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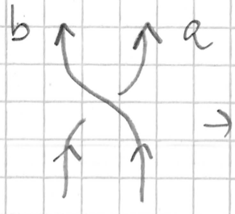
Lecture #14: Braiding of non-Abelian anyons & Hexagon equations

I. Braiding of (non-identical) anyons

A. $\hat{R}$-matrix.

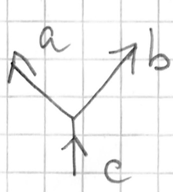
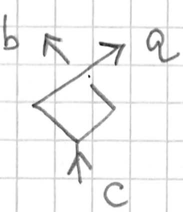
We want to ask what happens if one exchanges two different anyons (a & b)

→ clockwise (right-handed)



→ anti-clockwise (left-handed)

• Consider state $|(ab), c\rangle$ – a & b are fused to c

$\hat{R}_{ab}$ [] = 

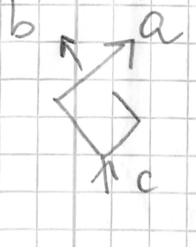
$\hat{R}_{ab} : V^{ab}_c \mapsto V^{ba}_c$

unitary operator

input space; output

ⓐ $a \times b = \sum_c N^c_{ab}\, c$ with $N^c_{ab} = \{0, 1\}$

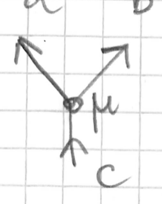
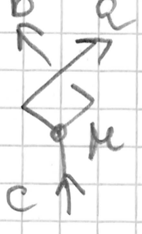
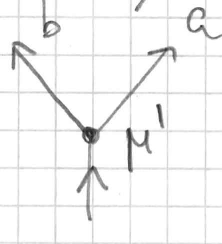
$\Rightarrow \dim(V^{ab}_c) = \dim(V^{ba}_c) = 1 \Rightarrow$ one picks up (Berry) phase

 $= R^{ba}_c$  , with $R^{ba}_c = e^{i\theta^{ab}_c}$

phase

ⓑ Let $N^c_{ab} \geq 2 \Rightarrow \dim(V^{ab}_c) = \dim(V^{ba}_c) = N^c_{ab}$

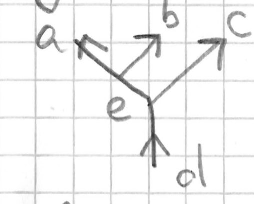
Here one deals with non-Abelian Berry connection

R_{ab} [] =  $= \sum_{\mu'}$  $\left(\hat{R}^{ba}_c\right)^{\mu'}_{\mu}$

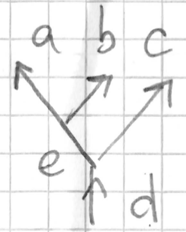
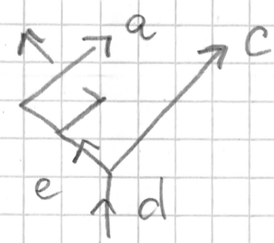
$\mu = 1, \ldots, N^{ab}_c$ | $\hat{R}^{ba}_c$ is unitary matrix $N^{ab}_c \times N^{ab}_c$

(2) (c) Anti-clockwise braiding → inverse, R^{+}_{ba}

(B) Braiding of <u>three</u> anyons (a, b, c)

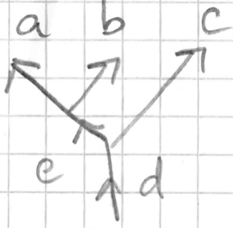
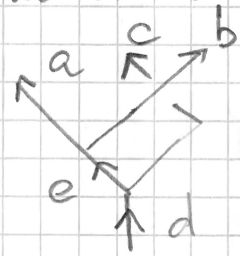
Input state  $= |(ab), e\rangle \otimes |(ec), d\rangle \in V^{abc}_{d}$

(In what follows no μ, μ' indexes, $N^{c}_{ab} = 0, 1$)

- $\hat{\sigma}_1 \Big[$  $\Big] =$  → exchange of 1^{st} and 2^{nd} anyons

$\sigma_1 : V^{abc}_{d} \mapsto V^{bac}_{d}$

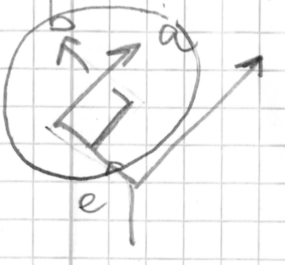
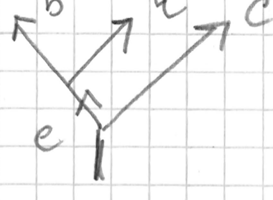
$\hat{\sigma}_1 \equiv \hat{B}_{ab}$ → just different notation

- $\hat{\sigma}_2 \Big[$  $\Big] =$  $\Big|\ \sigma_2 : V^{abc}_{d} \mapsto V^{acb}_{d}$

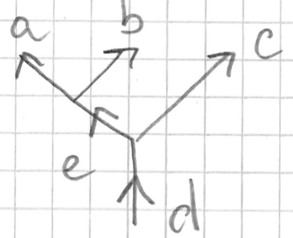
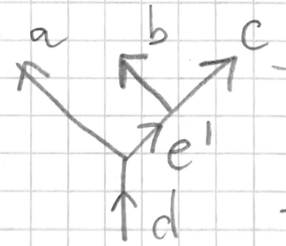
$\hat{\sigma}_2 \equiv B_{bc}$

Q: What are $\hat{B}$ operators in terms of $\hat{R}$?

A: (i) For $\sigma_1 = B_{ab}$ the answer is clear:

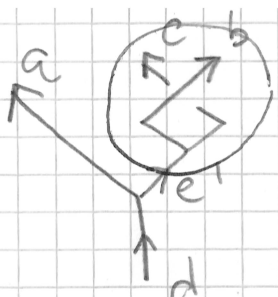
 $=$  R^{ba}_{e}

(ii) For $\hat{\sigma}_2 \equiv B_{bc}$ we may use F-moves.

$\hat{\sigma}_2 \Big[$  $\Big] = \sum_{e'} \hat{\sigma}_2 \Big[$  $\Big] \left(F^{abc}_{d}\right)^{e'}_{e}$

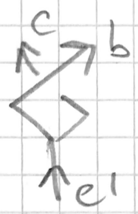
↑ F^{abc}_{d}

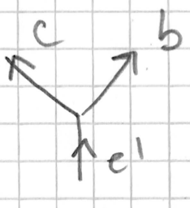
next page $\underline{\underline{*}}$

③ $\overset{*}{=} \sum_{e'}$ 

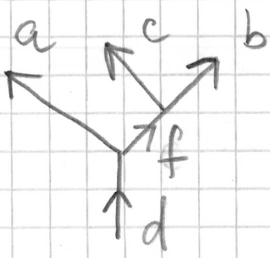
$\cdot \left(F^{abc}_{d}\right)^{e'}_{e} \overset{*}{=}$

[! The braiding $\sigma_2 : V^{bc}_{e'} \mapsto V^{cb}_{e'}$ is again known

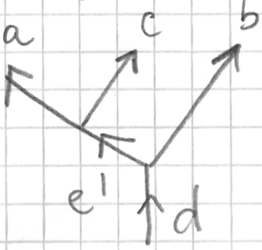


$=$ 

$R^{cb}_{e'}$, hence

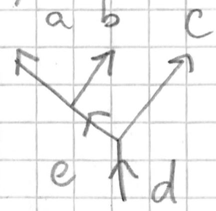
$\underset{(e'\to f)}{\overset{*}{=}} \sum_{f}$ 

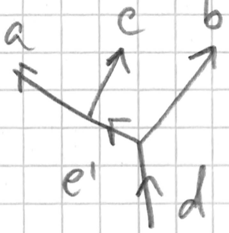
$R^{cb}_{f} \cdot \left(F^{abc}_{d}\right)^{f}_{e} \overset{\left(F^{acb}_{d}\right)^{-1}}{=}$

$= \sum_{fe'}$ 

$\left(F^{acb}_{d}\right)^{-1\,e'}_{\ f} \; R^{cb}_{f} \left(F^{abc}_{d}\right)^{f}_{e}$

$\Rightarrow$ Summary:

B_{bc} [

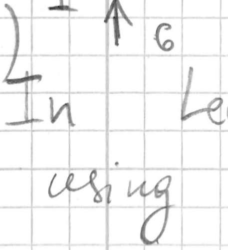
] $= \sum_{e'}$ 

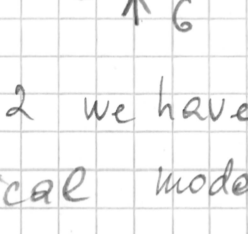
$\left(B_{bc}\right)^{e'}_{e}$

where $\left(B_{bc}\right)^{e'}_{e} = \sum_{f} \left(F^{acb}_{d}\right)^{-1\,e'}_{\ f} \; R^{cb}_{f} \left(F^{abc}_{d}\right)^{f}_{e}$

- Example: Ising anyons ⇝ Maj. fermions in p-wave SC

Consider fusion space V^{666}_{6} of $\dim(V^{666}_{6}) = 2$



$= |0\rangle$ 

$= |1\rangle$ | $6 \times 6 = I + \psi$

$6 \times \psi = 6$

(i) In Lecture #12 we have interpreted these states using a physical model with four Majoranas, $\hat{\gamma}_1 \ldots \hat{\gamma}_4$

(4) With $c_1 = \frac{1}{2}(\gamma_1 + i\gamma_2)$ and $c_2 = \frac{1}{2}(\gamma_3 + i\gamma_4)$
$[\{\gamma_i, \gamma_j\} = 2\delta_{ij}]$ one identifies

$$|0\rangle = \underbrace{|0\rangle_c}_{\text{physical vacuum for } c_1 \,\&\, c_2} \quad \text{and} \quad |1\rangle = c_1^\dagger c_2^\dagger |0\rangle_c$$

(ii) In Lecture #7 braiding matrices B_{12} & B_{23} were constructed:

$$B_{12} = e^{i\alpha} e^{\frac{\pi}{4}\gamma_2\gamma_1} = \begin{array}{c|cc} & |0\rangle & |1\rangle \\ |0\rangle & e^{-i\pi/4} & \emptyset \\ |1\rangle & \emptyset & e^{i\pi/4} \end{array} \, e^{i\alpha}$$

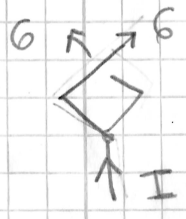
see below!

$$B_{23} = e^{i\alpha} e^{\frac{\pi}{4}\gamma_2\gamma_3} = \begin{array}{c|cc} & |0\rangle & |1\rangle \\ |0\rangle & 1 & -i \\ |1\rangle & -i & 1 \end{array} \, \frac{e^{i\alpha}}{\sqrt{2}}$$

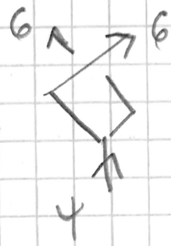
we restrict $\hat{B}$'s to even parity sector

(iii) The F-matrix is $F^{666}_{6} = H = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$
(see Lecture #12)

(iv) $\hat{R}$-matrix is equivalent to B_{12}, i.e.



$\rightarrow R^{66}_{I} = e^{i\alpha} e^{-i\pi/4}$;



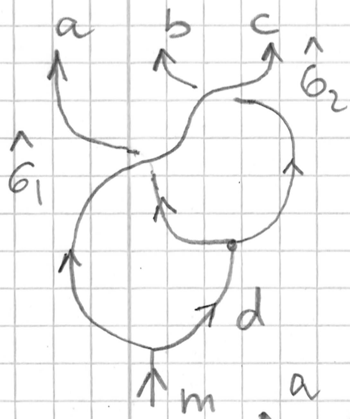
$\rightarrow R^{66}_{\psi} = e^{i\alpha} e^{i\frac{\pi}{4}}$

$\Rightarrow$ the relation $\boxed{B_{23} = H^{-1} R H}$, $R = \begin{pmatrix} R^{66}_{I} & \emptyset \\ \emptyset & R^{66}_{\psi} \end{pmatrix}$
holds (please check by yourself!)

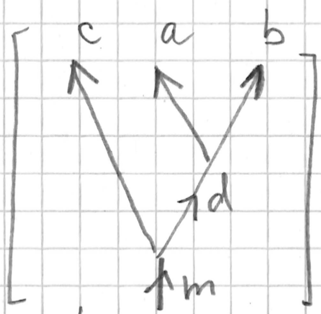
- In Majorana model Abelian phase α was left unidentified (projective representation was discussed)

Ising anyon model $\boxed{\alpha = \pi/8}$ $\Rightarrow$ it follows from Hexagon equations (see below)

⑤ II. Locality: fusion/splitting & braiding have to be compatible to each other



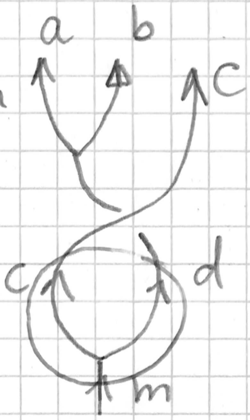
$= \hat{\sigma}_2 \hat{\sigma}_1$



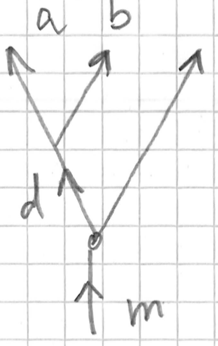
↓ two subseq. braidings

axiom

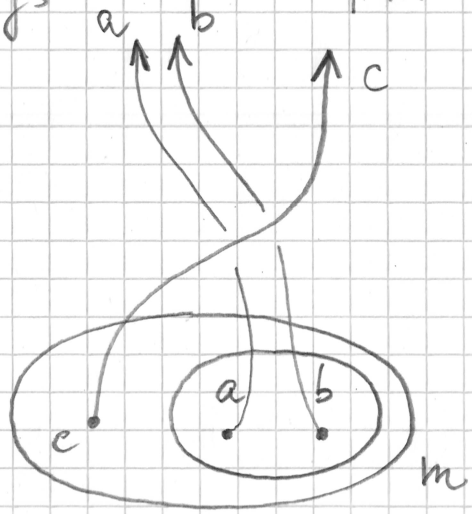
$\stackrel{!}{=}$



$=$



$\cdot R^{dc}_{m}$



* Braiding of c-anyon with the cluster (or pair) (a,b) has the same effect as braiding with a single anyon d (which is a collective quantum number of a pair (a,b)).