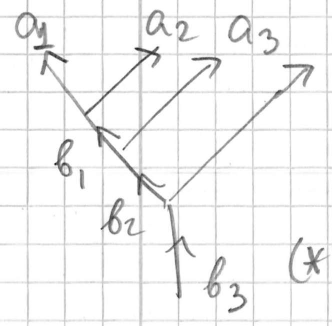
1.

Lecture #13 Quantum dimensions, d_a
Pentagon equations for F-matrix

I. Quantum dimensions

- Consider fusion of n anyons, $(a_1, \ldots, a_n)$

$\rightarrow$ $\mathcal{H}_{a_1,\ldots,a_n} \rightarrow$ resulting Hilbert space



$$(a_1 \times a_2) \times a_3 \ldots \times a_n \qquad N_{ab}^{c} \overset{!}{=} N_{ba}^{c}$$

Reminder: $a \times b = \sum_c N_{ab}^c \, c$

$$(*) \quad \dim(\mathcal{H}_{a_1,\ldots,a_n}) = \sum_{b_1,\ldots,b_{n-1}}' N_{a_1 a_2}^{b_1} N_{b_1 a_3}^{b_2} \ldots N_{b_{n-2} a_n}^{b_{n-1}}$$

- Particular situation: $a_1 = a_2 = \ldots = a_n = a$

$\mathcal{H}_a^{(n)} \rightarrow$ Hilbert space

Def: if $\dim(\mathcal{H}_a^{(n)}) \sim (d_a)^n$ at $n \to \infty$

then d_a is quantum dimension

a) $d_I = 1$ for vacuum state

b) $d_a = 1$ for Abelian (see below), $d_a > 1 \rightarrow$ non-Abelian

! • d_a can be fractional / irrational, not $\mathbb{Z}_+$.

Q: How to find d_a - ?

A. Lemma: define matrix $\hat{N}_a$ with matrix elements $(\hat{N}_a)^x{}_y = N_{ay}^x$

$\Rightarrow$ d_a – largest eigenvalue of $\hat{N}_a$

Proof: Use (i) $a_1 = \ldots = a_n = a$ (ii) reverse order in $(*)$

(iii) $N_{b_{n-2} a}^{b_{n-1}} = N_{a b_{n-2}}^{b_{n-1}}$ etc $\Rightarrow$

$$\dim(\mathcal{H}_a^{(n)}) = \sum_{b_{n-1},\ldots,b_1}' (N_a)^{b_{n-1}}{}_{b_{n-2}} (N_a)^{b_{n-2}}{}_{b_{n-3}} \ldots (N_a)^{b_1}{}_a$$

$$(b_{n-1} = b) \qquad = \sum_b' (\hat{N}_a^{n-1})^b{}_a \sim d_a^{\,n-1}$$

② To see the last assymptotic one expands over eigenbasis:

$$\left(N_a\right)^{n-1} = \sum_\mu |e_\mu\rangle \, \underbrace{\lambda_\mu}_{\text{eigenvalue}} \, \underbrace{\langle e_\mu|}_{\text{eigenstate}}$$

$$\Rightarrow \quad \sum_b (N_a^{n-1})^b{}_a = \sum_{\mu b} \langle b|e_\mu\rangle \lambda_\mu^{n-1} \langle e_\mu|a\rangle$$

and $\quad \ln \dim (H_a^{(n)}) = (n-1) \ln (\lambda_{max}) + O(1)$

$$= n \ln (d_a) + O(1) \quad \blacksquare$$

Let $\langle e_{max}| = (e_1, \dots, e_n)$ – left eigenstate, i.e.

$$\langle e_{max}| \hat{N}_a = d_a \langle e_{max}| \Rightarrow \sum_c e_c N_a{}^c{}_b = d_a e_b$$

(i) From Peron-Frobenius theorem $\Leftarrow \forall b \quad e_b > 0$

(ii) $N_{ac}^b = N_{ca}^b$ [see sec. 17.7 from Simon book]

$\Big\} \Rightarrow e_b \overset{!}{=} d_b$

$$\Rightarrow \quad \boxed{\sum_c N_{ab}^c d_c = d_a d_b} \rightarrow \text{eq. for } d_a$$

Example:

ⓐ Ising anyons:

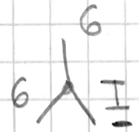
$\sigma \times \sigma = I + \psi$

$\sigma \times \psi = \sigma$

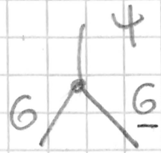
$\psi \times \psi = I$

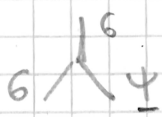
σ-matrix

$$N_\sigma = \begin{array}{c|ccc} & 1 & \sigma & \psi \\ \hline 1 & \varnothing & 1 & \varnothing \\ \sigma & 1 & \varnothing & 1 \\ \psi & \varnothing & 1 & \varnothing \end{array}$$

1^{st} column: 

2^{nd}: 

and 

3^d: 

$$\langle e_{max}| = (1, \sqrt{2}, 1) \quad ; \quad \langle e_{max}| \hat{N}_\sigma = \sqrt{2} \langle e_{max}|$$

$$\Rightarrow d_I = d_\psi = 1, \quad \boxed{d_\sigma = \sqrt{2}}$$

ψ-matrix: (exercise)

$$N_\psi = \begin{array}{c|ccc} & 1 & \sigma & \psi \\ \hline 1 & \varnothing & \varnothing & 1 \\ \sigma & \varnothing & 1 & \varnothing \\ \psi & 1 & \varnothing & \varnothing \end{array},$$

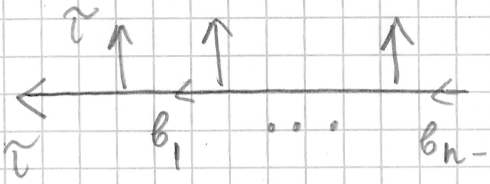
$$\Rightarrow \langle e_{max}| \hat{N}_\psi = \langle e_{max}|$$

(3.)

(b) Fibonacci anyons

$$\tau \times \tau = \tau + I$$
$$\tau \times I = \tau$$

Possible fusion of n τ particles



$b_{n-1} = I$ or τ

Info: Fibonacci sequence: $F_{n+1} \overset{\text{by def.}}{=} F_n + F_{n-1}$

$F_0 = F_1 = 1 \Rightarrow F_2 = 2$, $F_3 = 3$, etc.

(a) # of trees with $b_{n-1} = I$ is F_{n-2}

(b) # of trees with $b_{n-1} = \tau$ is F_{n-1}

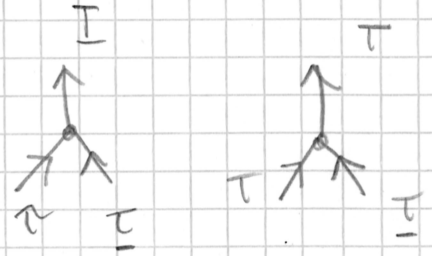
$\Rightarrow$ total # of trees $\boxed{\dim\left(H_{\tau}^{(n)}\right) = F_n}$

τ-matrix

$$N_\tau = \begin{array}{c|cc} & 1 & \tau \\ \hline 1 & 0 & 1 \\ \tau & 1 & 1 \end{array}$$

1^{st} column

2^{nd}:



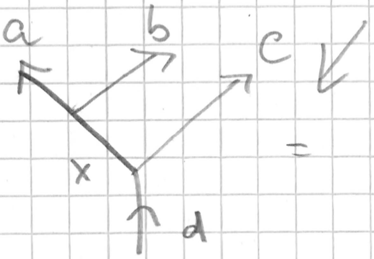
$$\begin{vmatrix} \lambda & -1 \\ -1 & \lambda-1 \end{vmatrix} = 0 \Rightarrow \lambda^2 - \lambda - 1 = 0 \Rightarrow$$

$$\boxed{d_\tau = \frac{1+\sqrt{5}}{2} = \varphi} \rightarrow \text{Golden ratio}$$

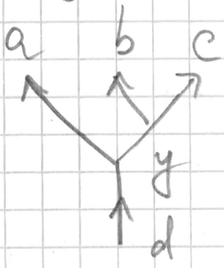
$\varphi \simeq 1,62$

II Pentagon equations "change of basis"

- F-matrix: (see Lecture # 12)



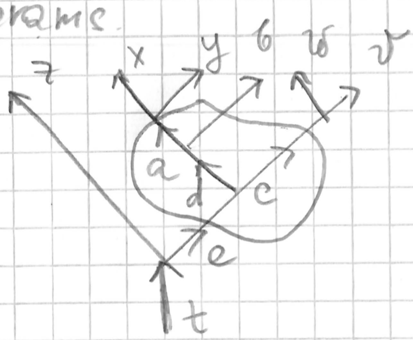
$= \sum_y$



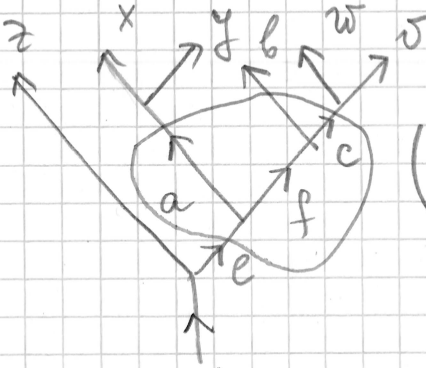
$\left(F^{abc}_{d}\right)^{y}_{x}$

- F-matrix can be applied to more complicated diagrams.

Example:



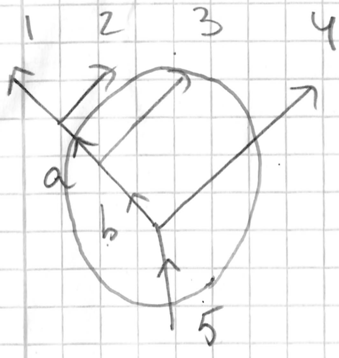
$= \sum_f$



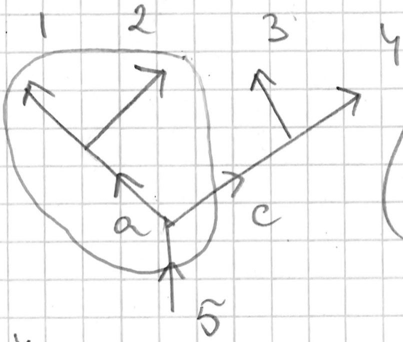
$\left(F^{abc}_{e}\right)^{d}_{f}$

- Pentagon eq. → consider fusion of four anyons (1, 2, 3, 4)

(I)

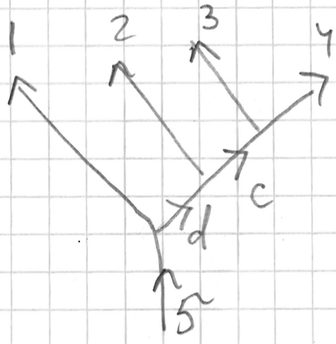


$\xrightarrow{F^{a34}_{5}} \sum_c$



$\left(F^{a34}_{5}\right)^{c}_{b}$

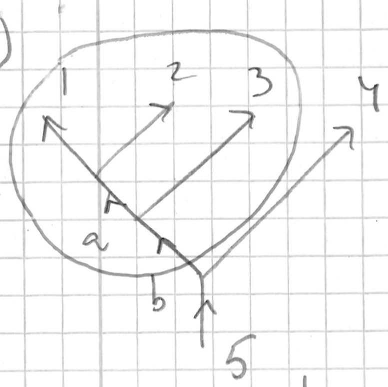
$\xrightarrow{F^{12c}_{5}} \sum_{cd}$



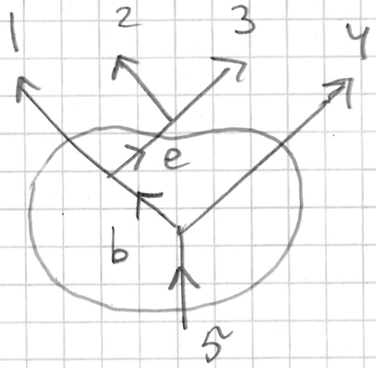
$\left(F^{12c}_{5}\right)^{d}_{a} \cdot \left(F^{a34}_{5}\right)^{c}_{b}$

(II) One can perform the same fusion in three steps (see below)

(5)

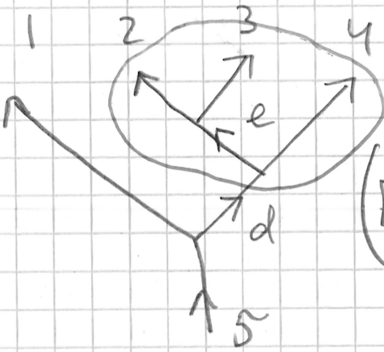


$\xrightarrow{F_b^{123}} \sum_e$



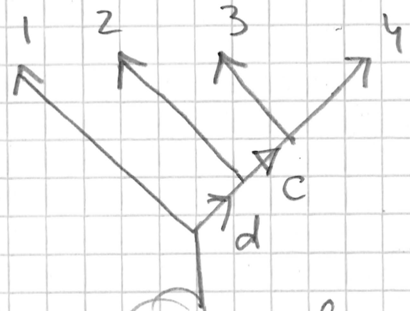
$\left(F_b^{123}\right)_a^e$

$\xrightarrow{F_5^{1e4}} \sum_{ed}'$



$\left(F_5^{1e4}\right)_b^d \left(F_b^{123}\right)_a^e$

$\xrightarrow{F_d^{234}} \sum_{ecd}$



$\left(F_d^{234}\right)_e^c \left(F_5^{1e4}\right)_b^d \left(F_b^{123}\right)_a^e$

Summary: (I) = (II) for $\forall\ (a,b)$ and (c,d)

$(a,b) \downarrow$ initial $\quad (c,d) \downarrow$ final

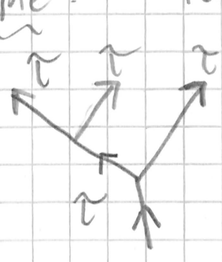
Pent. eq.

$$\Longrightarrow \quad \left(F_5^{12c}\right)_a^d \cdot \left(F_5^{a34}\right)_b^e = \sum_e \left(F_d^{234}\right)_e^c \cdot \left(F_5^{1e4}\right)_b^d \left(F_b^{123}\right)_a^e$$

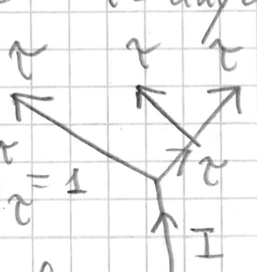
- Application: Pentagon equations can be used to find fusion matrix $\hat{F}$ for particular anyon model.

Example: fusion of three τ-anyons in Fibonacci model

(A)



$= |N\rangle \quad \left(F_I^{\tau\tau\tau}\right)_\tau^\tau = 1$

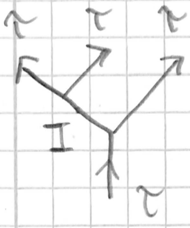


$= |N'\rangle$

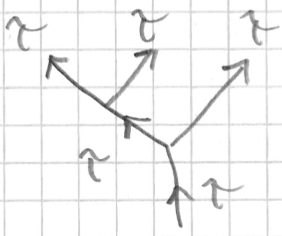
$F_1 = 1$

Trees with fusion outcome I

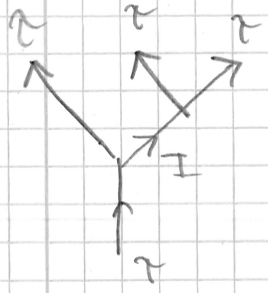
(6) (B) $F_2 = 2$ → trees with fusion outcome τ



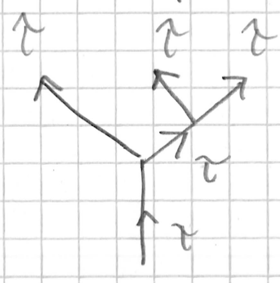
$= |0\rangle$



$= |1\rangle$



$= |0'\rangle$



$= |1'\rangle$

$$\begin{pmatrix} |0\rangle \\ |1\rangle \end{pmatrix} = \hat{F}^{\tau\tau\tau}_{\tau} \begin{pmatrix} |0'\rangle \\ |1'\rangle \end{pmatrix}, \quad \hat{F}^{\tau\tau\tau}_{\tau} = \begin{array}{c c} & \begin{array}{cc} I & \tau \end{array} \\ \begin{array}{c} I \\ \tau \end{array} & \begin{pmatrix} \varphi^{-1} & \varphi^{-1/2} \\ \varphi^{-1/2} & -\varphi^{-1} \end{pmatrix} \end{array}$$

where $\varphi = \frac{1+\sqrt{5}}{2}$

- This F-matrix can be obtained as the solution of Pentagon equations! (See problem set #4)

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* For any given set of fusion rules there is a finite (possibly empty) set of possible solutions to Pentagon equations – "Ocneanu rigidity"

* Mac Lane theorem: more complicated trees do not generate new identities for F-matrix beyond the pentagon diagram