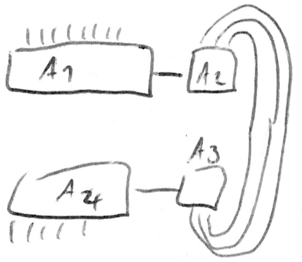
ρ_A is a product of 4 matrices

$A_1 : d^{N_A} \times D$

$A_2 : D \times d^{N_B}$

$A_3 : d^{N_B} \times D$

$A_4 : D \times d^{N_A}$



$$\mathrm{rank}(\rho_A) = \mathrm{rank}(A_1 A_2 A_3 A_4) \leq \min\{\mathrm{rank}\, A_1, \mathrm{rank}\, A_2, \mathrm{rank}\, A_3, \mathrm{rank}\, A_4\}$$
$$= D$$

$S \leq \log \mathrm{rank}\ D$: bond dimension puts an upper bound on entanglement

The higher the bond dimension, the more entanglement is possible.

Tensor network states

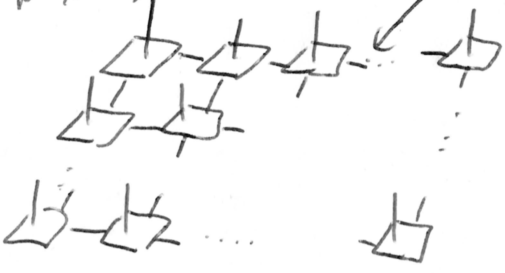
- 1D : matrix product states (MPS)

← physical indicis

virtual indicis

- 2D PEPS (projected entangled pair states)

physical → virtual



[2D grid, can be different lattice]

3D also 'PEPS'

- various other tensor network states

Remarks $\quad H=\sum_i h_i$ ↑local

$\updownarrow\Delta$, Δ finite if $N \to \infty$ (system size)

1. Ground states of local gapped Hamiltonians are well described by tensor network states.

2. PEPS (1D, 2D, 3D) fulfill an entanglement area law.

3. Many known models have exact, known tensor network ground states, for example the toric code.

4. Entanglement can be often calculated nicely with tensor network states.
   → toric code

5. Topological order is not characterized by strong entangled, but by long-range entanglement. → next lecture

   [The reduced density matrices are not very entropic, e.g. when compared to thermal states!]

6. Topological order is indicated by a topological correction to the area law:

$$S_A = |\partial A| c - \gamma + \dots$$

$|\partial A|$ ↑ boundary of A

γ ← topological correction (less entangled)

$\dots$ ↑ terms that vanish for infinite system size

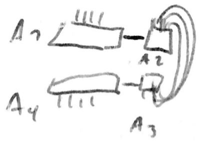
---

2. PEPS area law: $\quad |A| = N_A$: size of A, $|\partial A|$: size of boundary

$\rho_A = \mathrm{tr}_B |\psi\rangle\langle\psi|$; rank ρ_A = ?

$|\psi\rangle \hat{=}$

the same consideration as before:



matrices: A_1: $d^{|A|} \times D^{|\partial A|}$

A_2: $D^{|\partial A|} \times d^{|\bar{A}|}$

A_3: $d^{|\bar{A}|} \times D^{|\partial A|}$

A_4: $D^{|\partial A|} \times d^{|A|}$

$$\text{rank}(\rho_A) \leq \min\left(d^{|A|}, d^{|\bar{A}|}, D^{|\partial A|}\right) = D^{|\partial A|}$$

$$D^{|\partial A|} < d^{|A|}$$ when $|A|$ is large enough

boundary (length) $\sim L$; interior (area) $\sim L^2$

$$\Rightarrow \quad S_0 = \log \text{rank}(\rho_A) \leq \log D^{|\partial A|} = |\partial A| \log D$$

area law, $\log D$ is the constant.

Remark: generic PEPS saturate this bound. [non-trivial statement].

<u>Finally: How can a topological correction emerge?</u>

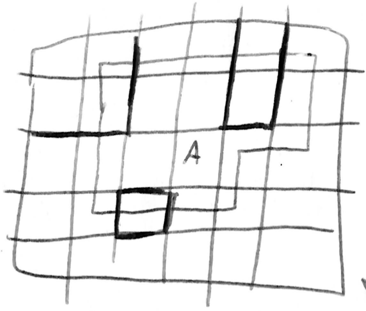
1) Toric code - elementary (easy, idea is clear)

2) Toric code PEPS ('deeper', new tool: tensor network, easy to generalize)

3) How can γ be interpreted?

1] Toric code ground state $|\psi\rangle = \sum'_{ij} |\psi_i\rangle_A |\psi_j\rangle_B = \sum'_e |\psi_e\rangle_A |\psi_e\rangle_B$

superposition →



'cut through spins', think of loop soup

e: all configurations on the boundary

↖ strings cross boundary an even number of times

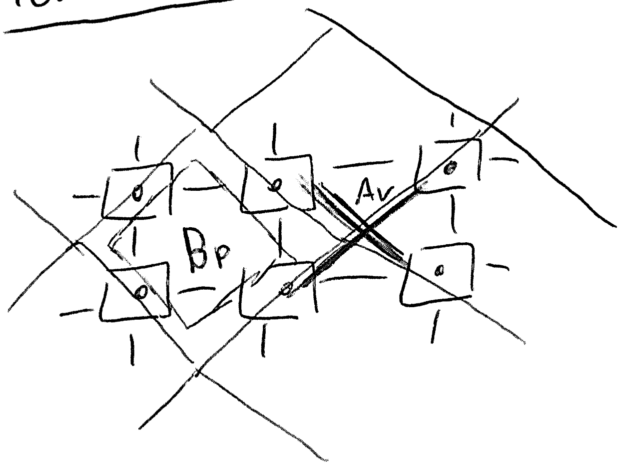
rank ρ_A is limited by the number of possible boundary configurations

OR: 'count' states [entropy]

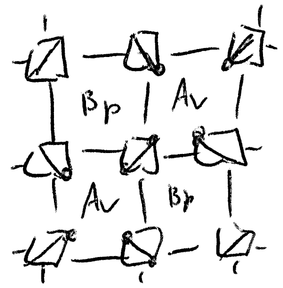
How many states? $2^{|\partial A|}/2 = 2^{|\partial A|-1}$ ← only configurations with an even number of 'strings'/spin-up-states at the boundary

$$S = \underbrace{|\partial A| \log 2}_{\text{area law}} - \underbrace{\log 2}_{\gamma:\ \text{top. correction}}$$

Toric Code PEPS



~> checkerboard of B_p and A_v



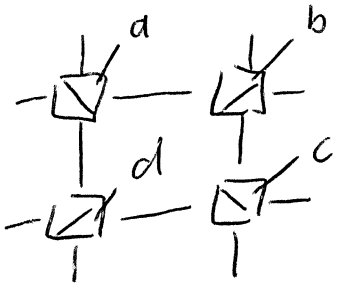
$A_v|GS\rangle = |GS\rangle$, $B_p|GS\rangle = |GS\rangle$

1) parity constraint

2) superposition of 'loop' + 'no loop'

encode 1) and 2) locally

1)



need to ensure that

$\hat{z}\hat{z}\hat{z}\hat{z}|abcd\rangle = |abcd\rangle$

think of parity as $(\mathbb{Z}_2, +)$

-> $a+b+c+d \pmod 2 = 0$

$d = 0, 1$

An easy way to achieve even parity:

$a = a' - b'$

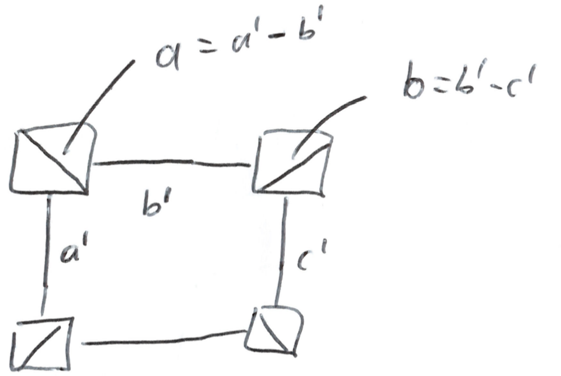
$b = b' - c'$

$c = c' - d'$

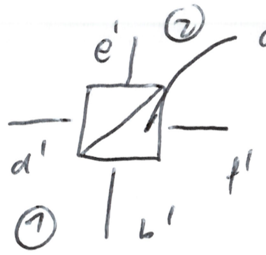
$d = d' - a'$

'define' spin states as differences of virtual indices

=> $a+b+c+d = 0$



What about the other two indices?



$a = a' - b'$ ← to satisfy vertex constraint ①

$a = e' - f'$ ← " ②

$\Rightarrow \quad a' - b' = e' - f'$

two options

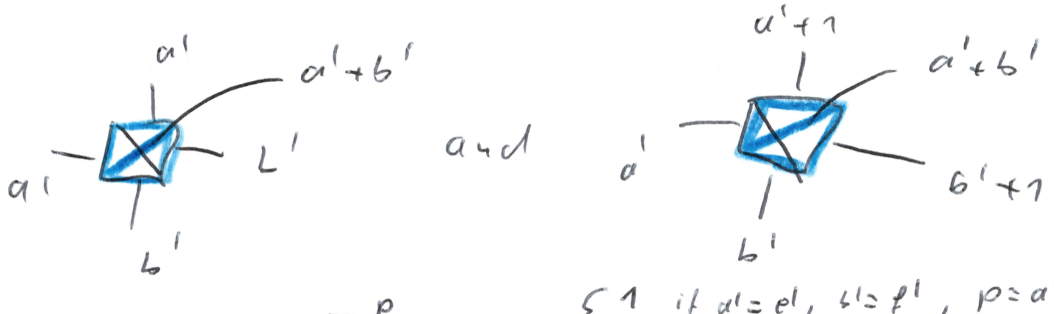
a'	b'	e'	f'	$a'-b'=e'-f'$
0	0	0	0	0
0	0	1	1	0
0	1	0	1	1
0	1	1	0	1

⋮

either (a', b') is equal to (e', f')

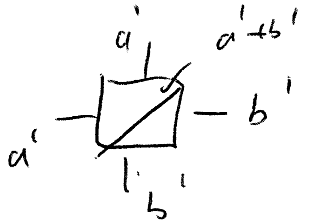
or $e' = a' + 1 \quad f' = b' + 1$

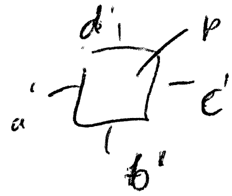
$\Rightarrow$ we have found two good tensors



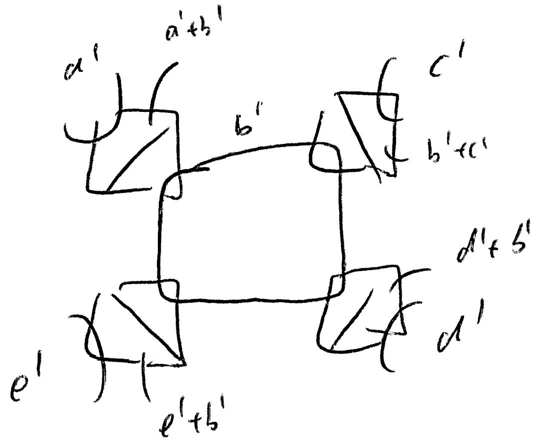
Short hand notation for $T^{p}_{a'b'e'f'} = \begin{cases} 1 & \text{if } a'=e',\ b'=f',\ p=a'+b' \\ 1 & \text{if } a'=e'+1,\ b'=f'+1,\ p=a'+b' \\ 0 & \text{otherwise} \end{cases} \quad /\mathbb{Z}_2$

The 'first one' is sufficient.

Take  : $T^{p}_{a'b'c'd'} = \begin{cases} 1 & \text{if } a'=d',\ b'=c',\ p'=a'+b' \\ 0 & \text{otherwise} \end{cases}$



show: $B_p|\psi\rangle = |\psi\rangle$



b': closed index line

$\to$ summation

Summing over b' generates superposition of

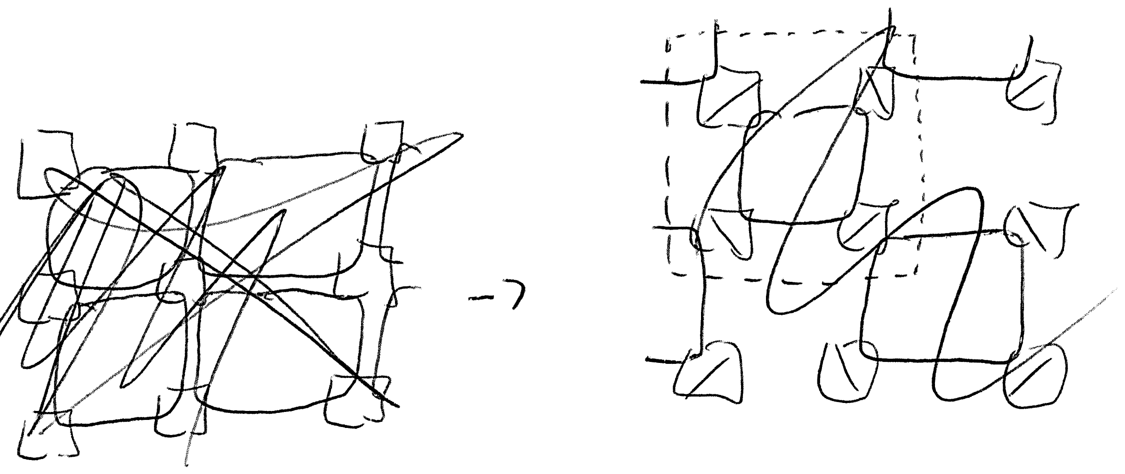
$\frac{1+xxxx}{2}$

Here: $b'=0 \to |a'\ c'\ d'\ e'\rangle$

$b'=1 \to |a'+1,\ c'+1,\ d'+1,\ e'+1\rangle$

effective spin flip on all four spins

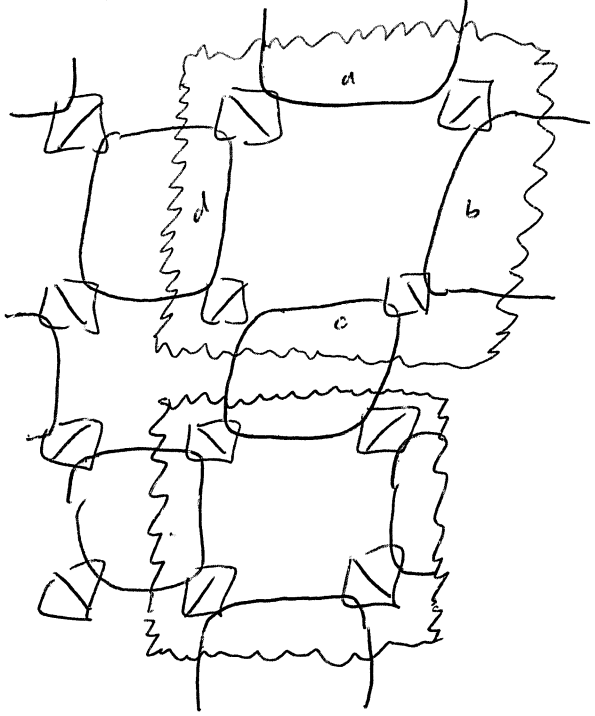
Blocking

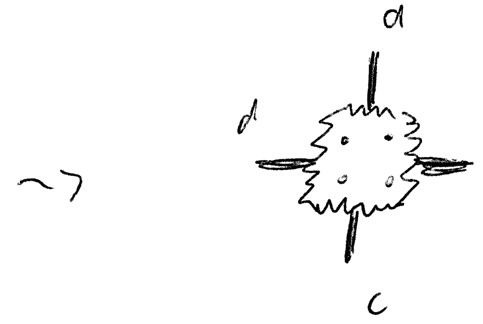


TC3

Blocking

- a + d
- a + b
- b + c
- c + d

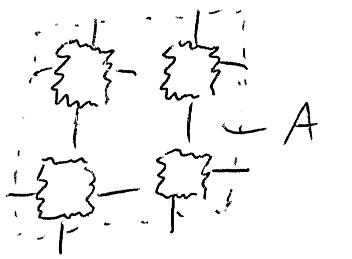




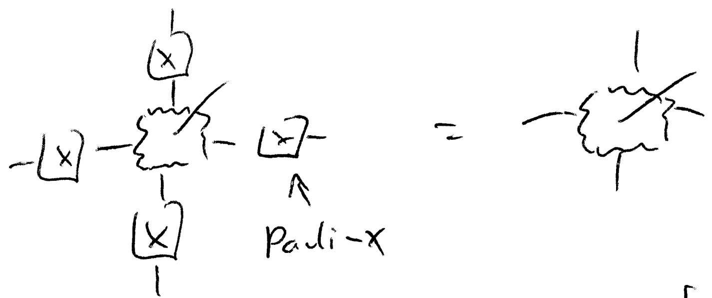
reduce redundancy and obtain translation invariance

entanglement entropy

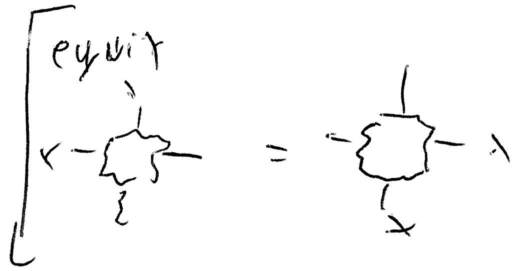
rank $\rho_A \le 2^{|\partial A|}$



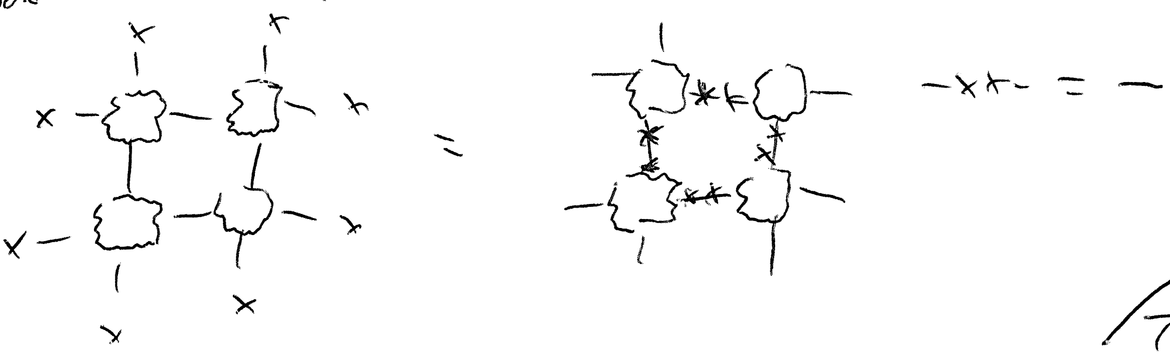
but actually, there is a symmetry in these tensors



equivit



Every path has this symmetry!



TC4

$$\text{rank } \rho_A = 2^{|\partial A|} / \underset{\substack{\uparrow \\ \text{two-to-one mapping}}}{2}$$

ρ has a flat spectrum $\rightarrow$ $S_A = \log \text{rank } \rho_A$

$$S_A = |\partial A| \log 2 - \underbrace{\log 2}_{\gamma \text{ : top correction}}$$

[This is a large detour for a simple result, but: the considerations generalize very straightforwardly for

$G = \mathbb{Z}_2 \rightarrow G$: any finite group!

$\log 2 = \log |\mathbb{Z}_2| \rightarrow \log |G|$

parity $\rightarrow$ $g_1 \cdot g_2 \cdot g_3 \cdot g_4 = e$

'loop sup' $\rightarrow$ g, h, j, k $\rightarrow \sum_x g \circ x, h \circ x, j \circ x, k \circ x$

[Kitaev group quantum double models

$\rightarrow$ include non-Abelian anyon models, e.g.

$D(S_3)$]
]