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Lecture # 12 (non)-Abelian anyons & their properties

I. Particle types:

- We have seen many anyon models
  - a) FQHE b) p-wave c) Toric-code

  ↗ all models have non-trivial braiding statistics of q.-particles (finite # of them!)

- General (algebraic) model

  ⓐ Particle types: (~anyons ~ q.-particles ~ top. charge) ↗ another names

  $I, a, b, \ldots$ → finite #

  ↙ vacuum

  ⓑ For each $a \to \bar{a}$ ↗ anti-particle; could be $a = \bar{a}$

  (e.g. $\gamma^{+} = \gamma \to$ Maj.)

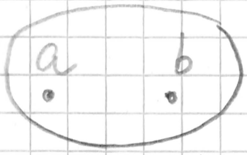
II. Fusion rules for anyons

↗ ! bring to anyons together & identify its collective behaviour

1) $a \times b = \sum_{c}' N_{ab}^{c}\, c$, $N_{ab}^{c} \in \mathbb{Z}_{+}$ ↗ positive integers

2) $a \times b = b \times a$ → commutativity 3) $a \times I = a$

Another way to write: $a \times b = N_{ab}^{c}\, c + N_{ab}^{d}\, d + \ldots$



c or d or...

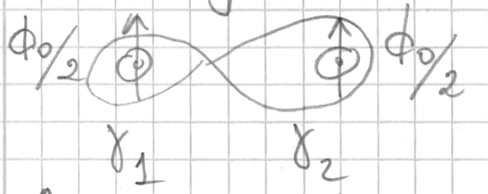
4) Motivation: (i) spin $1/2 \otimes 1/2 = 0 + 1$

proximity $\Rightarrow H = -J\, \hat{\vec{S}}_1 \cdot \hat{\vec{S}}_2$ $J \neq 0$

↓

gives energy splitting

② (ii) Majoranas in p-wave ($\phi_0/2$ vortex)



$c = \frac{1}{2}(\gamma_1 + i\gamma_2)$

$c^\dagger = \frac{1}{2}(\gamma_1 - i\gamma_2)$

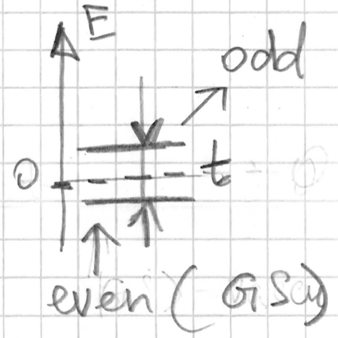
$\{\hat{\gamma}_i, \hat{\gamma}_j\} = 2\delta_{ij}$

$\gamma_i^2 = 1$

$P_{12} = -i\gamma_1\gamma_2 = 1 - 2c^\dagger c$

↓ parity operator

proximity $\Rightarrow$ $\hat{H} = -t P_{12} = it\gamma_1\gamma_2$
if $t>0$



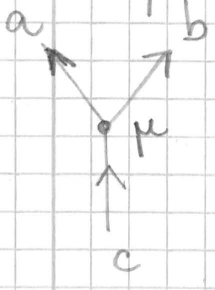
(iii) Toric code: $\varepsilon = e \times m$
(trivial, by def.)

5) Fusion Hilbert space of two anyons:

- spin $\frac{1}{2}$: $\frac{1}{2} \otimes \frac{1}{2} = 0 + 1 \Rightarrow 2 \times 2 = 1 + 3$
  ↓ dim 2
- anyons: $\dim(\mathcal{H}_{ab}) = \sum_c N_{ab}^c$

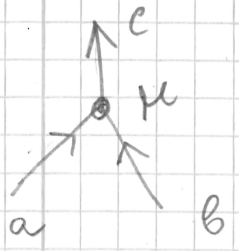
non-Abelian if $\dim(\mathcal{H}_{ab}) \geq 2$ for some (a,b)

- Quantum states in $\mathcal{H}_{ab}$; $\mathcal{H}_{ab} = \sum_c V_{ab}^c$



$= |(ab); c, \mu\rangle \in V_c^{ab}$ → ket / splitting

$\dim(V_c^{ab}) = \dim(V_{ab}^c) = N_{ab}^c$



$= \langle(ab); c, \mu| \in V_{ab}^c$ → bra / fusion

$\mu = 1, \ldots, N_{ab}^c$ → multiple outcomes

③ ! In what follows $N_{ab}^{c} = 0$ or $1 \Rightarrow$ no μ-index
(many simplest models are of this type)

Example: ⓐ Majoranas ≡ Ising anyons ; QHE $\nu = 5/2$ → Moore-Read

$\{I, \sigma, \psi\}$ — σ → Majorana, ψ → fermion

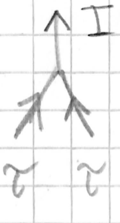
$\sigma \otimes \sigma = I + \psi$
$\sigma \otimes \psi = \sigma$
$\psi \otimes \psi = I$

$\equiv \langle 0|_c$ → even parity ;

$= \left(c^{+}|0\rangle\right)^{+} = \langle 0|c$ → odd parity state | $\dim(H_{\sigma\sigma}) = 2$

ⓑ Fibonacci anyons ; QHE at $\nu = 12/5$ → Read-Rezayi state

$\{I, \tau\}$ | $\tau \times \tau = I + \tau$



$= \langle 0|$

$= \langle 1|$ | $\dim(H_{\tau\tau}) = 2$

III. Associativity of fusion

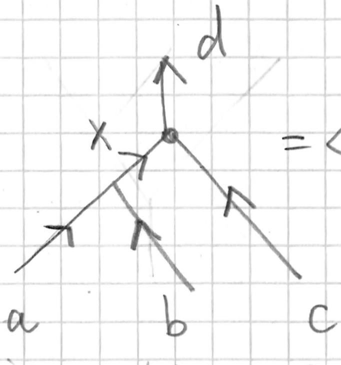
Let's bring 3 anyons together → $\mathcal{H}_{abc}$ → resulting Hilbert space

$(a \times b) \times c = a \times (b \times c)$

Ⓐ $(a \times b) = \sum_x N_{ab}^{x}\, x$

$x \times c = \sum_d N_{xc}^{d}\, d$

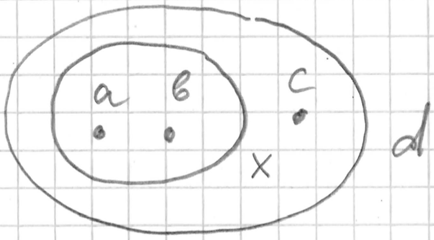
$(a \times b) \times c = \sum_{xd} N_{ab}^{x} N_{xc}^{y}\, d$

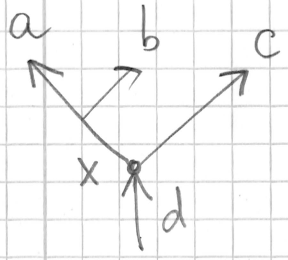


$= \langle bra|$

two-step fusion process

(4) Time-reversal → splitting





$$= |(ab), x\rangle \otimes |(xc), d\rangle \equiv |\mathrm{ket}\rangle$$

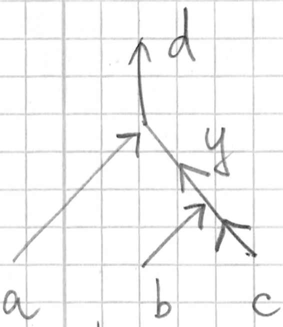
- tree $\equiv$ q. state in $H_{abc} = \sum_x V^{ab}_x \otimes V^{xc}_d$
- $|\mathrm{ket}\rangle \in V^{ab}_x \otimes V^{xc}_d \equiv V^{abc}_d$

  $\langle \mathrm{bra}| \in V^x_{ab} \otimes V^d_{xc} \equiv V^d_{abc}$ dual

  subspace with fixed outcome d

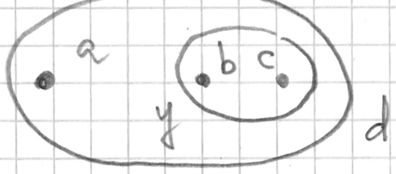
- $\dim\left(V^{abc}_d\right) = \dim\left(V^d_{abc}\right) = \sum_x N^x_{ab} N^{xc}_d$

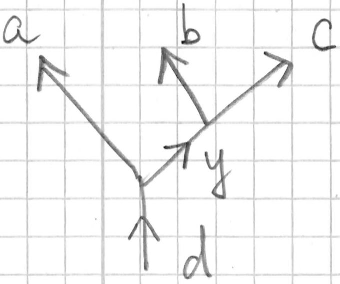
(B) We may choose another (but equivalent!) way:



$$\left[\begin{array}{l} b \times c = \sum_y N^y_{bc}\, y \\ a \times y = \sum_d N^d_{ay}\, d \end{array}\right.$$

$\langle \mathrm{bra}|$ state





$$= |(ay), d\rangle \otimes |(bc), y\rangle \in V^{ay}_d \otimes V^{bc}_y \equiv V^{abc}_d \Rightarrow$$

$|\mathrm{ket}\rangle = \left(\langle \mathrm{bra}|\right)^+$

time-reversal of world lines

→ (A) and (B) are just two different way to define basis in the space V^{abc}_d !

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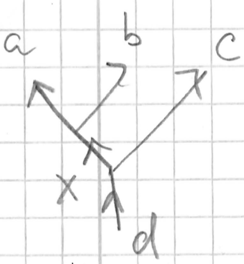
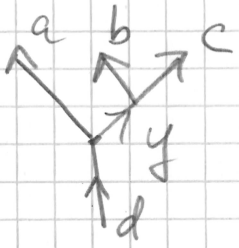
$\Rightarrow$ there is unitary transformation between Ⓐ & Ⓑ



$$= \sum_y \left(F^{abc}_d\right)^y_x$$



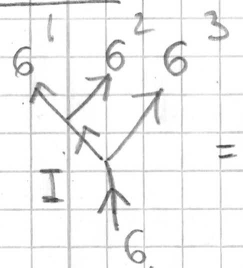
$$|(ab),x\rangle \otimes |(xc),d\rangle = \sum_y |(ay),d\rangle \otimes |(bc),y\rangle \left(F^{abc}_d\right)^y_x$$

- $\hat{F}^{abc}_d$ is known as fusion matrix

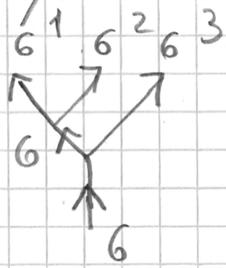
(i) $\left(F^{abc}_d\right)^\dagger = \left(F^{abc}_d\right)^{-1} \to$ unitary

(ii) In most model can be made orthogonal

Example: Ising anyons



$= |0\rangle$



$= |1\rangle$

Q: What is physical interpretation in terms of Majoranas

A: 3 Majoranas $(\gamma_1, \gamma_2, \gamma_3)$ + 1 more (γ_4) far away

- $c_1 = \frac{1}{2}(\gamma_1 + i\gamma_2)$ | $c_2 = \frac{1}{2}(\gamma_3 + i\gamma_4)$

  $c_1^\dagger = \frac{1}{2}(\gamma_1 - i\gamma_2)$ | $c_2^\dagger = \frac{1}{2}(\gamma_3 - i\gamma_4)$

- Let $|0\rangle_c \to$ vacuum for $c_{1,2} \Rightarrow c_1|0\rangle_c = c_2|0\rangle_c = 0$

- We choose to work in even parity sector

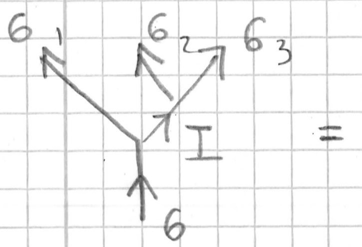
  $\curvearrowright$ our two states

  $|0\rangle_c$ and $c_1^\dagger c_2^\dagger |0\rangle$

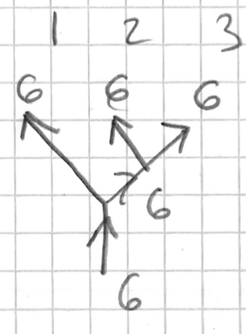
- Natural interpretation: $|0\rangle = |0\rangle_c$; $|1\rangle = c_1^\dagger c_2^\dagger |0\rangle_c$

$\left\{ \gamma_1 \,\&\, \gamma_2 \text{ fuse to vacuum} \right.$ $\qquad$ $\left\{ \gamma_1 \,\&\, \gamma_2 \text{ fuse to fermion } c_1 \right.$

Another basis



$= |0'\rangle$



$= |1'\rangle$

- $d_1 = \frac{1}{2}(\gamma_1 + i\gamma_4)$ $\qquad$ $d_2 = -\frac{i}{2}(\gamma_2 + i\gamma_3)$

$d_1^\dagger = \frac{1}{2}(\gamma_1 - i\gamma_4)$ $\qquad$ $d_2^\dagger = \frac{i}{2}(\gamma_2 - i\gamma_3)$

⤳ another set of fermions

- $\hat{c}$ & $\hat{d}$ are lineerly related (exersice!)

$$\begin{cases} d_1 = \frac{1}{2}(c_1 + c_1^\dagger + c_2 - c_2^\dagger) \\ d_2 = \frac{1}{2}(c_1^\dagger - c_1 + c_2 + c_2^\dagger) \end{cases}$$

$\Rightarrow$ $|0\rangle_d$ is not $|0\rangle_c$! ← different vacuums

- By definition $\quad d_1|0\rangle_d = d_2|0\rangle_d = 0$

$\Rightarrow$

(a) $|0\rangle_d = \frac{1}{\sqrt{2}}\left(|0\rangle_c + c_1^\dagger c_2^\dagger |0\rangle_c\right)$

exercise! (b) $d_1^\dagger d_2^\dagger |0\rangle_d = \frac{1}{\sqrt{2}}\left(|0\rangle_c - c_1^\dagger c_2^\dagger |0\rangle_c\right)$

- Similar to 1st choice we interprete

$|0\rangle_d = |0'\rangle$ $\qquad$ $d_1^\dagger d_2^\dagger |0\rangle_d = |1'\rangle$

$\left\{ \gamma_2, \gamma_3 \text{ fuse to vacuum} \right.$ $\qquad$ $\left\{ \gamma_2 \,\&\, \gamma_3 \text{ fuse to fermion } d_2 \right.$

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- Hence (a) & (b) give relation between two basis set

$$\begin{pmatrix} |0\rangle \\ |1\rangle \end{pmatrix} = \underbrace{\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}}_{= H} \begin{pmatrix} |0'\rangle \\ |1'\rangle \end{pmatrix}$$

$H \to$ Hadamard matrix.

$$\Rightarrow \boxed{F^{666}_{6} = H} \to$$ fusion matrix of 3 Ising anyons with total charge 6