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Entangled Phases of Matter, WS 2020/21

Exercise sheet 5

This exercise will be discussed on 11.2.2021

1. Topological entanglement entropy

We discussed in the lecture that the entanglement in a state with topological order fulfills an area law with a topological correction , i.e., the von Neumann entanglement entropy

$$S_A = -\operatorname{Tr}(\rho_A \log \rho_A) \tag{1}$$

of a simply connected[1] region A is given by

$$S_A = c|\partial A| - \gamma \ , \tag{2}$$

where $|\partial A|$ denotes the length of the boundary ∂A and

$$-\gamma = S_{\text{topo}} \tag{3}$$

is called the topological entanglement entropy. We discussed how to compute S_{topo} by adding and subtracting the entanglement entropies of appropriately chosen regions. In this exercise we will consider how to calculate S_{topo} from an alternative set of regions and show that

$$S_{\text{topo}} = \frac{1}{2}[(S_A - S_B) - (S_C - S_D)] \ , \tag{4}$$

where the regions A, B, C, D are defined below.

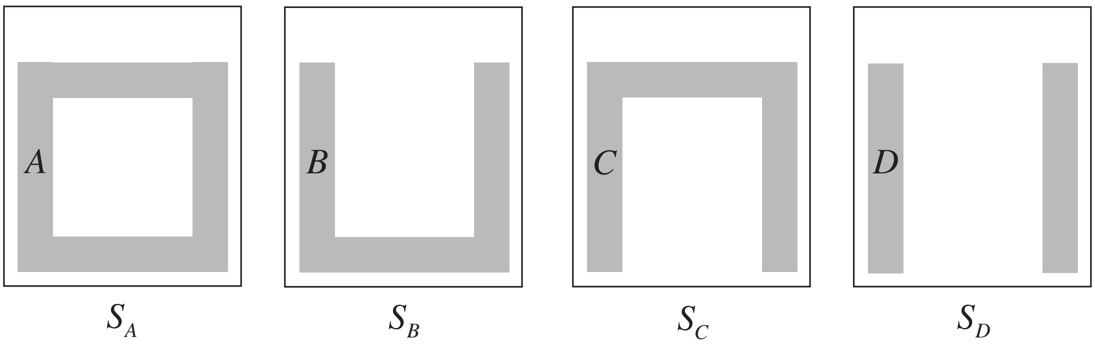

Figure 1: Definition of the regions A, B, C, D (from Fig. 9.6. of Ref. [1]).

To evaluate Eq. (4), we first need to consider how Eq. (2) is modified in the case, that the region A is not simply connected. Here we consider two ways to convince us of the fact that

$$S_A = c|\partial A| - n\gamma \ , \tag{5}$$

if ∂A consists of n disconnected parts.

[1] a disk-like region without any holes

a) Flat spectrum

We checked in the lecture that a tensor network patch of the toric code state on a simply connected region is invariant under applying the operator $V = \sigma_x \otimes \sigma_x \otimes \ldots \otimes \sigma_x$ to all virtual indices (see Fig. 2) and deduced that the rank of the reduced density matrix is reduced due to the presence of this symmetry by a fatctor 1/2. For a patch A with two disconnected boundaries similar considerations[2] lead to the fact that

$$\mathrm{rank}(\rho_A) = 2^{|\partial A|}/4 \, . \tag{6}$$

Assume that spectrum of ρ_A is flat, i.e., all its eigenvalues have the same magnitude. Show that in this case

$$\log \mathrm{rank}(\rho_A) = -\,\mathrm{Tr}(\rho_A \log \rho_A) \tag{7}$$

and conclude that

$$S_A = |\partial A| \log 2 - 2 \log 2 \, . \tag{8}$$

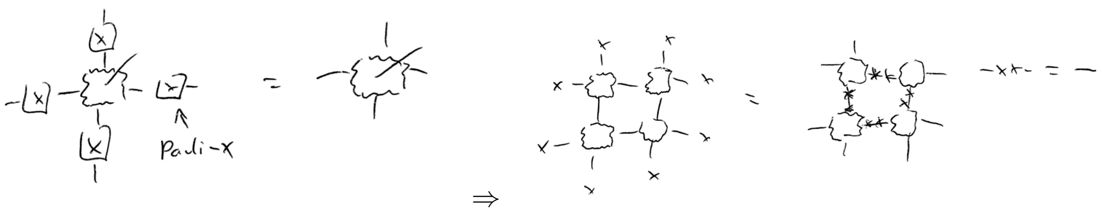

Figure 2: Virtual symmetries of the toric code PEPS. The symmetry of individual tensors (left) implies the symmetry on patches (right).

Solution

If the spectrum of the reduced density matrix ρ is flat and its rank is r, we can write as a diagonal matrix with r entries of magnitude $1/r$, such that the condition $\mathrm{Tr}(\rho) = 1$ is fulfilled. The von Neumann entropy is then calculated as

$$S(\rho) = -\,\mathrm{Tr}(\rho \log \rho) = -\sum_{i=1}^{r} \frac{1}{r} \log(1/r) = -\log(1/r) = \log r \, . \tag{9}$$

So we conclude that in the case of a flat spectrum, the von Neumann entanglement entropy is just the logarithm of the rank. For $\mathrm{rank}(\rho_A) = 2^{|\partial A|}/4$, we find

$$S_A = \log(2^{|\partial A|}/4) = \log(2^{|\partial A|-2}) = |\partial A| \log 2 - 2 \log 2 \, . \tag{10}$$

Additional information – non-simply connected region (toric code PEPS)

In the following we calculate the reduced density matrix in the tensor network formalism for the toric code for on a non-simply connected region. We checked in the lecture that a tensor network patch of the toric code state on a simply connected region is invariant under applying the operator $V = \sigma_x \otimes \sigma_x \otimes \ldots \otimes \sigma_x$ to all virtual indices (see Fig. 3) and used this argument to deduce that the rank of the reduced density matrix of a simply connected region $\mathcal{R}$ is

$$\mathrm{rank}(\rho_{\mathcal{R}}) = 2^{|\partial \mathcal{R}|}/2 \, , \tag{11}$$

because the symmetry halves the space.

This intuition is formalized by the following observations. Denote a single tensor from virtual to physical space by $A : V_v^4 \to V_p$ (four virtual to one physical index). For calculating the reduced

[2] A proof will be provided in the tutorial or on the solution sheet.

density matrix, we encounter the following operator $A \otimes A^\dagger$. Tracing out the physical indices, the operator $\mathrm{Tr}_p(A \otimes A^\dagger) : V_v^4 \to V_v^4$ defines a map from virtual space to virtual space. In agreement with the virtual symmetry of the A-tensor, we find

$$\mathrm{Tr}_p(A \otimes A^\dagger) = \mathbb{1} \otimes \mathbb{1} \otimes \mathbb{1} \otimes \mathbb{1} + \hat{x} \otimes \hat{x} \otimes \hat{x} \otimes \hat{x} \, , \tag{12}$$

where $\hat{x}$ denotes the Pauli matrix. Up to an unimportant normalization factor the operator on the right is a projector, which halves the space. We can check, that just like the symmetry, the projector property Eq. (12) holds on all patches. More formally, we write a tensor network patch on a simply connected region $\mathcal{R}$ as $A_\mathcal{R}$ and define the projector on its boundary as

$$P_{|\partial\mathcal{R}|} = \mathbb{1}^{\otimes|\partial\mathcal{R}|} + \hat{x}^{\otimes|\partial\mathcal{R}|} \, . \tag{13}$$

Eq. (12), the projector property for a single tensor, then generalizes to

$$\mathrm{Tr}_\mathcal{R}(A_\mathcal{R} \otimes A^\dagger_\mathcal{R}) = P_{|\partial\mathcal{R}|} \, , \tag{14}$$

where $\mathrm{Tr}_\mathcal{R}$ denotes the trace over all physical indices in the region $\mathcal{R}$. For calculating the reduced density matrix of a simply connected region $\mathcal{R}$, we consider a sphere that is divided into the region $\bar{\mathcal{R}}$ and its complement $\mathcal{R}$ and calculate

$$\rho_\mathcal{R} = \mathrm{Tr}_{\bar{\mathcal{R}}}(A_{\mathcal{R}\cup\bar{\mathcal{R}}} \otimes A^\dagger_{\mathcal{R}\cup\bar{\mathcal{R}}}) = (A_\mathcal{R} \otimes A^\dagger_\mathcal{R}) P_{\partial\bar{\mathcal{R}}} \, . \tag{15}$$

We note that the rank of $(A_\mathcal{R} \otimes A^\dagger_\mathcal{R})$ is upper bounded by $2^{|\partial\mathcal{R}|}$ (the number of virtual indices on the boundary is the bottleneck). The effect of $P_{\partial\bar{\mathcal{R}}}$ is to project out half the space and as a consequence we obtain Eq. (11).

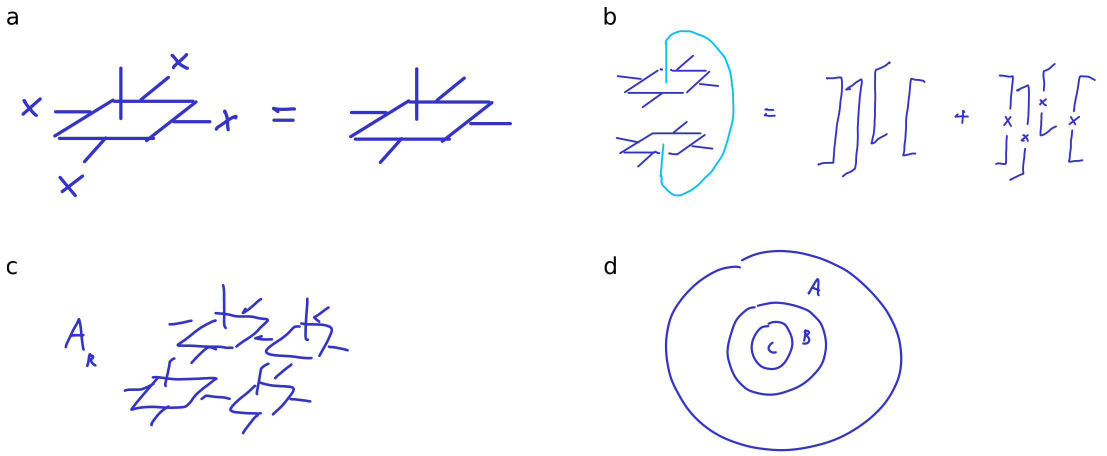

Figure 3: a) Virtual symmetry. b) Projector property for a single site (Eq. (12)). c) Definition of a patch $A_\mathcal{R}$ for the example of $\mathcal{R}$ a 2×2-square. d) Regions $\mathcal{A}, \mathcal{B}, \mathcal{C}$. The region $\mathcal{A} \cup \mathcal{B} \cup \mathcal{C}$ is a sphere (not a disk).

We can now apply Eq. (15) to calculate the reduced density matrix on a region with a topology of an annulus. To this end, we divide the sphere into regions $\mathcal{A}, \mathcal{B}, \mathcal{C}$ as shown in Fig. 3d and evaluate

$$\rho_\mathcal{B} = \mathrm{Tr}_{\mathcal{A}\cup\mathcal{C}}(A_{\mathcal{A}\cup\mathcal{B}\cup\mathcal{C}} \otimes A^\dagger_{\mathcal{A}\cup\mathcal{B}\cup\mathcal{C}}) \, . \tag{16}$$

We start by tracing out $\mathcal{C}$ and apply Eq. (15) to find

$$\begin{aligned} \rho_\mathcal{B} &= \mathrm{Tr}_\mathcal{A}(\mathrm{Tr}_\mathcal{C}(A_{\mathcal{A}\cup\mathcal{B}\cup\mathcal{C}} \otimes A^\dagger_{\mathcal{A}\cup\mathcal{B}\cup\mathcal{C}})) \\ &= \mathrm{Tr}_\mathcal{A}((A_{\mathcal{A}\cup\mathcal{B}} \otimes A^\dagger_{\mathcal{A}\cup\mathcal{B}}) P_{\partial\mathcal{C}}) \, . \end{aligned}$$

Next, we trace out $\mathcal{A}$

$$\begin{aligned}\rho_{\mathcal{B}} =& \operatorname{Tr}_{\mathcal{A}}((A_{\mathcal{A}\cup\mathcal{B}} \otimes A^{\dagger}_{\mathcal{A}\cup\mathcal{B}})P_{\partial\mathcal{C}}) \\ =& P_{\partial\mathcal{A}}(A_{\mathcal{B}} \otimes A^{\dagger}_{\mathcal{B}})P_{\partial\mathcal{C}} \ .\end{aligned}$$

The crucial observation is know that the two symmetry projectors $P_{\partial\mathcal{A}}$ and $P_{\partial\mathcal{C}}$ are independent and each reduce the $2^{|\partial\mathcal{B}|}$-dimensional space of virtual indices by a factor 2. Thus, we have

$$\operatorname{rank}(\rho_{\mathcal{B}}) = 2^{|\partial\mathcal{B}|}/4 \ . \tag{17}$$

b) Entropy of a non-simply connected region – counting states

Alternatively, we can convince ourselves that the rank of the reduced density matrix on a region A with n disconnected components of the boundary ∂A of the toric code state is

$$\operatorname{rank}(\rho_A) = 2^{|\partial A|}/2^n \tag{18}$$

by counting the number of allowed states. For a boundary of size $|\partial A|$, there are $2^{|\partial A|}$ boundary spins. Recall that the toric code ground state is a superposition of all closed loops of strings ('string' means $|1\rangle$, 'no string' means $|0\rangle$). Convince yourself that as a consequence the number of spins in state $|1\rangle$ on a connected component of the boundary is even. Do so by drawing pictures. Consider first the case, where the region A is simply connected and the boundary is just a single circle. Convince yourself that this 'mechanism' reduces the number of allowed boundary configurations by a factor of two for every boundary component and conclude that we obtain Eq. (18). As in the previous exercise we can conclude that Eq. (8) holds for a region with two disconnected boundary components.

Solution

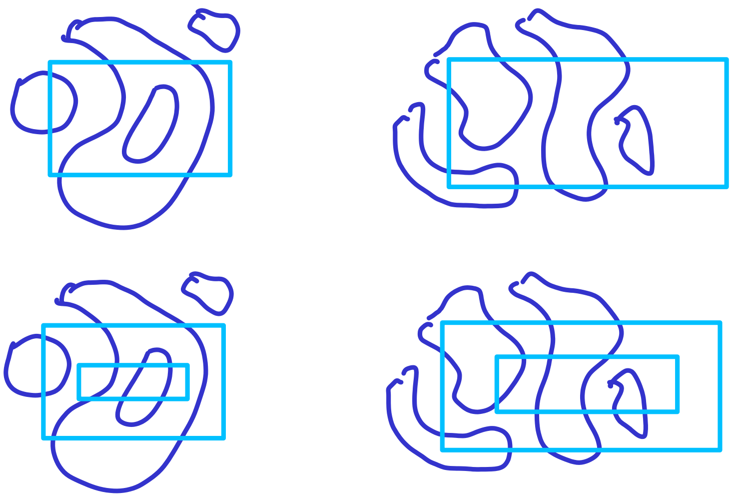

Figure 4: A simply vs non-simply connected region (top vs bottom) for different states in the loop soup superposition (left and right).

We consider the loop soup groundstate of the toric code. It is a superposition of all possible closed loop states. A state with only closed loops has the property that strings cross the boundary of a region an even number of times (see Fig. 4). This means, the number of spins on the boundary that is in the up state is even. Thus, the number of possible boundary configurations is reduced by a factor of two. If there are two disconnected boundaries, we have two independent constraints, because the number of spins in the up state needs to be even for each component separately. Since each boundary configuration leads to a unique bulk state (superposition of all closed loop patterns compatible with the boundary condition), we find that the rank of the reduced density matrix for a region $\mathcal{R}$ is $2^{|\partial \mathcal{R}|-n}$, where n is the number of disconnected boundary components. For a flat spectrum, this results in Eq. (5)where $c = \gamma = \log 2$.

The idea is due to Levin and Wen, who came up with interesting ways to generalize the toric code in so called string-net models (cf. Ref. [4]). Their construction for the topological entanglement entropy can be found in Ref. [3].

c) Calculating S_{topo}

We checked in the previous two exercises that the entanglement entropy for a region A with two boundary components is $S_A = |\partial A| \log 2 - 2 \log 2$ in the case of the toric code. This is consistent with Eq. (5). We are not ready to compute S_{topo}. Use Eq. (5) to calculate S_{topo} given in Eq. (4) similar to the calculation performed in the lecture. Comment: This exercise is independent of the previous two and can be done without solving a) and b).

Solution

We assume that Eq. (5) is valid and calculate S_{topo} using Eq. (4). The key observation is that the boundary contributions of $S_A - S_B$ cancels with the boundary components of $S_C - S_D$. The boundary can be expressed as two parallel lines of a square in horizontal and vertical direction, which we denote by ∂R_h and ∂R_h. We find $\partial A - \partial B = \partial C - \partial D = \partial R_h - \partial R_v$ and evaluate

$$\begin{aligned} S_{\text{topo}} =& \frac{1}{2}[(c|\partial A| - 2\gamma - c|\partial B| + \gamma) - (c|\partial C| - \gamma - c|\partial D| + 2\gamma)] \\ =& \frac{1}{2}[c(\partial R_h - \partial R_v) - \gamma) - (c(\partial R_h - \partial R_v) + \gamma)] = -\gamma \ . \end{aligned}$$

2. The tube algebra of the toric code

This exercise is just for fun. We introduced the tube algebra of the toric code in the lecture (see Fig. 5). In this exercise we will take a closer look at it. An algebra is a vector space on which one can define multiplication of elements. The most well known algebra for physicists are matrix algebras. For example, the algebra of 2×2 matrices, where we can add and multiply matrices. For the tube algebra, addition and scalar multiplication is defined formally by simply writing e.g., $1 + b$. Multiplication is defined by gluing to tubes together and using the string deformation rule (Fig.5). For example we have seen in the lecture that $b \cdot b = 1$.

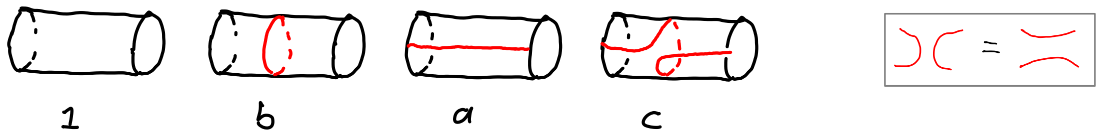

Figure 5: Left. Elemantary elements of the tube algebra of the toric code. Right. Deformation rule.

A good object to characterize algebras are the so-called *central idempotents*

$$P_i \cdot P_j = \delta_{ij} P_j \ . \tag{19}$$

These are orthogonal projectors which span the vector space.

We have already seen a vague correspondence between tube algebra elements and anyonic excitations in the lecture. In more detail, one can show that the anyons correspond to the central idempotents. In this exercise we calculate the central idempotents. We will convince ourselves that they correspond to the anyons of the toric code in the tutorial (if you are interested).

a) Multiplication table

To calculate the central idempotents, calculate the multiplication table of $1, a, b, c$. Hint: We impose that a tube with string ending on its boundary can not be glued to a tube without a string at the boundary, e.g., $1 \cdot c = 0$. Check, that the multiplication is commutative. Thus, it suffices to calculate only the six products $1 \cdot 1, 1 \cdot b, b \cdot b, a \cdot a, a \cdot c, c \cdot c$. All the others are obtained by commutativity or zero.

Solution

It is trivial to verify

$$1 \cdot 1 = 1, \quad 1 \cdot b = b \ , a \cdot a = a, a \cdot c = c$$

The identities

$$b \cdot b = 1, \quad c \cdot c = a \tag{20}$$

are proven using the deformation rule in Fig. 6.

b) Idempotents

Calculate the central idempotents, i.e, find linear combination $\alpha 1 + \beta b + \gamma a + \delta c$ of tubes which fulfill Eq. (19). Hint: We already know that multiplying a linear combinations of 1 and b with a linear combination of a and c yields 0. Thus, it makes sense to try to use the ansatz

$$P_1 = \alpha_1 1 + \beta_1 b \ , \quad P_2 = \alpha_2 1 + \beta_2 b \ , \quad P_3 = \gamma_1 a + \delta_1 c \ , \quad P_4 = \gamma_2 a + \delta_2 c \ . \tag{21}$$

Find P_1, P_2, P_3, P_4 using this ansatz, i.e., determine the coefficients $\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_1, \gamma_2, \delta_1, \delta_2$ such that Eq. (19) holds.

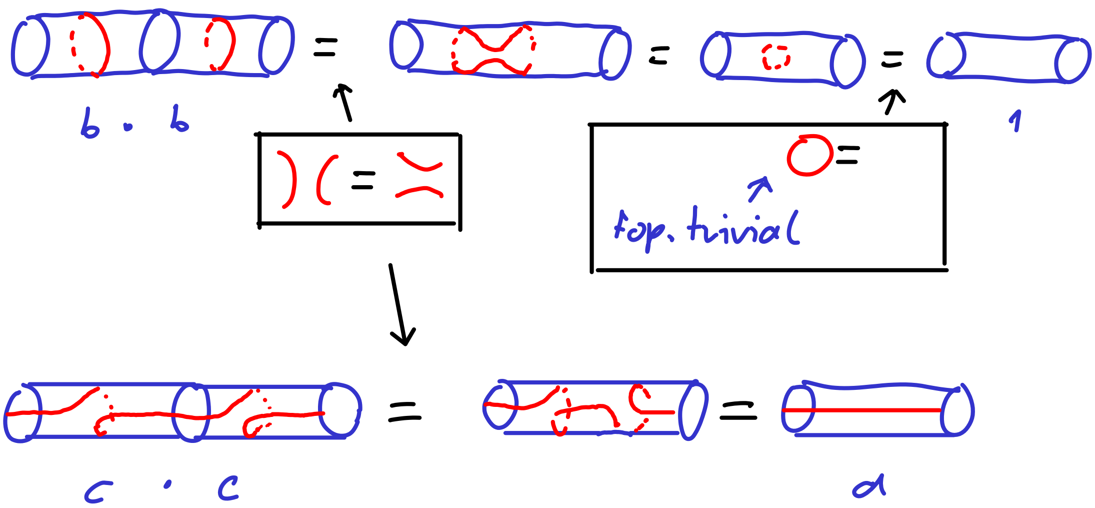

Figure 6: Proving the identities $b \cdot b = 1$ and $c \cdot c = a$.

Solution
We can think of 1 and b as $\mathbb{1}$ and $\mathbf{x}$, because they behave in the same way under multiplication. This suggests to use

$$P_1 = \frac{1}{2}(1+b)\ , \quad P_2 = \frac{1}{2}(1-b) \tag{22}$$

as the two central idempotents of the first 'block'. We check that

$$P_1 \cdot P_2 = \frac{1}{4}(1 \cdot 1 - 1 \cdot b + b \cdot 1 - b \cdot b) = \frac{1}{4}(1-1) = 0\ . \tag{23}$$

Similarly, we identify a and c with 1 and $\hat{x}$ (of an algebra independent from the one spanned by $1, b$, think of different blocks in a block diagonal matrix) and guess

$$P_3 = \frac{1}{2}(a+c)\ , \quad P_4 = \frac{1}{2}(a-c)\ . \tag{24}$$

We verify that

$$P_3 \cdot P_4 = 0\ . \tag{25}$$

The other relations follow from commutativity and the fact that products of the form $x \cdot y$ with $x = 1, b$ and $y = a, c$ are zero.
Relation to anyons (vertex and plaquette violations)
We can identify P_1 with the trivial particle (vacuum). This is because it corresponds to a superposition of 'no loop' and 'closed loop' with the right sign factors. The idempotent P_2 corresponds to a violation of the plaquette operator, because the configuration 'loop' carries a negative sign, which corresponds to the effect of a single z-operator like it occurs at the end of the familiar z-string. P_3 corresponds to a violation of an vertex operators, i.e., the presence of an open string (more formally, a superposition of a bare string and a 'string+loop' configuration). Lastly, P_4 is again a superposition of two variant of an open string, so it constitutes a violation of the vertex operator, but also, the superposition is with the 'wrong sign' – the 'no loop' vs. 'loop' configuration differs by a minus sign. Thus, the plaquette constraint is violated as well and we can identify P_4 with the compound particle $(e, m) = \epsilon$.
Some more insights in how to calculate with tubes is provided in Ref.[2], but this paper is formulated using tensor networks and it is not an easy read.

References

[1] Jiannis K. Pachos, *Introduction to Topological Quantum Computation*, (Cambridge University Press, 2012)

[2] Nick Bultinck, Michael Mariën, Dominic J. Williamson, Mehmet B. Sahinoğlu, Jutho Haegeman, Frank Verstraete, *Anyons and matrix product operator algebras*, https://arxiv.org/abs/1511.08090

[3] Michael Levin, Xiao-Gang Wen *Detecting topological order in a ground state wave function*, https://arxiv.org/abs/cond-mat/0510613.

[4] Michael Levin, Xiao-Gang Wen *String-net condensation: A physical mechanism for topological phases*, https://arxiv.org/abs/cond-mat/0404617