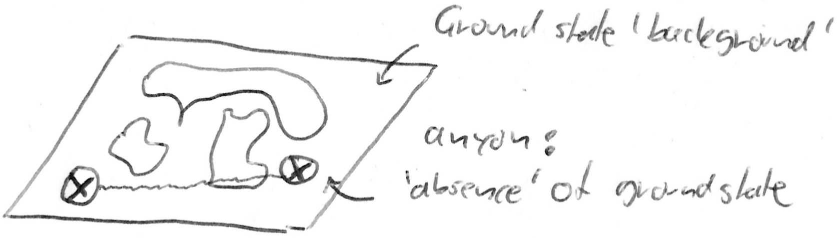
Entanglement and topological order

Observations so far:

- Non-Abelian Berry phases and anyons ← consequence
- Ground states are non-trivial superpositions
- sensitivity to topology (long-range 'detectors')
- absence of local order parameters

} long-range entanglement



The behaviour of anyons is a consequence of of 'punches / holes' move in a ground state background.

-> strong connection between GS properties and anyons

-> the special 'non-local superpositions' of GS is signified by interesting entanglement structure

Plan:

1) Entanglement and tensor network notation
   - density matrix
   - Penrose notation
   - von Neumann entropy
2) The area law and topological correction
   - toric code as tensor network
   - top. ent. entropy

Entanglement

- quantum many-body phenomenon
- beyond-classical correlations (Bell tests)
- resource for quantum computation and communication

Definition

A bipartite (two subsystems) state is entangled, if it can not be written as a product state.

Consider $|\psi\rangle = \sum_{i=1}^{d_1} \sum_{j=1}^{d_2} c_{ij} \, |i\rangle \otimes |j\rangle$

$|\psi\rangle$ is a product state iff there are vectors in $\mathcal{H}_1$ and $\mathcal{H}_2$,

such that $|\psi\rangle = |\psi_1\rangle \otimes |\psi_2\rangle$.

write $|\psi_1\rangle = \sum_i a_i |i\rangle$; $|\psi_2\rangle = \sum b_i |j\rangle$,

product state, if

$$|\psi\rangle = \sum c_{ij} |i\rangle |j\rangle = \sum a_i b_j |i\rangle |j\rangle$$

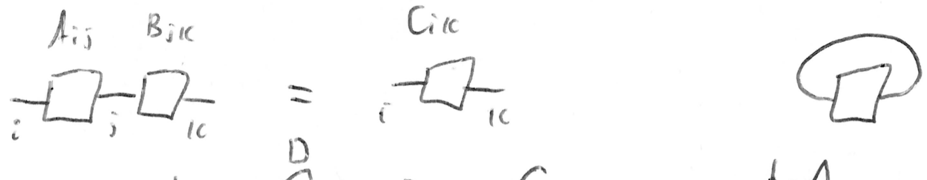
$\Rightarrow$ $c_{ij} = a_i b_j$ (1); 'c_{ij}' factorizes

Intuitive notation (Penrose)

A tensor (array of complex numbers) is written as a box with one line per index.

c_{ij} : matrix $\quad$ a_i : vector $\quad$ scalar $\quad$ A_{ijkem} : general tensor

Summation over an index (index contraction) is denoted by joining the index lines.



matrix multiplication $\sum_{j=1}^{D} A_{ij} B_{jk} = C_{ik}$ $\qquad$ tr A

Identity 'matrix': —— $\mathbb{1}$; index range $i=1,\dots,D$
bond dimension

bra: $|\psi\rangle = \sum_{ij} c_{ij} |i\rangle |j\rangle$:  (careful with complex conjugation)

$\updownarrow$

ket: $\langle\psi| = \sum_{ij} c_{ij}^* |i\rangle |j\rangle$: ;

scalar product: $\langle \psi | \psi \rangle =$ 

expectation value: $\langle \psi | \hat{O}_A | \psi \rangle =$ 

$\uparrow$

Product state in Penrose notation:

 $=$ 

c_{ij} $\qquad a_i \quad b_j$

Quantification of entanglement

- density matrix ρ describes ensemble

$$\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i| \quad , \qquad \rho \geq 0, \; \mathrm{Tr}\,\rho = 1, \; \rho^\dagger = \rho$$

example: thermal (Gibbs state)

$$\rho = e^{-\beta H} = \sum_i e^{-\beta \epsilon_i} |\psi_i\rangle\langle\psi_i|$$

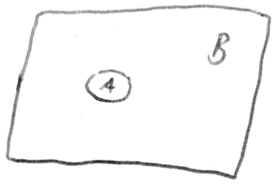
ensemble: incomplete knowledge of the system

origin? -> reduced density matrix

> $p_i = \begin{cases} 1 & \text{some } i \\ 0 & \text{others} \end{cases}$
> -> pure state
> otherwise:
> mixed state

- reduced density matrix

Large system AB, access only to A



$|\psi\rangle_{AB}$: is a pure state [the universe]

$$\rho_{AB} = |\psi\rangle_{AB}\,\langle\psi|_{AB}$$

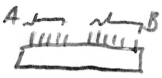
description of A: $\rho_A = \underset{\substack{\uparrow \\ \text{'trace out B' [average]}}}{\mathrm{tr}_B}\, \rho_{AB} = \sum_{j_B} \langle j_B | \psi \rangle_{AB}\, {}_{AB}\langle \psi | j_B \rangle$

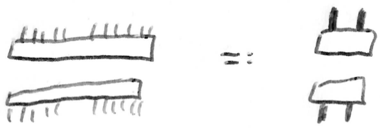
$\uparrow$ basis of B

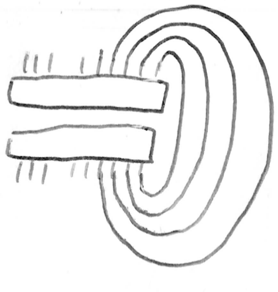
$$|\psi_{AB}\rangle = \sum_{i_A, j_B} \underset{\substack{\uparrow \\ \text{collection of many degrees of freedom}}}{c_{i_A j_B}} |i\rangle_A |j\rangle_B$$

$C_{i_A j_B} = C_{i_1 i_2 \dots i_{N_A} j_1 \dots j_{N_B}}$

in Penrose (tensor network) notation:

$|\psi_{AB}\rangle =$ 

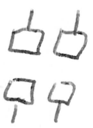
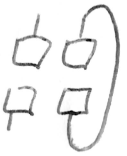
$\rho_{AB} =$  ← pure state [~~[illegible]~~]
(two vectors)

$\rho_A = \mathrm{tr}_B\, \rho_{AB} =$  $=$  $=$  ← mixed state (matrix)

sum

Idea: A and B are strongly entangled, if tracing out (average) B leaves us with a random state A.

Example: 1) product state 

ρ_{AB}(product) $=$  $\quad \rho_A =$  $=$   ↑ some scalar

pure state → pure state

'average' over B does not reduce the knowledge of A, because they are independent.

2) entangled state $|\psi\rangle =$  ← C_{ij} is δ_{ij} (identity matrix)

ρ_{AB}(ent.) $=$  $\quad \rho_A =$  $=$  $= \begin{pmatrix} 1/2 & \\ & 1/2 \end{pmatrix}$

maximum uncertainty

⌈normalization:

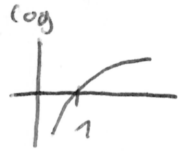
$|\psi\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) \quad \rho_{AB} = \frac{1}{2}(|00\rangle + |11\rangle)(\langle 00| + \langle 11|)$

$\rho_A = \langle 0|\rho_{AB}|0\rangle + \langle 1|\rho_{AB}|1\rangle = \frac{1}{2}(|0\rangle\langle 0| + |1\rangle\langle 1|)$ ⌋

expectation value of A is tied to expectation value of B

The probabilities p_i are 'generated' from the 'measurement' of [expectation values] on B. If all different expectation values on B lead to different states of A, ~~these a~~ strong we can say that B 'knows' a lot about A ~~and~~. ~~This~~ The resulting reduced density matrix is very mixed / ~~and~~ high entropy.

von Neumann entropy



$$S = -\mathrm{tr}\,\rho \log \rho$$ quantifies 'mixedness'

minimum: $S(\rho_{pure}) = 0$ $\qquad \rho = \begin{pmatrix} 1 & & & \\ & 0 & & \\ & & \ddots & \\ & & & 0 \end{pmatrix}$

$$S(\rho_{pure}) = -\sum_i p_i \log p_i$$ ~~[convention $\log 0 = 0$]~~

$$= -\log 1 = 0$$

maximum:

$$\int \rho(E) \log(\rho(E))\, dE = F[\rho] \quad \text{[continuum]}$$

$$\frac{\delta F}{\delta \rho} = 0 \Rightarrow \log \rho + \frac{1}{\rho}\rho = 0 \Rightarrow \log \rho = -1$$

$\Rightarrow \rho$ is a constant

[Better to do it with normalization -> Lagrange multiplier]

$$\rho = \frac{1}{\dim \mathcal{H}} \mathbb{1}$$: maximally mixed ; $\dim \mathcal{H} = d$

$\uparrow$ normalization

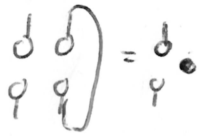
$$S[\rho_{mm}] = -\mathrm{tr}\left(\frac{1}{d}\mathbb{1} \log\left(\frac{1}{d}\mathbb{1}\right)\right) = -\sum_{i=1}^{d} \frac{1}{d} \log(1/d) = -\log(1/d)$$

$$= \log d = \log(\dim \mathcal{H})$$

Entanglement entropy

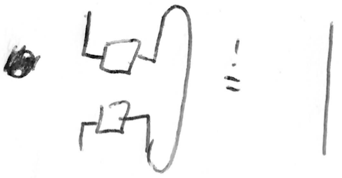
$$\boxed{S(\rho_A) :\equiv \text{entropy of reduced density matrix}}$$

Again: minimum entangled (product) $\rightarrow$ pure

 $S(\rho_A) = 0$

$\leftarrow$ definition of entanglement

maximally entangled $\rightarrow$ maximally mixed

 $S(\rho_A) = \log(\dim \mathcal{H})$

$\Rightarrow$  $= \sqcup$; $\boxed{\sqcup \text{ is a maximally entangled pair}}$

Cute: measuring A is like measuring B ; $|\psi\rangle = \dfrac{|00\rangle + |11\rangle}{\sqrt{2}} = |\text{Bell}\rangle = \sqcup$

$\langle\psi|\hat{O}_A|\psi\rangle =$  $=$  $= \langle\psi|\hat{O}_B|\psi\rangle$

Other useful measures to quantify entanglement

Renyi-entropy

$$S_\alpha = \frac{1}{1-\alpha} \log \operatorname{Tr} \rho^\alpha$$

; max-entropy:

$$\lim_{\alpha \to 0} S_\alpha = \log \operatorname{rank} \rho = S_0$$

very usefull rough estimate

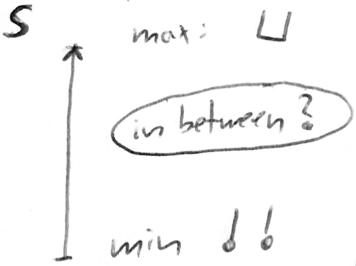
$$\lim_{\alpha \to 1} S_\alpha = S$$

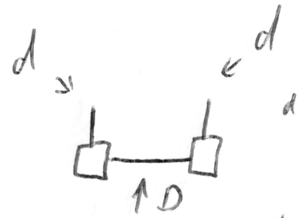
$\leftarrow$ von Neumann entropy

$$S \leq S_0 = \log \underset{\substack{\uparrow \\ \text{very easy to compute}}}{\operatorname{rank}} \rho$$

Tensor networks - parametrize entanglement

So far:

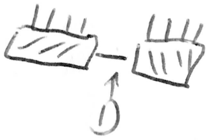


Consider:  

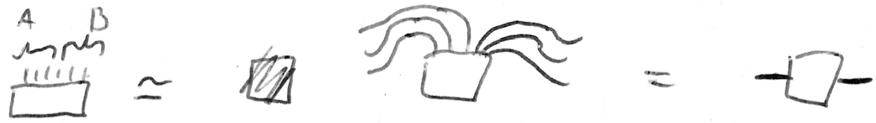
$\dim \mathcal{H}_1 = \dim \mathcal{H}_2 = d$

allow states to talk to each other, but restrict the shared information

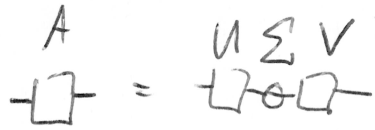
$\boxed{D < d}$

We can always write a state  as 

. and know the minimum value of D.

How? 

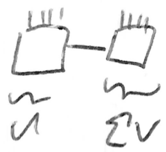
1) interpret state as matrix from A to B.

2) Perform a singular value decomposition 

Σ: diagonal rank $\Sigma = D$, U, V unitary

Σ: singular values (generalized eigenvalues)

3) interpret matrix as state:



What is the effect of D on the entanglement entropy?

$\rho =$  , $\rho_A =$  ; rank $\rho_A = ?$