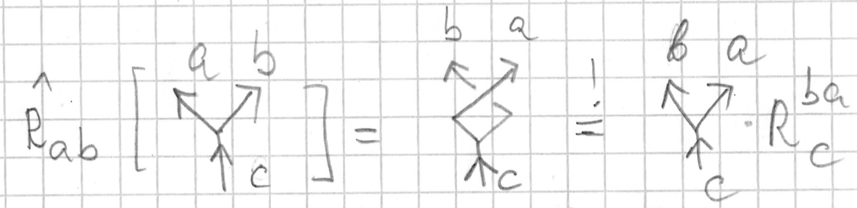
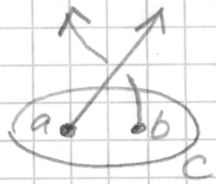
(1.a) Lecture #15 Braiding, hexagon equations

I. Reminder

- Two anyons:



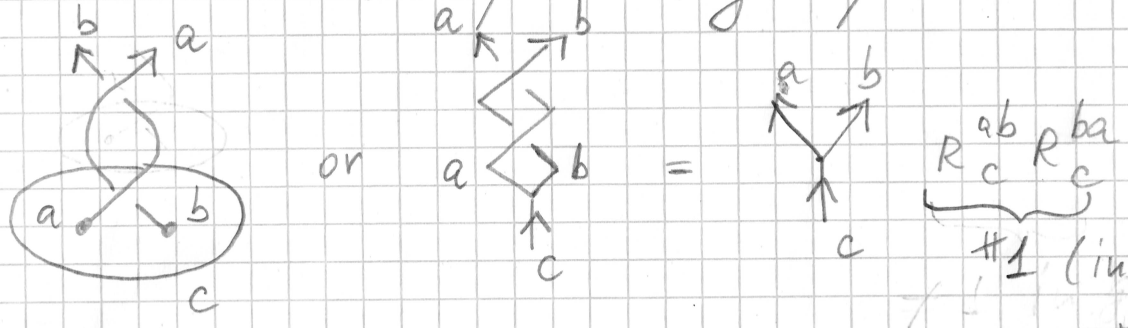


(i) Braiding is defined by R-matrix

(ii) For $N^c_{ab} = 1$ $\quad R^{ba}_c = e^{i\theta^{ab}_c}$ → phase

(iii) Double exchange =

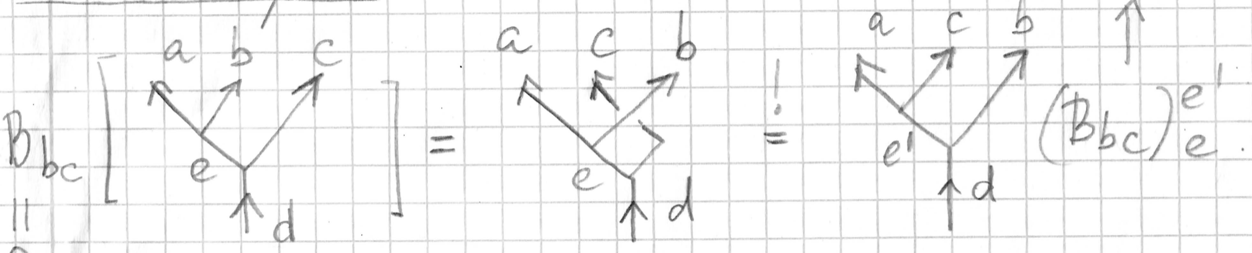
= full braid (a-anyon wrapping fully around b-anyon)



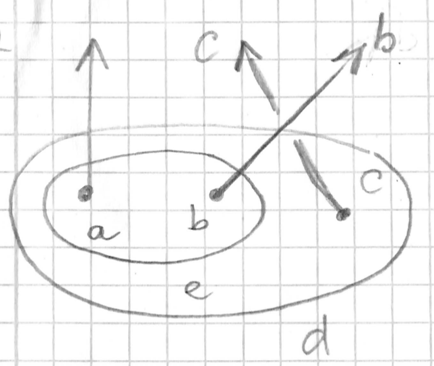
$\underbrace{R^{ab}_c R^{ba}_c}_{\neq 1 \text{ (in general)}}$

non-trivial matrix

- Three anyons:



$\| \quad \hat{\sigma}_2$



(i) $B \sim F^{-1} R F$ → schematically (see Lecture #14 for details)

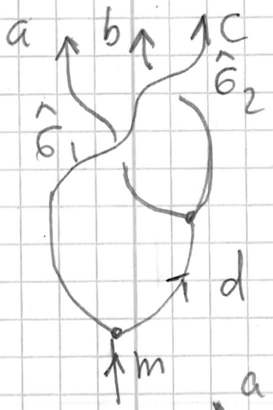
(ii) F → fusion matrix

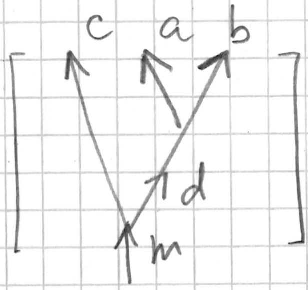
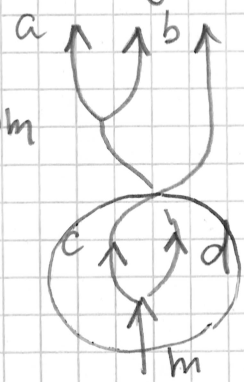
* Braidings induces non-trivial unitary transformations on the anyon Hilbert space: V^{abc}_d

(! total quantum # d remains unchanged!) $\quad \mapsto V^{acb}_d$

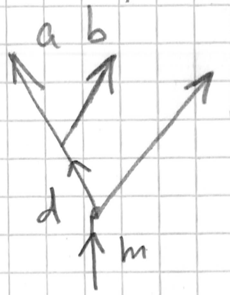
1.b

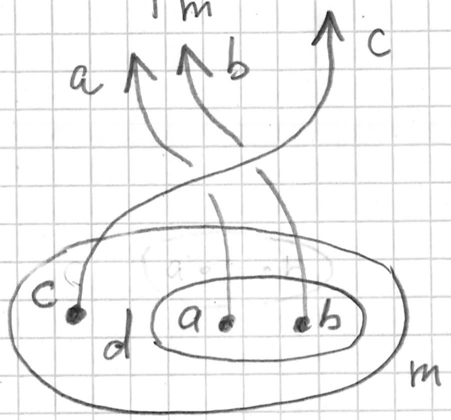
II Locality: fusion/splitting & braiding have to be compatible to each other



$= \hat{\sigma}_2 \hat{\sigma}_1$ [] $\overset{\text{axiom IV}}{=}$  $=$

two subsequent braidings

$=$  $\cdot\ R^{dc}_{m}$

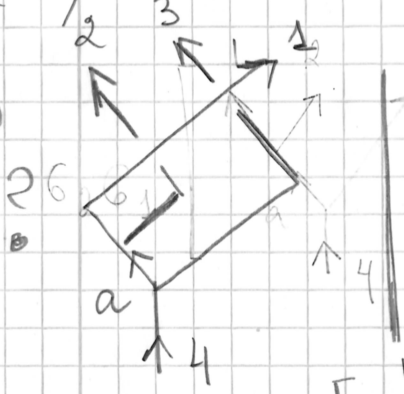
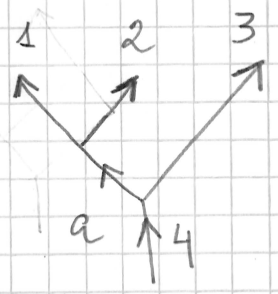


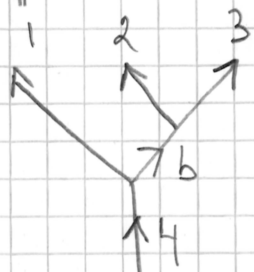
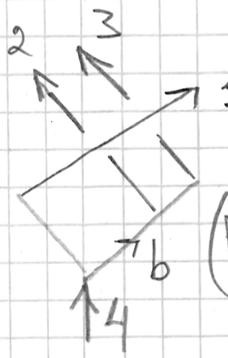
with

* Braiding of c-anyon with the cluster (or pair) (a, b) has the same effect as with a single anyon d (which is a collective q. number of pair (a, b))

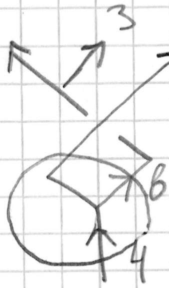
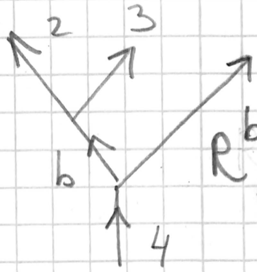
III
- Pentagon equations define fusion matrix F.
- Let's derive Hexagon equation for R-matrix

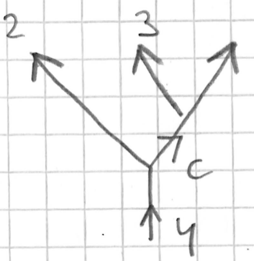
(they establish compatibility of fusion & braiding)

A  $= \sigma_2 \sigma_1$ [] $\overset{F^{123}_{4}}{=}$

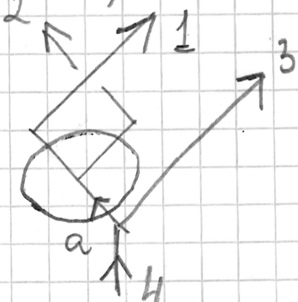
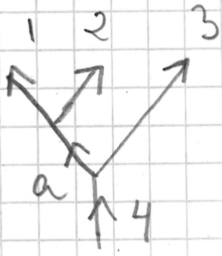
$= \sigma_2 \sigma_1 \sum_b$ [ $\left(F^{123}_{4}\right)^{b}_{a}$] $= \sum_b{}'$  $\left(F^{123}_{4}\right)^{b}_{a}$

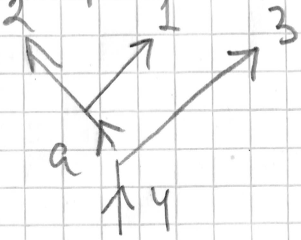
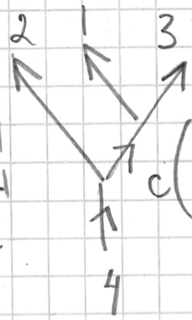
= (we use locality axiom!) = (next page)

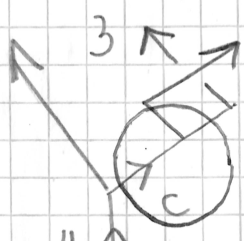
② ... $= \sum_b$  $\left(F^{123}_4\right)^b_c = \sum_b$  $R^{b1}_4 \left(F^{123}_4\right)^b_c$

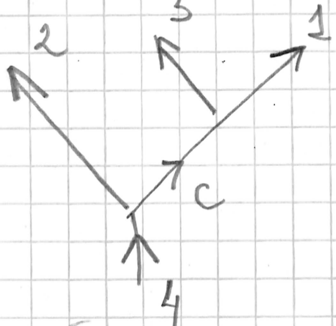
$\overset{F^{231}_4}{=} \sum_{bc}'$  $\left(F^{231}_4\right)^c_b \; R^{b1}_4 \left(F^{123}_4\right)^b_c$

Ⓑ Let's try another way to arrive to the same final state

$6_2 \Big[$  $\Big] = 6_2 \hat{R}_{12} \Big[$  $\Big] =$

$= 6_2 \Big[$  $R^{21}_a \Big] \overset{F^{213}_4}{=} 6_2 \Big[\sum_c'$  $\left(F^{213}_4\right)^c_a R^{21}_a \Big]$

$= \sum_c$  $\left(F^{213}_4\right)^c_a \cdot R^{21}_a =$

$= \sum_c'$  $R^{31}_c \cdot \left(F^{213}_4\right)^c_a \cdot R^{21}_a$

<u>Result:</u> Ⓐ = Ⓑ for $\forall (a,c)$ =>

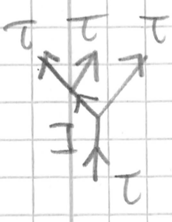
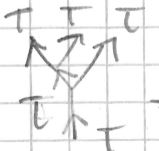
$$\sum_b \left(F^{231}_4\right)^c_b \; R^{b1}_4 \cdot \left(F^{123}_4\right)^b_c = R^{31}_c \cdot \left(F^{213}_4\right)^c_a \cdot R^{21}_a$$

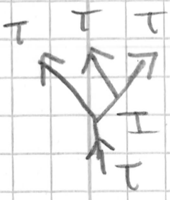
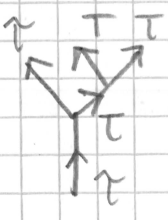
known as Hexagon equation

③ III. Application # 1 : Fibonacci anyons

- Consider splitting space $V^{\tau\tau\tau}_{\tau}$, dim = 2

$\Rightarrow$ two basis sets (see problem set #4)

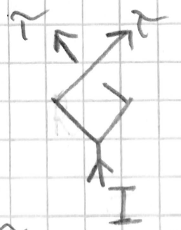
 $= |0\rangle$;  $= |1\rangle$

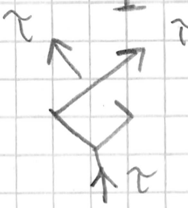
 $= |0'\rangle$;  $= |1'\rangle$

$$\begin{pmatrix} |0\rangle \\ |1\rangle \end{pmatrix} = \hat{\mathcal{F}}^{\tau\tau\tau}_{\tau} \begin{pmatrix} |0'\rangle \\ |1'\rangle \end{pmatrix}$$

Fusion matrix $\mathcal{F}^{\tau\tau\tau}_{\tau} \equiv F = \begin{pmatrix} \varphi^{-1} & \varphi^{-1/2} \\ \varphi^{-1/2} & -\varphi^{-1} \end{pmatrix}$, $\varphi = \frac{1+\sqrt{5}}{2}$

- We search for two phases

 $=$  $R^{\tau\tau}_{I}$

 $=$  $R^{\tau\tau}_{\tau}$

$$\hat{R} := \begin{pmatrix} R^{\tau\tau}_{I} & \emptyset \\ \emptyset & R^{\tau\tau}_{\tau} \end{pmatrix}$$

$\boxed{R^{I\tau}_{\tau} = R^{\tau I}_{\tau} = 1}$ trivial braidings

- Hexagon equation assume the form $(1=2=3=4=\tau)$

$$R^{\tau\tau}_{c} \, F^{c}_{b} \cdot R^{\tau\tau}_{a} = \sum_{b=\{I,\tau\}} F^{c}_{b} \, R^{b\tau}_{\tau} \, F^{b}_{a} \qquad a,b,c \in \{I,\tau\}$$

If we define another matrix $\tilde{R} = \begin{pmatrix} 1 & \emptyset \\ \emptyset & R^{\tau\tau}_{\tau} \end{pmatrix}$

$b = I \Rightarrow R^{I\tau}_{\tau} = 1$

then we get 2x2 matrix equation

$$\boxed{R \cdot F \cdot R = F \tilde{R} F} \Rightarrow \left(R^{\tau\tau}_{I} = x \, , \quad R^{\tau\tau}_{\tau} = y \right)$$

$$\begin{pmatrix} x^2\varphi^{-1} & xy\varphi^{-1/2} \\ xy\varphi^{-1/2} & -y^2\varphi^{-1} \end{pmatrix} = \begin{pmatrix} y\varphi^{-1} + \varphi^{-2} & \varphi^{-3/2}(1-y) \\ \varphi^{-3/2}(1-y) & y\varphi^{-2} + \varphi^{-1} \end{pmatrix}$$

$\rightsquigarrow$ three equations for two unknown (x, y)

(4)

- However, the solution does exist!

$$x_1 = R^{\tau\tau}_{I} = e^{4i\pi/5} \; ; \; y_1 = R^{\tau\tau}_{\tau} = e^{-3i\pi/5} \rightarrow$$

"left-handed" Fib. anyons

$$x_2 = R^{\tau\tau}_{I} = e^{-4i\pi/5} \; ; \; y_2 = R^{\tau\tau}_{\tau} = e^{+3i\pi/5}$$

$\rightarrow$ "right-handed" Fib. anyons

One may "Mathematica" package to check the above statements.