Entangled Phases of Matter

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Meetings via Zoom - the same link as of today!
Tuesdays & Thursdays, 16:00-17:30

Tutorial sessions (6LP), each 2nd week, either Tue or Thu

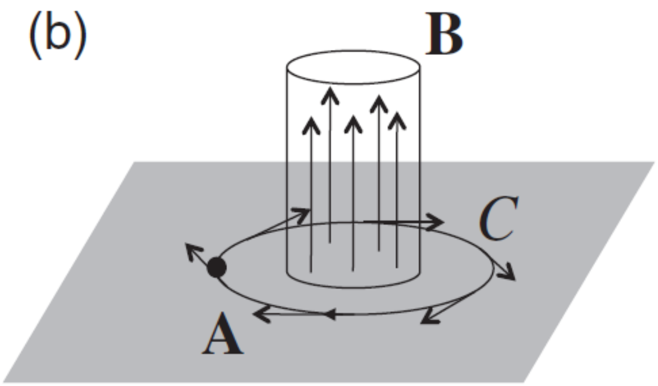
[1] Jiannis K. Pachos, "Introduction to Topological Quantum Computation"
[2] David Tong, "Lectures on the Quantum Hall Effect"
[3] Steven H. Simon, "Topological Quantum: Lecture Notes and Proto-Book"
[4] Xiao-Gang Wen, "Quantum Field Theory of Many-body Systems"

Anyons and Aharonov-Bohm effect

We discuss the physics of 2D world

Wave-function acquires the phase:

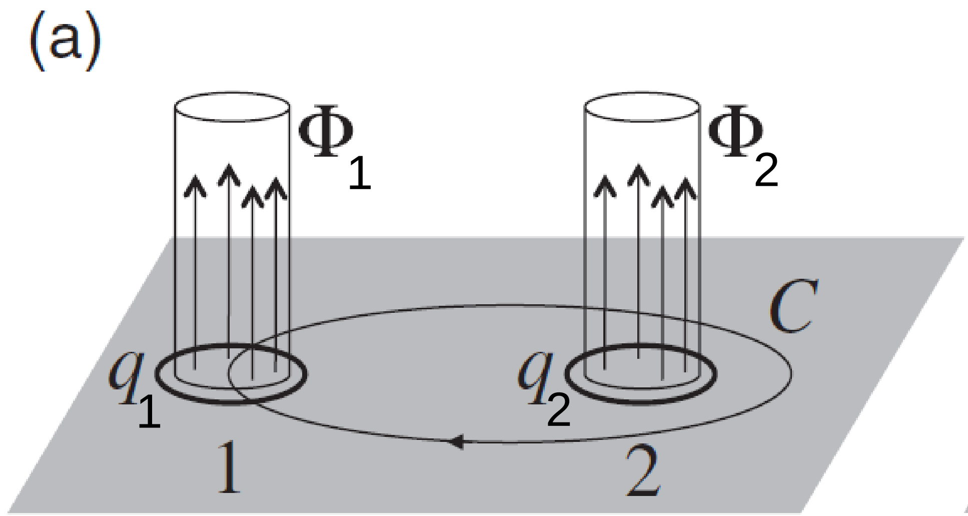
$$\varphi = \frac{q}{c\hbar} \oint_C \mathbf{A} \cdot d\mathbf{r}.$$



Employing Stokes' theorem

$$\varphi = \frac{q}{c\hbar} \iint_{S(C)} \nabla \times \mathbf{A} \cdot d\mathbf{s} = \frac{q}{c\hbar} \iint_{S(C)} \mathbf{B} \cdot d\mathbf{s} = \frac{q}{c\hbar} \Phi.$$

Anyons and Aharonov-Bohm effect



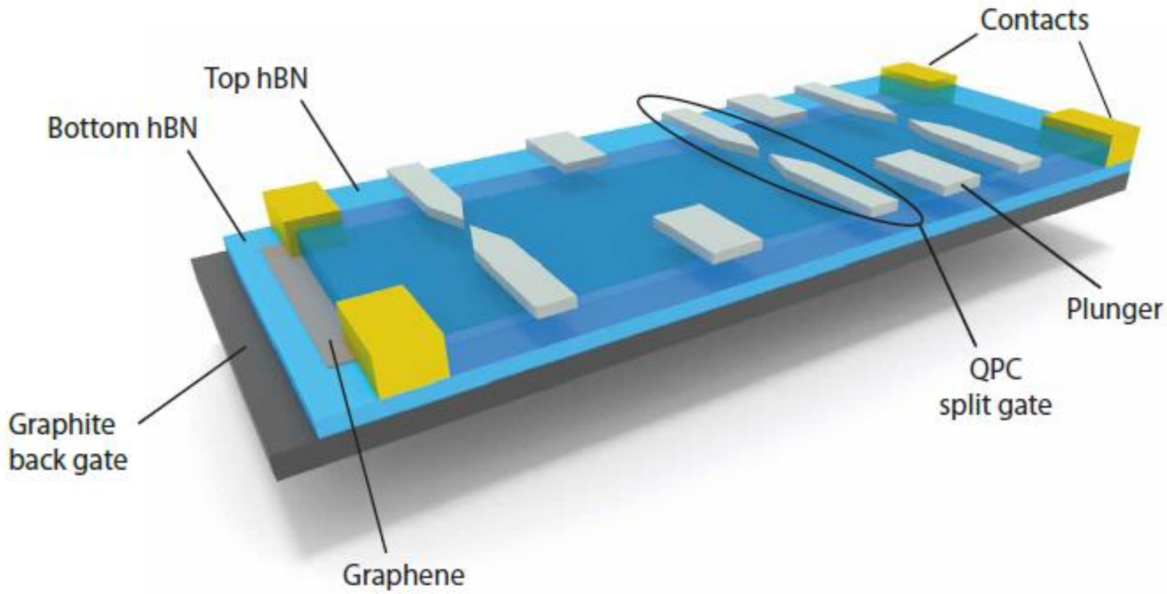
The phase acquired by the 1st particle is $\varphi = q_1\Phi_2 + q_2\Phi_1$

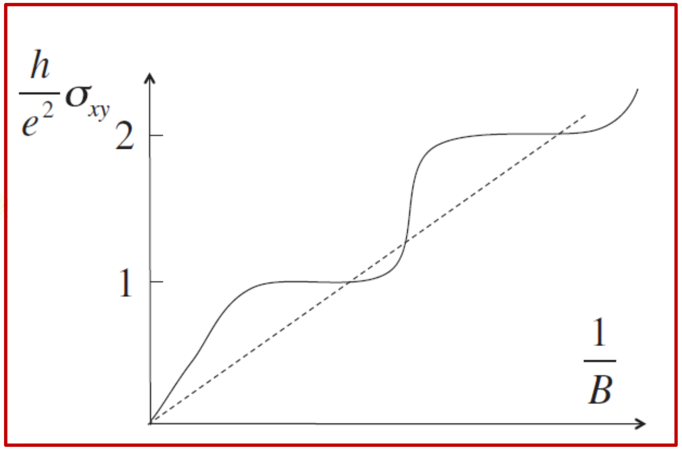
The phase under exchange of two identical particles is

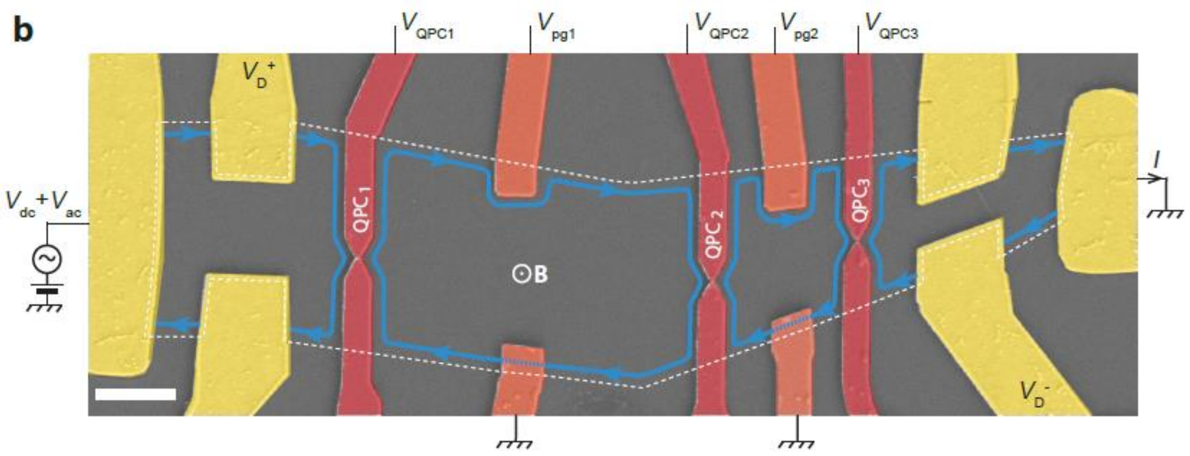
$$\varphi = q\Phi$$

Quantum Hall effect (QHE)

Realization: 2DEG or graphene







$$\sigma_{xy} = \nu \frac{e^2}{h}$$

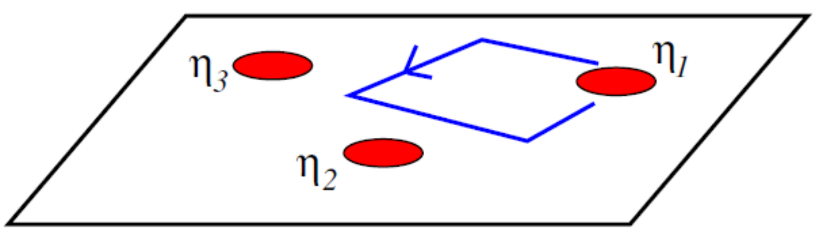
$$\nu=2,\ 1,\ \frac{2}{3},\ \frac{1}{3},\ \frac{1}{5},\ \ldots$$

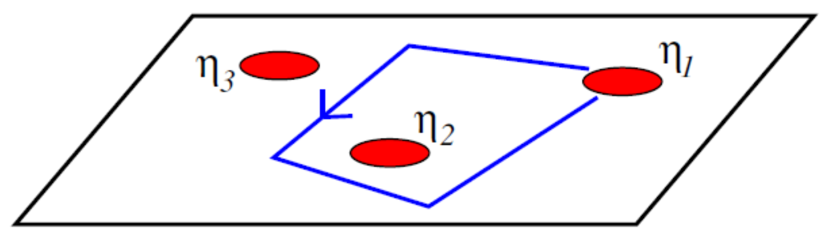
Fractional QHE

$$\nu = \frac{1}{m}, \quad \mathrm{m} = (2\mathrm{p}+1), \mathrm{p\ being\ integer}$$

FQHE state is described by many-body (!) Laughlin wave function

$$\psi_{\text{hole}}(z;\eta) = \prod_{i=1}^{N}(z_i - \eta)\prod_{k<l}(z_k - z_l)^m \, e^{-\sum_{i=1}^{n}|z_i|^2/4l_B^2}$$





Charge $q = \frac{e}{m}$

Statistical phase $\varphi = \frac{\pi}{m}$

2D spinless p+ip superconductor

Realization: FQHE at $\nu=5/2$

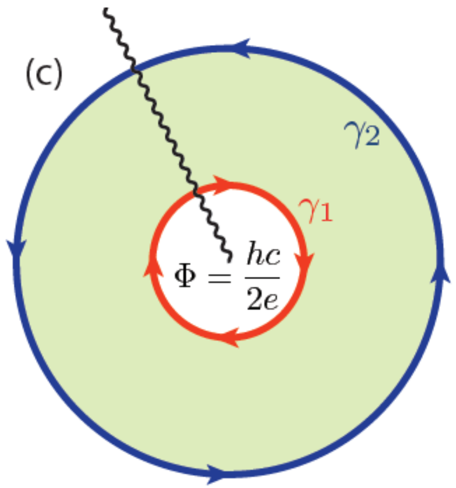
Bogoliubov-de Gennes Hamiltonian:

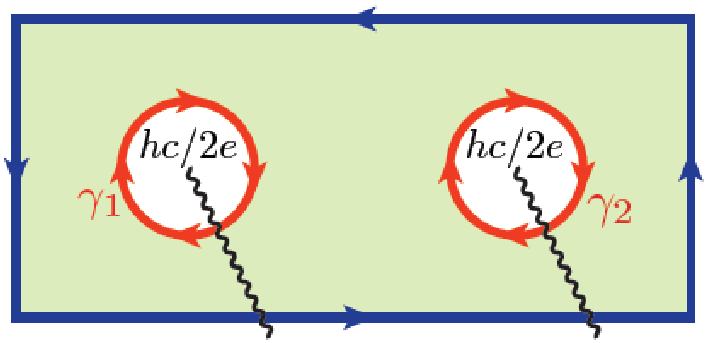
$$H = \int d^2\mathbf{r}\Big\{\psi^\dagger\left(-\frac{\nabla^2}{2m}-\mu\right)\psi + \frac{\Delta}{2}\left[e^{i\phi}\psi(\partial_x+i\partial_y)\psi + H.c.\right]\Big\}$$

p+ip order parameter

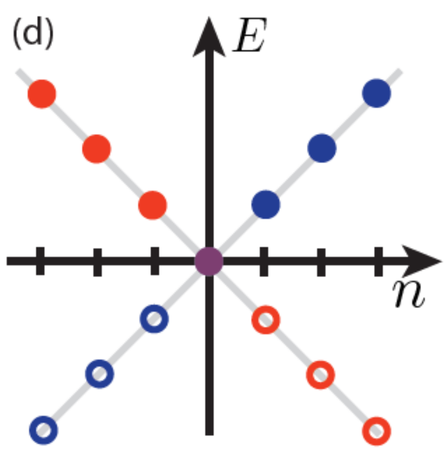
Vorticies in p+ip superconductor

Vorticies host the so-called zero-modes = Majorana fermions





$$\{\gamma_i, \gamma_j\} = 2\delta_{ij}$$



Complex fermion -> two Majoranas

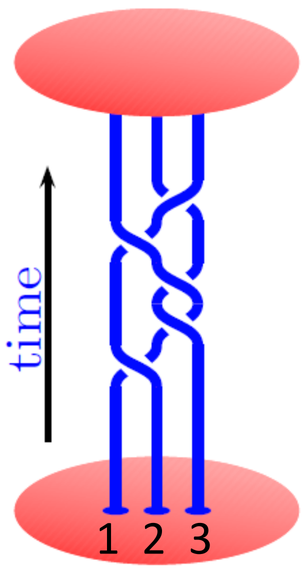
$$c = \frac{1}{2}(\gamma_1 + i\gamma_2)$$

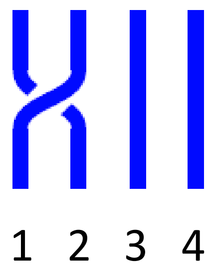
$$c^+ = \frac{1}{2}(\gamma_1 - i\gamma_2)$$

Braiding of Majoranas

Majoranas are non-Abelian anyons !

Consider 4 Majoranas = 2 complex fermions



$\sigma_1 =$ 

$$\begin{pmatrix} e^{i\pi/4} & & & \\ & e^{-i\pi/4} & & \\ & & e^{i\pi/4} & \\ & & & e^{-i\pi/4} \end{pmatrix}$$

$\sigma_2 =$ 

$$\frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 & -i \\ 0 & 1 & -i & 0 \\ 0 & -i & 1 & 0 \\ i & 0 & 0 & 1 \end{pmatrix}$$

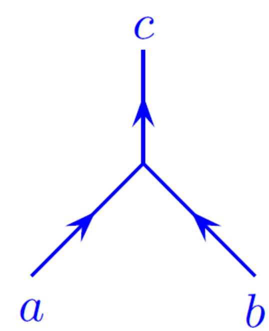
Basis: $|0\rangle, c_1^+|0\rangle, c_2^+\,|0\rangle, c_1^+c_2^+|0\rangle$

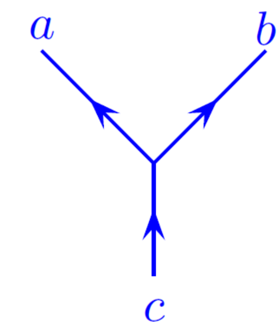
Entangling/non-Abelian transformation

Fusion of non-Abelian anyons

Fusion rules define a structure of Hilbert space of N anyons

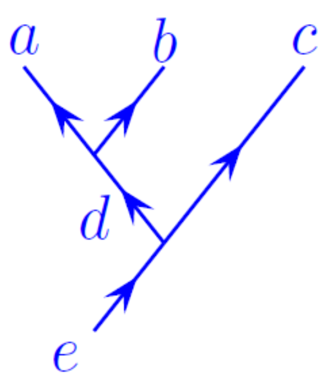
$$a \times b = \sum_c N_{ab}^c \; c$$



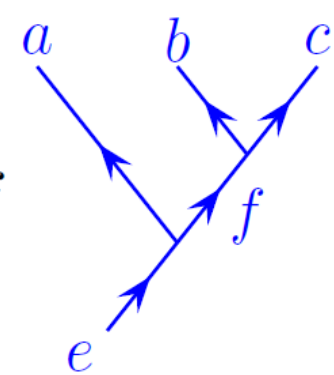


$\langle bra| = \langle ab, c|$ $\qquad$ $|ket\rangle = |ab, c\rangle$

Change of basis (associativity), F-matrix



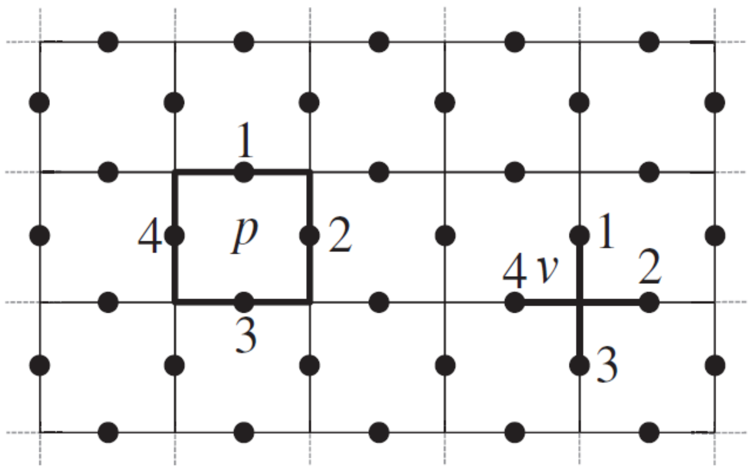
$$= \sum_f \left[F_e^{abc}\right]_{df}$$



Toric code

Alexei Kitaev, 2003

Spins are located on links



- **Vertex:** $A(v) = \sigma^x_{v,1}\sigma^x_{v,2}\sigma^x_{v,3}\sigma^x_{v,4}$
- **Plaquette :** $B(p) = \sigma^z_{p,1}\sigma^z_{p,2}\sigma^z_{p,3}\sigma^z_{p,4}.$

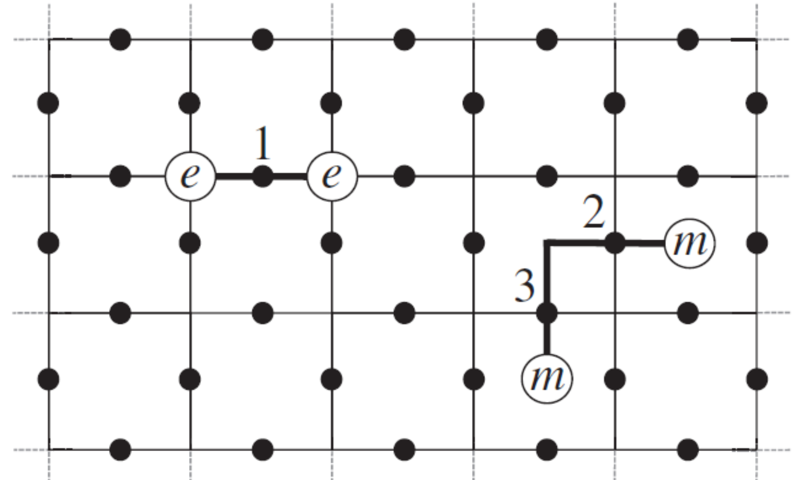
$$H = -\sum_v A(v) - \sum_p B(p).$$

- **Vertex & plaquette mutually commute:** $[A(v), B(p)] = 0$
- **Ground state:** $|\xi\rangle = \prod_v \frac{1}{\sqrt{2}}\big(\mathbb{1} + A(v)\big)|00\ldots0\rangle$ $\quad A(v) = B(p) = +1$

Anyons of toric code

Alexei Kitaev, 2003

- **e-particle:** $|e,e\rangle = \sigma_1^x |GS\rangle$
- **m-particle:** $|m,m\rangle = \sigma_2^z \sigma_3^z |GS\rangle$
- **ε-particle:** $|\varepsilon,\varepsilon\rangle = i\sigma_1^y |GS\rangle$

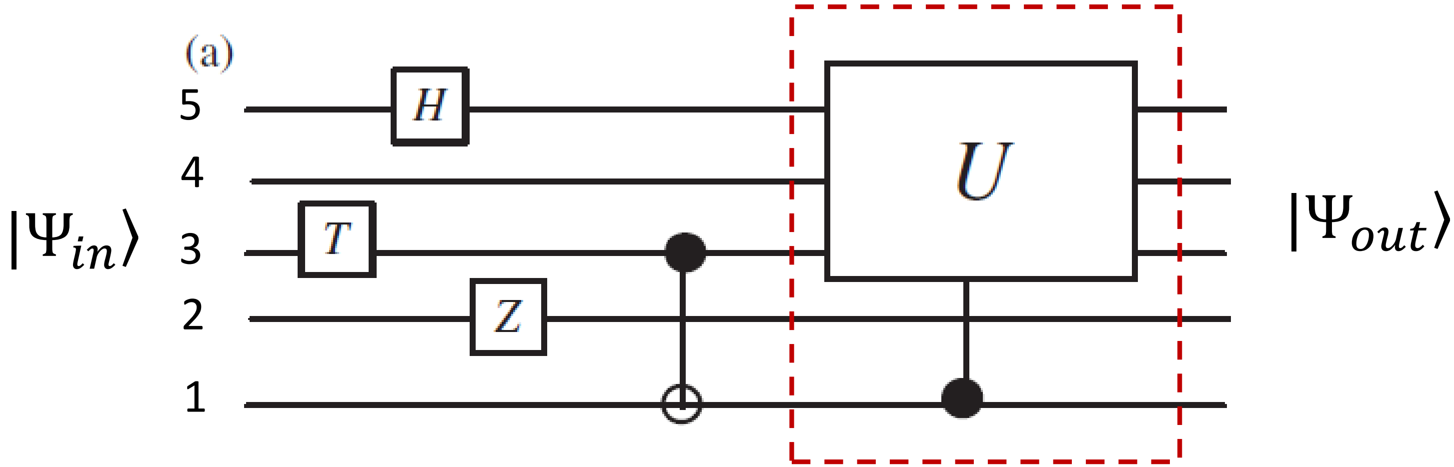


Fusion rules:

$$e \times e = m \times m = \epsilon \times \epsilon = 1,\ e \times m = \epsilon,\ \epsilon \times e = m \text{ and } \epsilon \times m = e$$

When put on a torus (g=1), toric code can be used to encode q-info

Quantum computer (software)



Three-qubit gate contolled by the 1st qubit

Single-qubit gates:

$$\text{Hadamard}: H = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

$$\pi/8: \ T = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\pi/4} \end{pmatrix}$$

Advanced quantum algorithms:

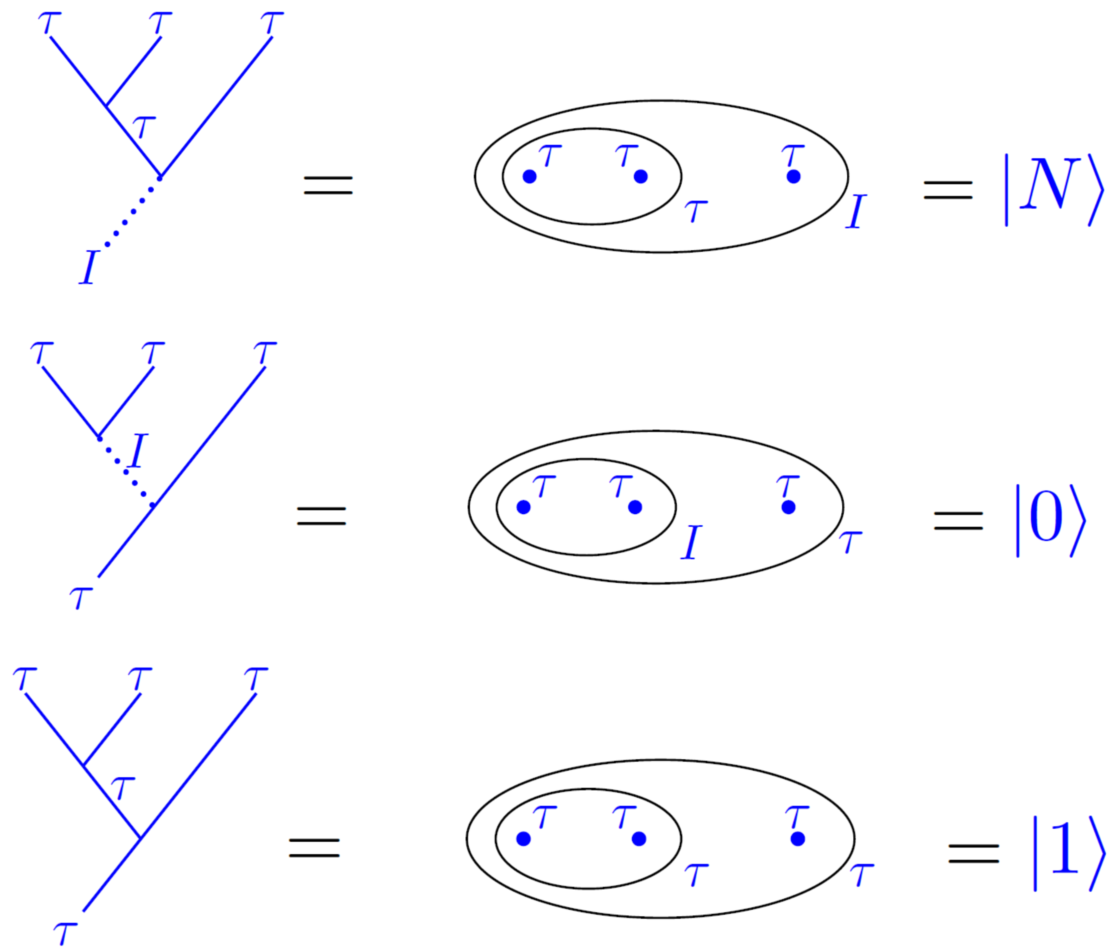
- Quantum Fourier transform
- Grover's quantum search

Fibonacci anyons

Realization (?): FQHE at ν=12/5

- Particle types = {I,τ}
- Fusion rules are:

$$
\begin{aligned}
I \times I &= I \\
I \times \tau &= \tau \\
\tau \times \tau &= I + \tau
\end{aligned}
$$



Fibonacci sequence {1,1,2,3,5,8,13, …} = degeneracy of a state

Quantum computer (hardware)

Exploiting Fibonacci anyons

Covariant distance measure between gates:

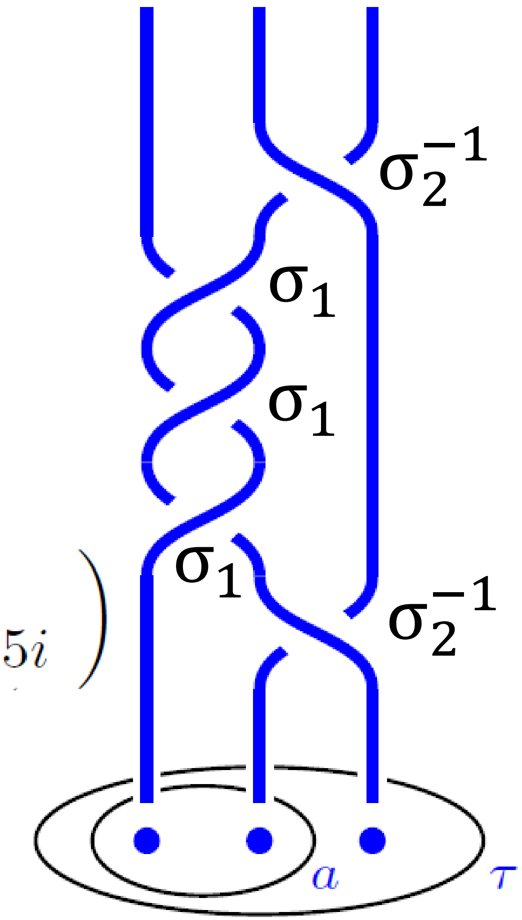
$$\mathrm{dist}(U;V) = \sqrt{1 - \frac{|\mathrm{Tr}[U^\dagger V]|}{D}}$$

Braid (right) as approximate single qubit X-gate

$$U_{\mathrm{approx}} \approx e^{-3\pi i/5} \begin{pmatrix} 0.073 - 0.225i & 0.972 \\ 0.972 & -0.073 - 0.225i \end{pmatrix}$$

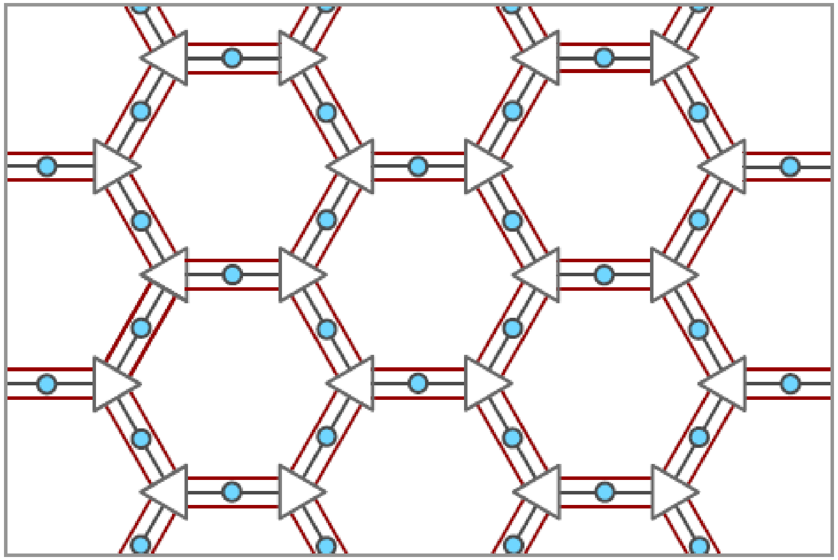
$$U_{\mathrm{approx}} = \hat{\sigma}_2^{-1}\, \hat{\sigma}_1^{3}\, \hat{\sigma}_2^{-1}$$

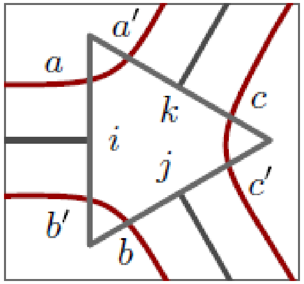
dist(U, X) = 0.17



Other topics

- Topological entanglement entropy & top. phases
- Tensor networks:





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