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Entangled Phases of Matter, WS 2020/21

Exercise sheet 5

This exercise will be discussed on 11.2.2021

1. Topological entanglement entropy

We discussed in the lecture that the entanglement in a state with topological order fulfills an area law with a topological correction , i.e., the von Neumann entanglement entropy

$$S_A = -\operatorname{Tr}(\rho_A \log \rho_A) \tag{1}$$

of a simply connected[1] region A is given by

$$S_A = c|\partial A| - \gamma \ , \tag{2}$$

where $|\partial A|$ denotes the length of the boundary ∂A and

$$-\gamma = S_{\text{topo}} \tag{3}$$

is called the topological entanglement entropy. We discussed how to compute S_{topo} by adding and subtracting the entanglement entropies of appropriately chosen regions. In this exercise we will consider how to calculate S_{topo} from an alternative set of regions and show that

$$S_{\text{topo}} = \frac{1}{2}[(S_A - S_B) - (S_C - S_D)] \ , \tag{4}$$

where the regions A, B, C, D are defined below.

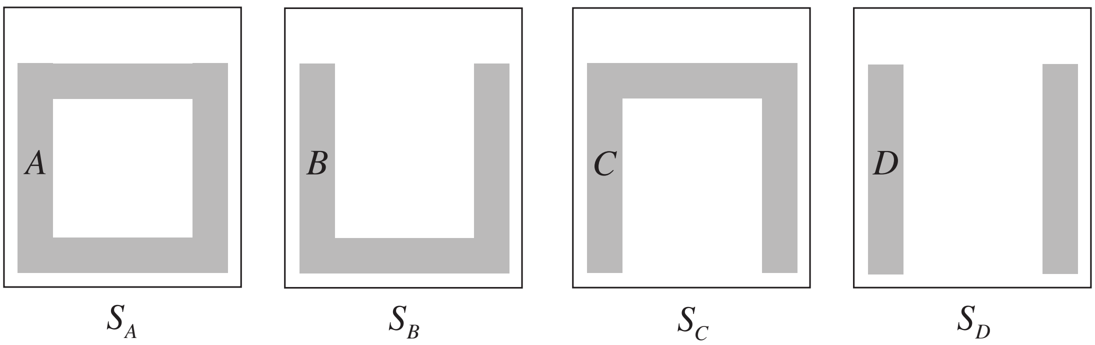

Figure 1: Definition of the regions A, B, C, D (from Fig. 9.6. of Ref. [1]).

To evaluate Eq. (4), we first need to consider how Eq. (2) is modified in the case, that the region A is not simply connected. Here we consider two ways to convince us of the fact that

$$S_A = c|\partial A| - n\gamma \ , \tag{5}$$

if ∂A consists of n disconnected parts.

[1] a disk-like region without any holes

a) Flat spectrum

We checked in the lecture that a tensor network patch of the toric code state on a simply connected region is invariant under applying the operator $V = \sigma_x \otimes \sigma_x \otimes \ldots \otimes \sigma_x$ to all virtual indices (see Fig. 2) and deduced that the rank of the reduced density matrix is reduced due to the presence of this symmetry by a fatctor 1/2. For a patch A with two disconnected boundaries similar considerations[2] lead to the fact that

$$\mathrm{rank}(\rho_A) = 2^{|\partial A|}/4 \; . \tag{6}$$

Assume that spectrum of ρ_A is flat, i.e., all its eigenvalues have the same magnitude. Show that in this case

$$\log \mathrm{rank}(\rho_A) = -\operatorname{Tr}(\rho_A \log \rho_A) \tag{7}$$

and conclude that

$$S_A = |\partial A| \log 2 - 2 \log 2 \; . \tag{8}$$

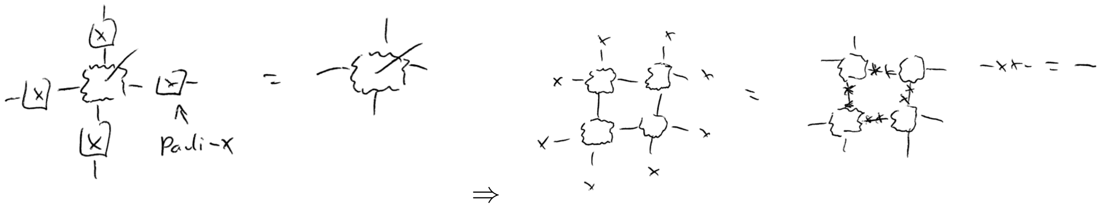

Figure 2: Virtual symmetries of the toric code PEPS. The symmetry of individual tensors (left) implies the symmetry on patches (right).

b) Entropy of a non-simply connected region – counting states

Alternatively, we can convince ourselves that the rank of the reduced density matrix on a region A with n disconnected components of the boundary ∂A of the toric code state is

$$\mathrm{rank}(\rho_A) = 2^{|\partial A|}/2^n \tag{9}$$

by counting the number of allowed states. For a boundary of size $|\partial A|$, there are $2^{|\partial A|}$ boundary spins. Recall that the toric code ground state is a superposition of all closed loops of strings ('string' means $|1\rangle$, 'no string' means $|0\rangle$). Convince yourself that as a consequence the number of spins in state $|1\rangle$ on a connected component of the boundary is even. Do so by drawing pictures. Consider first the case, where the region A is simply connected and the boundary is just a single circle. Convince yourself that this 'mechanism' reduces the number of allowed boundary configurations by a factor of two for every boundary component and conclude that we obtain Eq. (9). As in the previous exercise we can conclude that Eq. (8) holds for a region with two disconnected boundary components.

c) Calculating S_{topo}

We checked in the previous two exercises that the entanglement entropy for a region A with two boundary components is $S_A = |\partial A| \log 2 - 2 \log 2$ in the case of the toric code. This is consistent with Eq. (5). We are not ready to compute S_{topo}. Use Eq. (5) to calculate S_{topo} given in Eq. (4) similar to the calculation performed in the lecture. Comment: This exercise is independent of the previous two and can be done without solving a) and b).

[2] A proof will be provided in the tutorial or on the solution sheet.

2. The tube algebra of the toric code

This exercise is just for fun. We introduced the tube algebra of the toric code in the lecture (see Fig. 3). In this exercise we will take a closer look at it. An algebra is a vector space on which one can define multiplication of elements. The most well known algebra for physicists are matrix algebras. For example, the algebra of 2×2 matrices, where we can add and multiply matrices. For the tube algebra, addition and scalar multiplication is defined formally by simply writing e.g., $1 + b$. Multiplication is defined by gluing to tubes together and using the string deformation rule (Fig.3). For example we have seen in the lecture that $b \cdot b = 1$.

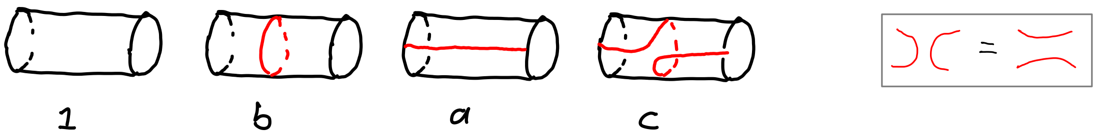

Figure 3: Left. Elemantary elements of the tube algebra of the toric code. Right. Deformation rule.

A good object to characterize algebras are the so-called *central idempotents*

$$P_i \cdot P_j = \delta_{ij} P_j \ . \tag{10}$$

These are orthogonal projectors which span the vector space.

We have already seen a vague correspondence between tube algebra elements and anyonic excitations in the lecture. In more detail, one can show that the anyons correspond to the central idempotents. In this exercise we calculate the central idempotents. We will convince ourselves that they correspond to the anyons of the toric code in the tutorial (if you are interested).

a) Multiplication table

To calculate the central idempotents, calculate the multiplication table of $1, a, b, c$. Hint: We impose that a tube with string ending on its boundary can not be glued to a tube without a string at the boundary, e.g., $1 \cdot c = 0$. Check, that the multiplication is commutative. Thus, it suffices to calculate only the six products $1 \cdot 1, 1 \cdot b, b \cdot b, a \cdot a, a \cdot c, c \cdot c$. All the others are obtained by commutativity or zero.

b) Idempotents

Calculate the central idempotents, i.e, find linear combination $\alpha 1 + \beta b + \gamma a + \delta c$ of tubes which fulfill Eq. (10). Hint: We already know that multiplying a linear combinations of 1 and b with a linear combination of a and c yields 0. Thus, it makes sense to try to use the ansatz

$$P_1 = \alpha_1 1 + \beta_1 b \ , \quad P_2 = \alpha_2 1 + \beta_2 b \ , \quad P_3 = \gamma_1 a + \delta_1 c \ , \quad P_4 = \gamma_2 a + \delta_2 c \ . \tag{11}$$

Find P_1, P_2, P_3, P_4 using this ansatz, i.e., determine the coefficients $\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_1, \gamma_2, \delta_1, \delta_2$ such that Eq. (10) holds.

References

[1] Jiannis K. Pachos, *Introduction to Topological Quantum Computation*, (Cambridge University Press, 2012)