(1) Problem set # 4

(I) Braiding of Maj fermions

(a) $B_{12} = e^{\frac{\pi}{4}\gamma_2\gamma_1}$, Let $P_{12} = i\gamma_2\gamma_1$

$P_{12}^2 = -\gamma_2\gamma_1\gamma_2\gamma_1 = \mathbb{1} \Rightarrow$

$B_{12} = e^{-i\frac{\pi}{4}P_{12}} = \cos\frac{\pi}{4} - i\sin\frac{\pi}{4}P_{12} =$

$= \frac{1}{\sqrt{2}}(1 - iP_{12}) = \frac{1}{\sqrt{2}}(1+\gamma_2\gamma_1)$

$\Rightarrow \quad B_{i\,i+1} = e^{\frac{\pi}{4}\gamma_{i+1}\gamma_i} = \frac{1}{\sqrt{2}}(1+\gamma_{i+1}\gamma_i)$

(b) $|0\rangle = |0\rangle_c$ and $|1\rangle = c_1^+c_2^+|0\rangle$

1) $B_{12} = e^{-i\frac{\pi}{4}P_{12}}$, with $P_{12} = i\gamma_2\gamma_1$

$c_1^+ = \frac{1}{2}(\gamma_1 - i\gamma_2) \qquad c_1 = \frac{1}{2}(\gamma_1 + i\gamma_2)$

$1 - 2c_1^+c_1 = 1 - \frac{1}{2}(\gamma_1 - i\gamma_2)(\gamma_1 + i\gamma_2)$

$= 1 - \frac{1}{2}(1 + i\gamma_1\gamma_2 - i\gamma_2\gamma_1 + 1) = i\gamma_2\gamma_1$ !

$\Rightarrow \quad B_{12} = e^{-i\frac{\pi}{4}(1-2c_1^+c_1)} \Rightarrow \begin{cases}(1-2c_1^+c_1)|0\rangle = |0\rangle \\ (1-2c_1^+c_1)|1\rangle = -|1\rangle\end{cases}$

$B_{12} = \begin{pmatrix} e^{-i\frac{\pi}{4}} & \emptyset \\ \emptyset & e^{i\frac{\pi}{4}} \end{pmatrix}$

2) $B_{34} = e^{-i\frac{\pi}{4}P_{34}}$ with $P_{34} = i\gamma_4\gamma_3$

$\Rightarrow \quad B_{34} = \begin{pmatrix} e^{-i\frac{\pi}{4}} & \emptyset \\ \emptyset & e^{i\frac{\pi}{4}} \end{pmatrix}$

3) $B_{23} = \frac{1}{\sqrt{2}}(1+\gamma_3\gamma_2) = ?$ $\quad \begin{cases} c_1 = \frac{1}{2}(\gamma_1 + i\gamma_2) \\ c_2 = \frac{1}{2}(\gamma_3 + i\gamma_4)\end{cases} \Rightarrow$

$= \frac{1}{\sqrt{2}}(1 + i(c_2 + c_2^+)(c_1^+ - c_1))$ $\quad \begin{cases}\gamma_3 = c_2 + c_2^+ \\ \gamma_2 = i(c_1^+ - c_1)\end{cases}$

2

$$B_{23}|0\rangle = \frac{1}{\sqrt2}\left(1 + i\cancel{c_2 a_1^+} - i\cancel{c_2 a_1} + i c_2^+ a_1^+ - i\cancel{c_2^+ a_1}\right)|0\rangle$$

$$= \frac{1}{\sqrt2}\left(1 - i\, a_1^+ c_2^+\right)|0\rangle = \frac{1}{\sqrt2}\left(|0\rangle - i|1\rangle\right)$$

$$B_{23}|1\rangle = \frac{1}{\sqrt2}\left(1 + i\cancel{c_2 a_1^+} - i c_2 a_1 + i\cancel{c_2^+ a_1^+} - i\cancel{c_2^+ a_1}\right)a_1^+ c_2^+|0\rangle$$

$$= \frac{1}{\sqrt2}\left(1 - i c_2 a_1\right) a_1^+ c_2^+ |0\rangle$$

$$= \frac{1}{\sqrt2}\left(|1\rangle - i\, c_2 c_2^+ a_1 a_1^+ |0\rangle\right) =$$

$$= \frac{1}{\sqrt2}\left(|1\rangle - i|0\rangle\right) \Rightarrow$$

$$B_{23} = \frac{1}{\sqrt2}\begin{pmatrix}1 & -i\\ -i & 1\end{pmatrix} \to \text{ok.}$$

c) 1) Matrix: $B_{12}\, B_{23}\, B_{12} \doteq B_{23}\, B_{12}\, B_{23}$

$$\frac{1}{\sqrt2}\begin{pmatrix}e^{-i\pi/4} & 0\\ 0 & e^{i\pi/4}\end{pmatrix}\overset{\begin{pmatrix}1 & -i\\ -i & 1\end{pmatrix}}{\begin{pmatrix}e^{-i\pi/4} & -ie^{-i\pi/4}\\ -ie^{i\pi/4} & e^{i\pi/4}\end{pmatrix}}\overset{\begin{pmatrix}e^{-i\pi/4} & 0\\ 0 & e^{i\pi/4}\end{pmatrix}}{\begin{pmatrix}-i & -i\\ -i & i\end{pmatrix}} = -\frac{i}{\sqrt2}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}$$

$$\frac{1}{2}\begin{pmatrix}1 & -i\\ -i & 1\end{pmatrix}\overset{\begin{pmatrix}e^{-i\pi/4} & 0\\ 0 & e^{i\pi/4}\end{pmatrix}}{\begin{pmatrix}e^{-i\pi/4} & -ie^{i\pi/4}\\ -ie^{-i\pi/4} & e^{i\pi/4}\end{pmatrix}}\overset{\begin{pmatrix}1 & -i\\ -i & 1\end{pmatrix}}{\begin{pmatrix}-i\sqrt2 & -i\sqrt2\\ -i\sqrt2 & i\sqrt2\end{pmatrix}} = -\frac{i}{\sqrt2}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}$$

$\boxed{\phantom{x}}$

$$e^{-i\pi/4} - e^{+i\pi/4} = -2i\sin\pi/4 = -i\sqrt2$$
$$-ie^{-i\pi/4} - ie^{i\pi/4} = -2i\cos\pi/4 = -i\sqrt2$$

③ 2) Operator approach

(i) $$B_{12}\,B_{23}\,B_{12} = \frac{1}{2\sqrt{2}}\left(1+\gamma_2\gamma_1\right)\left(1+\gamma_3\gamma_2\right)\left(1+\gamma_2\gamma_1\right)$$

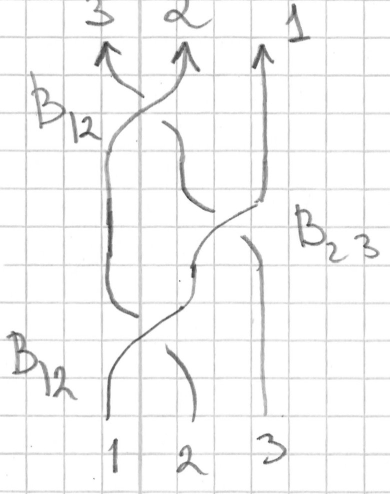
$$= \frac{1}{2\sqrt{2}}\left(1+\gamma_3\gamma_2+\gamma_2\gamma_1+\gamma_1\gamma_3\right)\left(1+\gamma_2\gamma_1\right)$$

$$= \frac{1}{2\sqrt{2}}\Big(\cancel{1}+\gamma_2\gamma_1+\gamma_3\gamma_2+\cancel{\gamma_3\gamma_1}+\gamma_2\gamma_1-\cancel{1}$$
$$+\cancel{\gamma_1\gamma_3}+\gamma_3\gamma_2\Big) = \frac{1}{\sqrt{2}}\left(\gamma_3\gamma_2+\gamma_2\gamma_1\right)$$

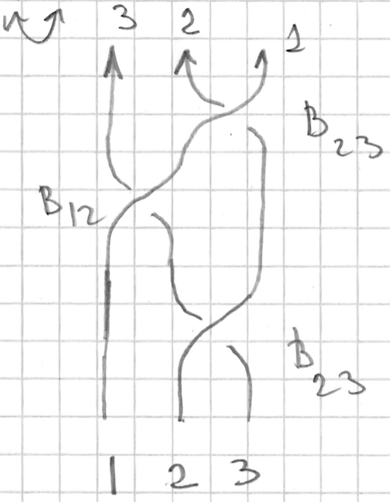
(ii) $$B_{23}\,B_{12}\,B_{23} = \frac{1}{2\sqrt{2}}\left(1+\gamma_3\gamma_2\right)\left(1+\gamma_2\gamma_1\right)\left(1+\gamma_3\gamma_2\right)$$

$$= \frac{1}{2\sqrt{2}}\left(1+\gamma_2\gamma_1+\gamma_3\gamma_2+\gamma_3\gamma_1\right)\left(1+\gamma_3\gamma_2\right) =$$

$$= \frac{1}{2\sqrt{2}}\Big(\cancel{1}+\gamma_3\gamma_2+\gamma_2\gamma_1+\cancel{\gamma_1\gamma_3}+\gamma_3\gamma_2-\cancel{1}+$$
$$+\cancel{\gamma_3\gamma_1}-\gamma_1\gamma_2\Big) = \frac{1}{\sqrt{2}}\left(\gamma_3\gamma_2+\gamma_2\gamma_1\right) \boxtimes$$




(i)




(ii)

(4)

(d)

$$\begin{cases} c_1 = \frac{1}{2}(\gamma_1 + i\gamma_2) \\ c_1^+ = \frac{1}{2}(\gamma_1 - i\gamma_2) \end{cases} \Rightarrow \quad \gamma_1 = c_1 + c_1^+ \quad \gamma_2 = i(c_1^+ - c_1)$$

$$\begin{cases} c_2 = \frac{1}{2}(\gamma_3 + i\gamma_4) \\ c_2^+ = \frac{1}{2}(\gamma_3 - i\gamma_4) \end{cases} \Rightarrow \quad \gamma_3 = c_2 + c_2^+ \quad \gamma_4 = i(c_2^+ - c_2)$$

If we define $d_1 = \frac{1}{2}(\gamma_1 + i\gamma_4)$ and $d_2 = -\frac{i}{2}(\gamma_2 + i\gamma_3)$

then
$$d_1 = \frac{1}{2}(c_1 + c_1^+ + c_2 - c_2^+)$$
$$d_2 = \frac{1}{2}(c_1^+ - c_1 + c_2 + c_2^+)$$

- Let $|0\rangle_d = |0\rangle_c \alpha + c_1^+ c_2^+ |0_c\rangle \beta \Rightarrow$

(i) $d_1 |0\rangle_d = 0 \Rightarrow \frac{1}{2}(c_1^+ - c_2^+)|0\rangle_c \alpha +$

$$+ \frac{1}{2}(c_1 + \cancel{c_1^+} + c_2 - \cancel{c_2^+}) c_1^+ c_2^+ |0\rangle_c \beta =$$

$$= \frac{1}{2}(c_1^+ - c_2^+)|0\rangle_c \alpha + \frac{1}{2}(c_2^+ - c_1^+)|0\rangle_c \beta = 0 \Rightarrow$$

$\alpha = \beta$ & normalization $\Rightarrow$ $$\boxed{|0\rangle_d = \frac{1}{\sqrt{2}}(|0\rangle_c + c_1^+ c_2^+ |0\rangle_c)}$$

(ii) $d_2 |0\rangle_d = 0 \Rightarrow \frac{1}{2}(c_1^+ + c_2^+)|0\rangle_c \alpha +$

$$+ \frac{1}{2}(c_2 - c_1) c_1^+ c_2^+ |0\rangle_c \beta = \frac{1}{2}(c_1^+ + c_2^+)|0\rangle_c \alpha$$

$$+ \frac{1}{2}(-c_1^+ - c_2^+)|0\rangle_c \beta \Rightarrow \alpha = \beta\ !$$

- $\sqrt{2}\, d_1^+ d_2^+ |0\rangle_d = \frac{1}{4}(c_1 + c_1^+ - c_2 + c_2^+)(-c_1^+ + \cancel{c_1} + \cancel{c_2} + c_2^+)|0\rangle_c$

$$+ \frac{1}{4}(c_1 + c_1^+ - c_2 + c_2^+)(-\cancel{c_1^+} + c_1 + c_2 + \cancel{c_2^+}) c_1^+ c_2^+ |0\rangle_c$$

$$= \frac{1}{4}(-|0\rangle_c + c_1^+ c_2^+ |0\rangle_c - |0\rangle_c - c_2^+ c_1^+ |0\rangle_c$$

$$+ \frac{1}{4}(c_1 + c_1^+ - c_2 + c_2^+)(+ c_2^+ |0\rangle_c - c_1^+ |0\rangle_c) =$$

$= \frac{1}{2}\left(-|0\rangle_c + c_1^\dagger c_2^\dagger |0\rangle_c\right)$

$+ \frac{1}{4}\left(-|0\rangle_c + c_1^\dagger c_2^\dagger |0\rangle_c - |0\rangle_c - c_2^\dagger c_1^\dagger |0\rangle_c\right)$

$= \left(-|0\rangle_c + c_1^\dagger c_2^\dagger |0\rangle_c\right)$

Result of (ii) $d_1^\dagger d_2^\dagger |0\rangle_d = \frac{1}{\sqrt{2}}\left(|0\rangle_c - c_1^\dagger c_2^\dagger |0\rangle_c\right)$

Combining (i) & (ii) we can say that

$$\begin{cases} |0'\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle + |1\rangle\right) \\ |1'\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle - |1\rangle\right) \end{cases} \Rightarrow \begin{pmatrix} |0'\rangle \\ |1'\rangle \end{pmatrix} = \underset{\substack{\uparrow \\ \text{Hadamard matrix}}}{H} \begin{pmatrix} |0\rangle \\ |1\rangle \end{pmatrix}$$

or $\left(H^2 = 1 \ ; \ H = H^{-1}\right)$ $\quad \begin{pmatrix} |0\rangle \\ |1\rangle \end{pmatrix} = H \begin{pmatrix} |0'\rangle \\ |1'\rangle \end{pmatrix}$

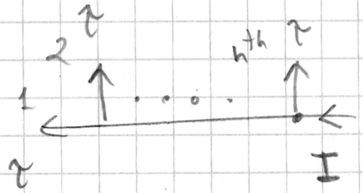
$\Rightarrow \boxed{F = H} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$

## II. Fibonacci anyons

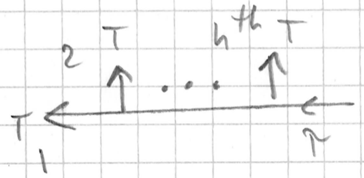
$\tau \times \tau = I + \tau \ , \ \tau \times I = \tau$

(a) • Let at $n^{th}$ step

$\boxed{\mathcal{H}_b^n \to \text{Hilbert space of } n \ \tau\text{-anyons fused to } b}$

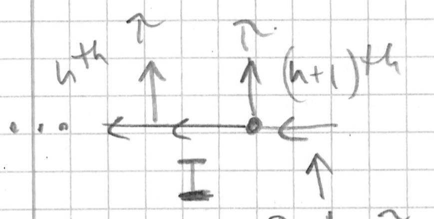
$\#$ 

$= F_{n-2}$

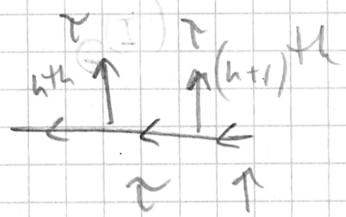
$\#$ 

$= F_{n-1}$

• Consider $(n+1)^{th}$ step



Only $\tau$



either $I$ or $\tau$

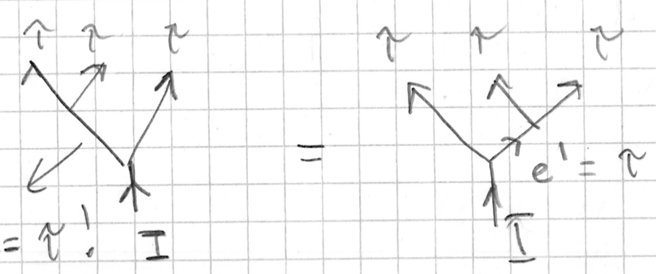
$\dim\left(\mathcal{H}_I^{(n+1)}\right) = F_{n-1}$

$\Rightarrow \dim\left(\mathcal{H}_\tau^{(n+1)}\right) = F_{n-2} + F_{n-1} = F_n$ !

(6) Initial step of induction: $\{|0\rangle, |1\rangle\} \to$ basis in $H^{(3)}_{\tau}$, $F_2 = 2$

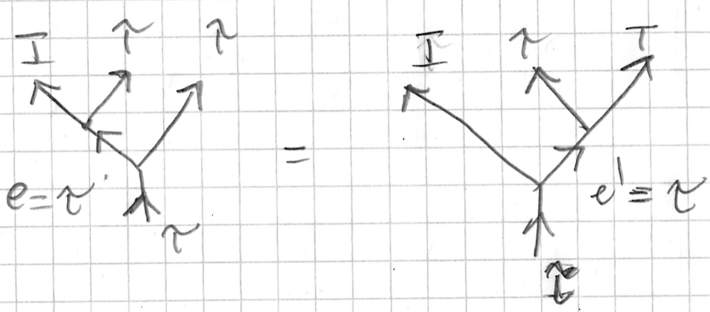
$|N\rangle \to$ basis in $H^{(3)}_{I}$, $F_1 = 1$

(b) $V^{\tau\tau\tau}_{I}$:




$\Rightarrow \left(F^{\tau\tau\tau}_{I}\right)^{\tau}_{\tau} = 1$

$V^{I\tau\tau}_{\tau}$




$\Rightarrow \left(F^{I\tau\tau}_{\tau}\right)^{\tau}_{\tau} = 1$

Pentagon equations:

$$\left(F^{\tau\tau c}_{\tau}\right)^{d}_{a} \cdot \left(F^{a\tau\tau}_{\tau}\right)^{c}_{b} = \sum_e \left(F^{\tau\tau\tau}_{d}\right)^{c}_{e} \cdot \left(F^{\tau e\tau}_{\tau}\right)^{d}_{b} \cdot \left(F^{\tau\tau\tau}_{b}\right)^{e}_{a}$$

We set $a = d = I$ & $b = c = \tau$

$$\Rightarrow \left(F^{\tau\tau\tau}_{\tau}\right)^{I}_{I} \underbrace{\left(F^{I\tau\tau}_{\tau}\right)^{\tau}_{\tau}}_{=1} = \sum_e \underbrace{\left(F^{\tau\tau\tau}_{I}\right)^{\tau}_{e}}_{=1} \cdot \quad \Downarrow e = \tau \,!$$

$$\cdot \left(F^{\tau\tau\tau}_{\tau}\right)^{I}_{\tau} \left(F^{\tau\tau\tau}_{\tau}\right)^{\tau}_{I} \Rightarrow$$

$$\left(F^{\tau\tau\tau}_{\tau}\right)^{I}_{I} = \left(F^{\tau\tau\tau}_{\tau}\right)^{I}_{\tau} \cdot \left(F^{\tau\tau\tau}_{\tau}\right)^{\tau}_{I}$$

→ see opposite site

(7) Let $F = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ & $F^T = F^{-1}$ (b)

with $\det F = -1$ (a)

$\Rightarrow$ (a) $ad - bc = -1$

(b) $F^{-1} = \begin{pmatrix} -d & b \\ c & -a \end{pmatrix} = F^T = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$

$\Rightarrow$ $b = c$ and $a = -d$ $\Rightarrow$

$F = \begin{pmatrix} a & b \\ b & -a \end{pmatrix} \Rightarrow \begin{cases} -a^2 - b^2 = -1 & \leftarrow \det = -1 \\ a = b^2 & \leftarrow \text{Pent. eq.} \end{cases}$

$\Rightarrow$ $a^2 + a - 1 = 0$ $\quad \Delta = 1 + 4 = 5$

$a = \cancel{\frac{-1 \pm \sqrt{5}}{2}}$ $\Rightarrow$ $a = \frac{-1+\sqrt{5}}{2} > 0$ since $a = b^2$

$\parallel$ $\varphi^{-1}$

$\Rightarrow F^{TTT}_{T} = \begin{pmatrix} \varphi^{-1} & \varphi^{-1/2} \\ \varphi^{-1/2} & -\varphi^{-1} \end{pmatrix}$ $\qquad \Rightarrow b = \sqrt{\varphi}$

$\underset{a}{\frac{-1+\sqrt{5}}{2}} \cdot \underbrace{\frac{1+\sqrt{5}}{2}}_{\varphi} = \frac{5-1}{4} = 1$ $\quad \Rightarrow a = \frac{-1+\sqrt{5}}{2} = \varphi^{-1}$