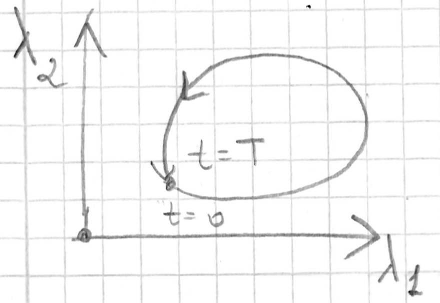
(1)

(I) Berry phase

Lecture #1

Sections 1.5.1, 1.5.2 & 1.5.4 from Tong

- $\hat{H}(\lambda)$ $\to$ Hamiltonian, depends on set of $\vec{\lambda}$, which are slow functions of time $\lambda_1(t)$, $\lambda_2(t)$, ...



- Let $|m(\lambda)\rangle$ instant. eigenstates, i.e.

$$H(\lambda)|m(\lambda)\rangle = \varepsilon_m(\lambda)|m(\lambda)\rangle \quad \&$$

$|n(\lambda)\rangle$ with $\varepsilon_n(\lambda)$ is ground state

- Initial condition $|\psi(t=0)\rangle = |n(\lambda(0))\rangle$
- Q: What is $|\psi(t=T)\rangle = ?$
- A: $$|\psi(t=T)\rangle = \underbrace{e^{-i\int_0^T \varepsilon_n(\lambda(t))dt}}_{\text{dynamical phase}} \; \underset{\text{Berry phase}}{e^{i\gamma}} \; |\psi(t=0)\rangle$$

$|\psi(t)\rangle$ evolves accord. to Schröd. eq. $i\partial_t|\psi(t)\rangle = H(\lambda(t))|\psi(t)\rangle$

- Adiabatic evolution means $\dot{\lambda}_k \ll |\underset{\text{ground state}}{\varepsilon_n} - \underset{\text{excited}}{\varepsilon_m}|$, $m \neq n$

"adiab. theorem"

$\Rightarrow$ $|\psi(t)\rangle$ remains in the ground state, i.e.

$$|\psi(t)\rangle = |n(\lambda(t))\rangle \, C(t), \quad \text{with } |C(t)| = 1$$

$\swarrow$ only phase changes

(2.) Use Schröd. eq. $\rightarrow$

$$\underbrace{\hat{H}(\lambda(t))\,|n(\lambda(t))\rangle\, c(t)} = i\,|\dot{n}\rangle\, c(t) + i\,|n(\lambda(t))\rangle\, \dot{c}(t)$$

$$\varepsilon_n(\lambda(t))\,|n(\lambda(t))\rangle\, c(t) = i\,|\dot{n}\rangle\, c(t) + i\,|n(\lambda(t))\rangle\, \dot{c}(t)$$

$\langle n|(\ldots) \Rightarrow$ (I suppress further depend. on λ)

project

$$\varepsilon_n(\lambda(t))\, c(t) = i\langle n|\dot{n}\rangle\, c(t) + i\dot{c}(t)$$

Let $c_n(t) = e^{-i\int_0^t \varepsilon_n(\lambda(t))dt}\, e^{i\gamma(t)} \Rightarrow$

dynam. phase

$$0 = i\langle n|\dot{n}\rangle\, c(t) - \dot{\gamma}(t)\, c(t) \Rightarrow$$

$$\dot{\gamma}(t) = i\langle n(\lambda(t))|\, \vec{\partial}_t\, |n(\lambda(t))\rangle \quad \text{, or}$$

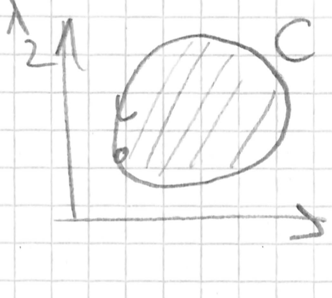
$$\dot{\gamma}(t) = \sum_k i\langle n(\lambda)|\, \frac{\vec{\partial}}{\partial \lambda_k}\, |n(\lambda)\rangle\, \dot{\lambda}_k(t)$$

- We define Berry connection

Result

$$a_k^{(\lambda)} = i\langle n(\lambda)|\, \frac{\vec{\partial}}{\partial \lambda_k}\, |n(\lambda)\rangle \Rightarrow$$

$$\gamma = \int_0^T a_k(\lambda(t))\, \dot{\lambda}_k(t)\, dt = \oint_C a_k(\lambda)\, d\lambda_k$$



Let $C = \partial D \rightarrow$ boundary

Stokes' theorem

$$b_3 \equiv (\partial_1 a_2 - \partial_2 a_1)$$

$$\gamma = \oint_C (a_1\, d\lambda_1 + a_2\, d\lambda_2) = \int b_3\, d\lambda_1\, d\lambda_2$$

(3) Example : spin $1/2$ in magnetic field

$$\vec{B} = (B_x, B_y, B_z)$$

$$\left|\begin{array}{l} B_x = B\sin\theta\cos\varphi \\ B_y = B\sin\theta\sin\varphi \\ B_z = B\cos\theta \end{array}\right\} \text{ with } \theta(t) \text{ and } \varphi(t) \text{ slowly variyng}$$

$$|\vec{B}| = B = \text{Const} \; ; \qquad \hat{H} = \sum_{k=1}^{3} \left(B_k \hat{\sigma}_k\right) + B \cdot \mathbb{1}_{2\times 2}$$

$\to$ Pauli matricies

Explicitely (exercise)

$$\hat{H} = B \begin{pmatrix} \cos\theta + 1 & \sin\theta e^{-i\varphi} \\ \sin\theta e^{i\varphi} & -\cos\theta + 1 \end{pmatrix} \qquad |\dot{\theta}, \dot{\varphi}| \ll B$$

- Eigenvalues (instant.) : $\varepsilon_{\downarrow}(t) = 0$; $\varepsilon_{\uparrow}(t) = 2B$
- Eigenvectors

$$|\downarrow(t)\rangle = \begin{pmatrix} e^{-i\varphi(t)} \sin \theta(t)/2 \\ -\cos\theta(t)/2 \end{pmatrix}$$

$$|\uparrow(\theta,\varphi)\rangle = \begin{pmatrix} e^{-i\varphi} \cos\theta/2 \\ \sin\theta/2 \end{pmatrix}$$

- Berry connection

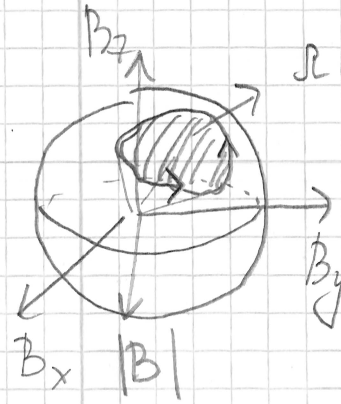
(a) $$a_\theta = i\langle\downarrow|\vec{\partial}_\theta|\downarrow\rangle = \frac{i}{2}\left(\sin\theta/2 \cos\theta/2\right) - \frac{i}{2}\left(\cos\theta/2 \sin\theta/2\right) = 0.$$

(b) $$a_\varphi = i\,(-i)\sin^2\theta/2 = \tfrac{1}{2}(1-\cos\theta)$$

Hence Berry phase is

$$\gamma = \oint_C \tfrac{1}{2}(1-\cos\theta)\,d\varphi = \tfrac{1}{2}\int_D \sin\theta\, d\theta\, d\varphi$$

$$b = \partial_\theta a_\varphi - \partial_\varphi a_\theta = \tfrac{1}{2}\sin\theta \,\Big| = \tfrac{1}{2}\Omega(C)$$



$\Omega(C) \rightarrow$ solid angle which is spanned by loop C on a sphere.

$$\boxed{e^{i\gamma} = e^{\frac{1}{2}\Omega(C)}}$$

Remark: if one chooses complimentary angle $\Omega' = 4\pi - \Omega$,

then $e^{-\frac{i}{2}\Omega'} = e^{-2i\pi} e^{+i\Omega/2} = e^{i\Omega/2}$

(orientation of surface in Stokes' theorem is opposite)

II Non-abelian Berry connection

- Let we now have N-degenerate ground state with $E_g \equiv 0$

$|n_a(\lambda)\rangle \rightarrow$ basis with $a = 1, \dots, N$ and
$\vec{\lambda}(t) = (\lambda_1(t), \lambda_2(t), \dots, ..) \rightarrow$ parameters

By definition $\hat{H}(\lambda)\,|n_a(\lambda)\rangle = 0.$ for $\forall a$

- Consider states $|\psi_a(t)\rangle$, such that

$$i\partial_t |\psi_a(t)\rangle = H(\lambda(t))\,|\psi_a(t)\rangle \quad \text{and}$$

$$|\psi_a(t=0)\rangle = |n_a(\lambda(0))\rangle \leftarrow \text{initial condition}$$

(5)

- By virtue of adiab. theorem $|\Psi(t)\rangle$ remains in the linear subspace spanned by $|n_b(\lambda(t))\rangle \Rightarrow$

$$|\Psi_a(t)\rangle = \sum_{b=1}^{N} |n_b(\lambda(t))\rangle\, \mathcal{U}_{ba}(t) \quad , \text{where}$$

$\mathcal{U}(t) \in \mathcal{U}(N) \to$ unitary matrix.

- We define $|\dot{n}_a\rangle \equiv \frac{\partial}{\partial t}|n_a(\lambda(t))\rangle \Rightarrow$ take into account!

$$i\partial_t|\Psi_a(t)\rangle = H(\lambda(t))\,|\Psi_a(t)\rangle \Rightarrow \quad \left(H|n_b\rangle = 0\right)$$

$$i\sum_b\left(|\dot{n}_b\rangle\, \mathcal{U}_{ba} + i|n_b\rangle\, \dot{\mathcal{U}}_{ba}(t)\right) = 0 \quad \wedge\ \forall a$$

valid for

$$\sum_a(\ldots)\times \mathcal{U}^{+}_{ac} \Rightarrow \quad \left(\mathcal{U}_{ba}\mathcal{U}^{+}_{ac} = \delta_{bc}\right)$$

$$i|\dot{n}_c\rangle + i\sum_{ba}|n_b\rangle\, \dot{\mathcal{U}}_{ba}(t)\, \mathcal{U}^{+}_{ac}(t) = 0.$$

We project this result on $\langle n_b|(\ldots) = 0$

$$\Rightarrow \sum_{a=1}^{N} \left\| \dot{\mathcal{U}}_{ba}(t)\, \mathcal{U}^{+}_{ac}(t) = -\langle n_b|\dot{n}_c\rangle \right\| \quad \forall\, b,c \to \text{fixed}$$

This is intermediate result. | N^2 relations for matrix $\mathcal{U}^{N\times N}(t)$

- We define $A^{k}_{bc} \equiv i\langle n_b(\lambda)|\frac{\partial}{\partial\lambda_k}|n_c(\lambda)\rangle$

A^{k} is from Lie algebra of group $\mathcal{U}(N)$.

Then

$$(*)\ \sum_{a=1}^{N}\dot{\mathcal{U}}_{ba}(t)\, \mathcal{U}^{+}_{ac}(t) = i\sum_k A^{k}_{bc}(\lambda(t))\, \dot{\lambda}_k(t)$$

Here we have N^2 relation

(6)

Introdue $\mathcal{U}(t) = \sum_{a,b} |a\rangle \mathcal{U}_{ab}(t) \langle b|$

In the operator form (*) reads

& $A^k = \sum_{ab} |a\rangle A^k_{ab}(\lambda) \langle b|$

$$\hat{\dot{\mathcal{U}}}(t)\hat{\mathcal{U}}^+(t) = i\,\hat{A}^k(t)\,\dot{\lambda}_k(t) \quad , \text{or}$$

$$\partial_t \hat{\mathcal{U}}(t) = i\,\hat{A}^k(t)\,\hat{\mathcal{U}}(t)\,\dot{\lambda}_k(t)$$

Formal solution of this equation is

$$\mathcal{U}(T) = P_c \exp\left[i \oint_C A^k(\lambda)\, d\lambda_k \right]$$

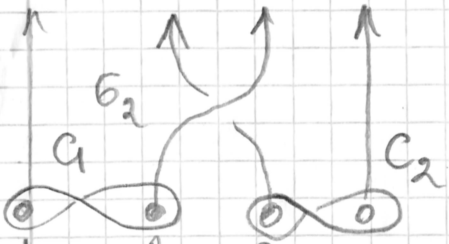
path ordering along contour C (Willson's loop).

- Result of adiabatic evolution

$$|\psi_a(t=T)\rangle = \mathcal{U}_{ab}(T)\, |n_b(\lambda(0))\rangle \quad , \text{i.e.}$$

initial basis $|n_b(\lambda(0))\rangle$ is rotated !

- Example : brading of Majoranas



fermion

$(\gamma_1, \gamma_2) \to c_1$

$(\gamma_3, \gamma_4) \to c_2$

4 vortexes in the p-wave SC

States: $|0\rangle$, $c_1^+|0\rangle$, $c_2^+|0\rangle$, $c_1^+c_2^+$

vaccuum

$$\sigma_2 = \begin{pmatrix} 1 & 0 & 0 & -i \\ 0 & 1 & -i & 0 \\ 0 & -i & 1 & 0 \\ i & 0 & 0 & 1 \end{pmatrix} \to \text{braiding operation}$$

$$|\psi(t=0)\rangle = |0\rangle\psi_{00} + c_1^+|0\rangle\psi_{10} + c_2^+|0\rangle\psi_{01} + c_1^+c_2^+ \psi$$