+1} = \frac{1}{\sqrt{2}}(1 + \gamma_{i+1}\gamma_i). \tag{2}$$

b) Check that in the even parity basis $\{|0\rangle, |1\rangle\}$ the matrices of braiding operators are

$$B_{12} = B_{34} = R = \begin{pmatrix} e^{-i\pi/4} & 0 \\ 0 & e^{i\pi/4} \end{pmatrix}, \qquad B_{23} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix}. \tag{3}$$

c) Check the defining relation of braiding group

$$B_{12}B_{23}B_{12} \stackrel{!}{=} B_{23}B_{12}B_{23}. \tag{4}$$

You may perform a calculation either in a matrix or in an operator form (the latter is more general, since it holds for both even and odd parity sectors). Draw related brading diagrams for world lines of three Majoranas.

As it was discussed in a lecture, one may pair 4 Majoranas into 2 fermions in a different way by introducing operators

$$d_1 = \frac{1}{2}(\gamma_1 + i\gamma_4), \qquad d_2 = -\frac{i}{2}(\gamma_2 + i\gamma_3) \tag{5}$$

and the corresponding basis states, $|0'\rangle \equiv |0\rangle_d$ and $|1'\rangle \equiv d_1^\dagger d_2^\dagger |0\rangle_d$, where $d_i|0\rangle_d = 0$ is the definition of a vacuum in d-basis and which is not the same as $|0\rangle_c$.

d) Check that two choices of basis states are related by the F-matrix

$$\begin{pmatrix} |0\rangle \\ |1\rangle \end{pmatrix} = F \begin{pmatrix} |0'\rangle \\ |1'\rangle \end{pmatrix}, \qquad F = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}. \tag{6}$$

Hint: One may start by assuming that $|0'\rangle = |0\rangle\alpha + |1\rangle\beta$ and find two unknown constants from the defining relation $d_i|0'\rangle = 0$ and the normalization by expressing d-operators in terms of c_i and $c_i^\dagger$. After that one may evaluate $|1'\rangle$ in terms of c-basis.

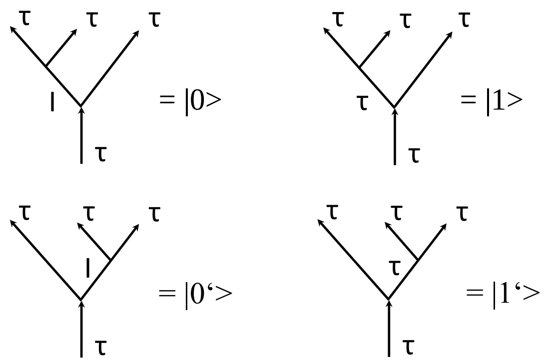

Figure 1: Two different basis sets of the splitting space $V_\tau^{\tau\tau\tau}$.

e) Verify the general relation $B_{23} = FRF^{-1}$ derived in a lecture, which relates B_{23} in the basis of c-operators to its diagonal form in d-basis.

2. Fibonacci anyons

Consider the Fibonacci anyonic system with two particle types $\{\mathbf{I}, \tau\}$ and fusion rules

$$\mathbf{I} \times \tau = \tau \times \mathbf{I} = \tau, \qquad \tau \times \tau = \mathbf{I} + \tau. \tag{7}$$

a) Verify that a dimension of the Hilbert space of n anyons of type τ equals to $\dim(\mathcal{H}_\tau^{(n)}) = F_n$, where F_n are Fibonacci numbers.

Hint: You may check that a number of all trees with a overall fusion channel being $\mathbf{I}$ is F_{n-2} while the number of those with a final result being τ is F_{n-1}. Since Fibonacci numbers are defined by a recurrence relation $F_n = F_{n-2} + F_{n-1}$ with $F_0 = F_1 = 0$, it will bring you the required result. The proof can be done by induction.

Consider now a two-dimensional Hilbert space of three anyons $V_\tau^{\tau\tau\tau}$ with the overall fusion channel τ. We denote its orthogonal basis in terms of left standard trees as $\{|0\rangle, |1\rangle\}$ and in terms of right standard trees as $\{|0'\rangle, |1'\rangle\}$, see Fig. 1. Similar to Eq. (6), these two basis sets are related by the 2×2 orthogonal fusion matrix $F_\tau^{\tau\tau\tau}$ satisfying to the pentagon equation

$$(F_\tau^{\tau\tau c})_a^d (F_\tau^{a\tau\tau})_b^c = \sum_e (F_d^{\tau\tau\tau})_a^c (F_\tau^{\tau e\tau})_b^d (F_b^{\tau\tau\tau})_a^e. \tag{8}$$

b) Check that splitting spaces $V_{\mathrm{I}}^{\tau\tau\tau}$ and $V_\tau^{\mathrm{I}\tau\tau}$ are one-dimensional by drawing the corresponding left and right fusion trees. Then set $a = d = \mathrm{I}$ and $c = b = \tau$ and derive a simpler equation for fusion matrix

$$(F_\tau^{\tau\tau\tau})_{\mathrm{I}}^{\mathrm{I}} = (F_\tau^{\tau\tau\tau})_\tau^{\mathrm{I}} (F_\tau^{\tau\tau\tau})_{\mathrm{I}}^\tau. \tag{9}$$

You may assume that F-matrices in one-dimensional splitting spaces just equal to 1.

c) Solve Eq. (9) under the assumption that matrix $F_\tau^{\tau\tau\tau}$ is orthogonal with det $= -1$, which corresponds to a particular gauge choice. The result you'll obtain should read

$$F_\tau^{\tau\tau\tau} = \begin{pmatrix} \phi^{-1} & \phi^{-1/2} \\ \phi^{-1/2} & -\phi^{-1} \end{pmatrix}, \qquad \phi = (1 + \sqrt{5})/2. \tag{10}$$