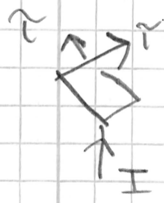
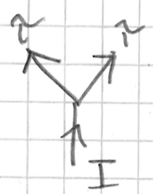
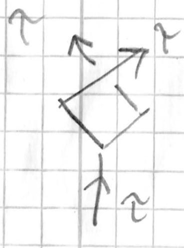
(4.0) **Lecture #16**

Part I: Braiding Part II – Topol. q. comput.

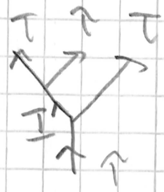
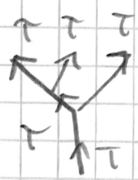
(I) Reminder: previous lecture → we've derived & then solved hexagon eq. for R-matrix on the example of Fib. anyons.

- Fib. anyons: $\{I, \tau\}$, $\tau \times \tau = I + \tau$
- R-matrix ("left-handed")

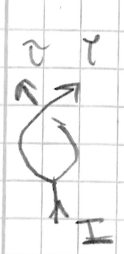
 =  $\left(R^{\tau\tau}_{I} = e^{4i\pi/5}\right)$

 =  $\left(R^{\tau\tau}_{\tau} = e^{-3i\pi/5}\right)$

- Basis set in Hilbert space $V^{\tau\tau\tau}_{\tau}$

 $= |0\rangle$  $= |1\rangle$

⇝ Continue page (4)

 and  → just better drawings used in other books

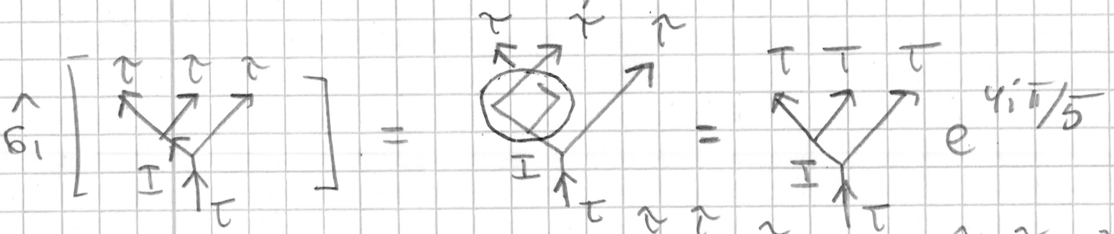
④ ②

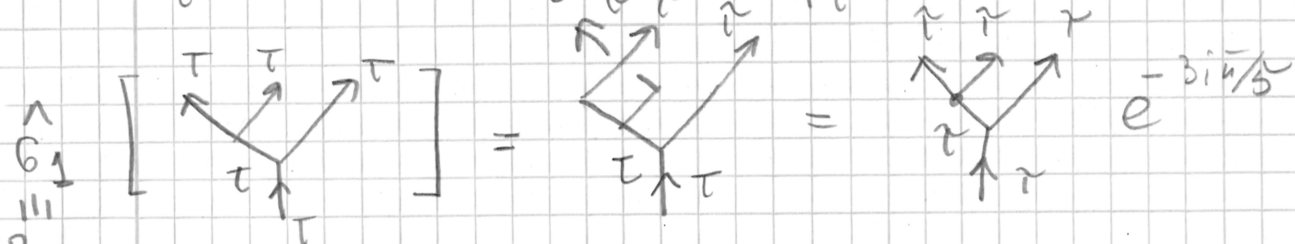
- Result: $\vee$ $R^{\tau\tau}_{I} = e^{4i\pi/5}$, $R^{\tau\tau}_{\tau} = e^{-3i\pi/5}$ → "left-handed"

  $R^{\tau\tau}_{I} = e^{-4i\pi/5}$, $R^{\tau\tau}_{\tau} = e^{3i\pi/5}$ → "right-handed"

- Braiding matrices in Hilbert space $V^{\tau\tau\tau}_{\tau}$

ⓐ $\hat{\sigma}_1$ → braids anyons 1 & 2 ("left-handed")

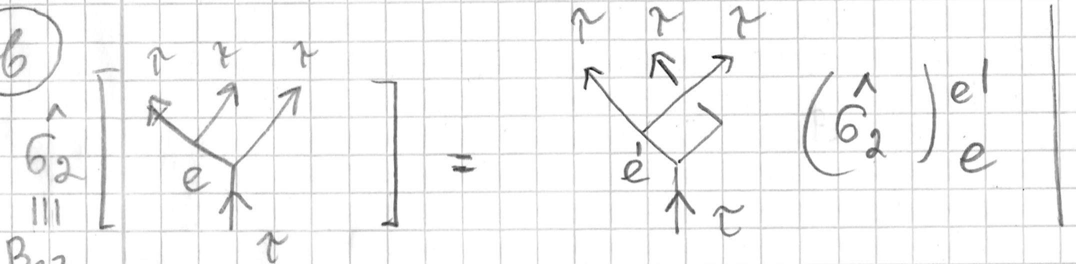




$\hat{\sigma}_1 \equiv B_{12}$

⇒ $\hat{\sigma}_1 = \mathrm{diag}\left(e^{4i\pi/5}, e^{-3i\pi/5}\right)$ in basis $\{|0\rangle, |1\rangle\}$

ⓑ



$\hat{\sigma}_2 \equiv B_{23}$

$$\sigma_2 = F^{-1} \underset{\sigma_1}{\underset{\parallel}{R}} F$$ → see Lecture # 14

↓

braiding of anyons 2 & 3 ("left-handed")

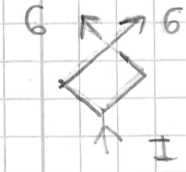
Result:

$$\hat{\sigma}_2 = \begin{pmatrix} e^{-4i\pi/5}\varphi^{-1} & e^{3i\pi/5}\varphi^{-1/2} \\ e^{3i\pi/5}\varphi^{-1/2} & -\varphi^{-1} \end{pmatrix}$$

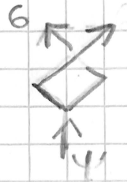
⇝ analog of non-diagonal braiding matrix B_{23} of Majoranas in the basis of c-fermions

5) Application #2: Ising anyons ~ Majorana fermions

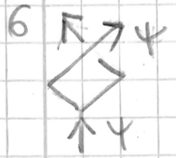
- Result for R-matrix (from hexagon equations)



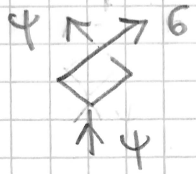
$\rightarrow R^{\sigma\sigma}_{I} = e^{-i\pi/8}$



$\rightarrow R^{\sigma\sigma}_{\psi} = e^{i\,3\pi/8}$

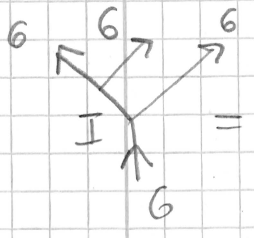


$\rightarrow R^{\sigma\psi}_{\psi} = i$;

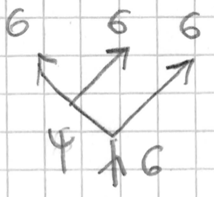


$\rightarrow R^{\psi\sigma}_{\psi} = i$

- Braiding matrices: $c_1 = \frac{1}{2}(\gamma_1 + i\gamma_2)$; $c_2 = \frac{1}{2}(\gamma_3 + i\gamma_4)$



$= |0\rangle \overset{!}{=} |0\rangle_c$



$= |1\rangle = c_1^\dagger c_2^\dagger |0\rangle_c$

This is a basis in splitting space $V^{\sigma\sigma\sigma}_{\sigma}$; $\dim = 2$

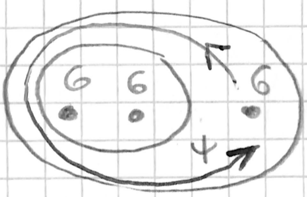
$$\hat{B}_{12} = e^{i\alpha}\, e^{\frac{\pi}{4}\gamma_2\gamma_1} = e^{i\alpha}\begin{pmatrix} e^{-i\pi/4} & \emptyset \\ \emptyset & e^{i\pi/4} \end{pmatrix} \overset{!}{=} \begin{pmatrix} R^{\sigma\sigma}_{I} & \emptyset \\ \emptyset & R^{\sigma\sigma}_{\psi} \end{pmatrix}$$

$$\hat{B}_{23} = e^{i\alpha}\, e^{\frac{\pi}{4}\gamma_3\gamma_2} = \frac{e^{i\alpha}}{\sqrt{2}}\begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix} \quad \Big| \quad \alpha = \pi/8$$

$\downarrow$

"AB phase"

- Checking locality constrain



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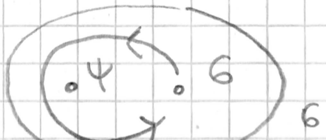
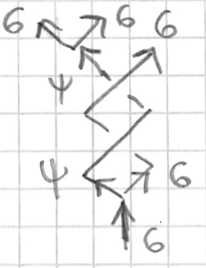
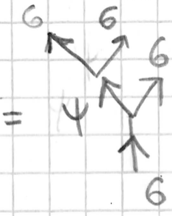
Let $|i\rangle = |1\rangle \rightarrow$ initial state

ⓐ





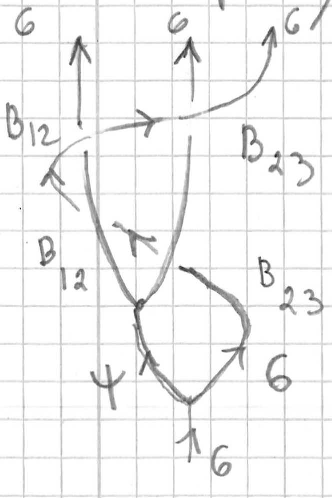


$= \quad \left(R^{\psi\sigma}_{\sigma}\, R^{\sigma\psi}_{\sigma} = i^2 = -1\right)$

We braid 3^d σ-anyon around "composite" fermion ψ

$\Rightarrow$ outcome: $|f\rangle = |1\rangle(-1) \rightarrow$ final state

(b) We braid 3d 6-anyon around 1st & 2nd step by step.



$$B = B_{23} B_{12}^2 B_{23}$$

full braiding matrix

(i) Matrix calculations:

$$B = \begin{pmatrix} -i & 0 \\ 0 & +i \end{pmatrix} e^{4i\alpha} \Rightarrow$$

$$B|1\rangle = |1\rangle i e^{4i\alpha}$$

If $\alpha = \pi/8$, then we get $|f\rangle = |1\rangle(-1) \rightarrow$ this checks with the braiding around 4!

(ii) Operator approach: $B_{12} = e^{\frac{\pi}{4}\gamma_2\gamma_1}$, $B_{23} = e^{\frac{\pi}{4}\gamma_3\gamma_2}$

$P_{12} = +i\gamma_2\gamma_1 \rightarrow$ parity operator for fermion $c = \frac{1}{2}(\gamma_1 + i\gamma_2)$

$\stackrel{!}{=} (1 - 2c^+c)$; $P_{12}^2 = 1$

$$\Rightarrow B_{12}^2 = e^{2i\alpha} e^{-i\frac{\pi}{2}P_{12}} = -i P_{12} e^{2i\alpha} = e^{2i\alpha}\gamma_2\gamma_1$$

$$\Rightarrow B = e^{\frac{\pi}{4}\gamma_3\gamma_2} \gamma_2\gamma_1 e^{\frac{\pi}{4}\gamma_3\gamma_2} e^{4i\alpha} \stackrel{(*)}{=} \gamma_2\gamma_1 e^{4i\alpha} = e^{4i\alpha}(-iP_{12})$$

(*) $\left[e^a b = b e^{-a} \text{ if } \{b,a\} = 0 \right]$

$$\Rightarrow B|1\rangle = e^{4i\alpha}(-iP_{12})|1\rangle = |1\rangle \underbrace{i e^{4i\alpha}}$$

the same as in matrix approach

$e^a := \sum_{n=0}^{+\infty} \frac{1}{n!} a^n \rightarrow$ definition

Info:

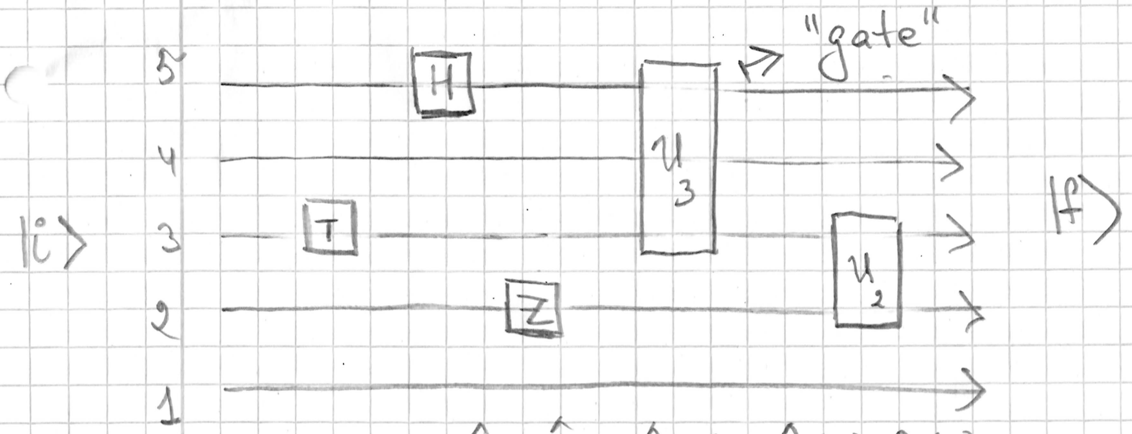
$$\gamma_3\gamma_2\gamma_2\gamma_1 = \gamma_3\gamma_1$$

$$\gamma_2\gamma_1\gamma_3\gamma_2 = \gamma_1\gamma_3\gamma_2^2 = -\gamma_3\gamma_1$$

$$\Rightarrow \{\underbrace{\gamma_3\gamma_2}_{=a}, \underbrace{\gamma_2\gamma_1}_{=b}\} = 0$$

Part II: Quantum Computing

- To build up quantum computer we need Hilbert space build from small pieces, q-dits
  $\to$ $\dim H = q^N$ for N q-dits
  typically $q = 2, 3$
- Given Hilbert space, the model of q. computer has three steps for q. computation
  (1) Initialize Hilbert space in some know state $|i\rangle$
  (2) Perform controlled unitary evolution, $|f\rangle = \hat{\mathcal{U}}(t)|i\rangle$
  (3) Measure some degrees of freedom in $|f\rangle$ – final state
- If $\hat{\mathcal{U}}$ is implemented as a series of $\hat{\mathcal{U}}_k$, where each acts on only small parts of Hilb. space (just few q-dits at a time) $\Rightarrow$ then we call this scheme "quantum circuit model"



Example

$\mathcal{U}_3^{\dagger} = \mathcal{U}_3^{-1}$

$\mathcal{U}_2^{\dagger} = \mathcal{U}_2^{-1}$

$\dim(\mathcal{U}) = 2^5 = 32$
$\downarrow$
Hilbert space

$$|f\rangle = \underbrace{\hat{\mathcal{U}}_2\, \hat{\mathcal{U}}_3}_{\text{three q-bit gate}}\, \hat{Z}^{(2)}\, \hat{H}^{(2)}\, \underbrace{\hat{T}^{(3)}}_{\text{single qubit gate}}$$

Single q-bit gates:

(a) Hadamard $H = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ (b) $\hat{X}, \hat{Y}, \hat{Z} \equiv \sigma_{x,y,z}$

Pauli matrices
X-gate, etc.

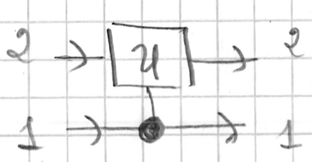
(c) Phase gate: $\hat{S} = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}$

(d) "$\pi/8$": $T = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\pi/4} \end{pmatrix} \underset{(*)}{\sim} \begin{pmatrix} e^{-i\pi/8} & 0 \\ 0 & e^{i\pi/8} \end{pmatrix}$

(*) Global $U(1)$ phase never matters!

Two-qubit gates:

(a) Controlled U-gate



1st $\otimes$ 2nd	
1.	$\|0\rangle \otimes \|0\rangle \to \|0\rangle \otimes \|0\rangle$
2.	$\|0\rangle \otimes \|1\rangle \to \|1\rangle \otimes \|0\rangle$
3.	$\|1\rangle \otimes \|0\rangle \to \|1\rangle \otimes (U\|0\rangle)$
4.	$\|1\rangle \otimes \|1\rangle \to \|1\rangle \otimes (U\|1\rangle)$

$$U_{CNOT} = \left(\begin{array}{cc|cc} 1 & & & \\ & 1 & & \emptyset \\ \hline & \emptyset & U_{00} & U_{01} \\ & & U_{10} & U_{11} \end{array}\right)$$

(b) CNOT-gate $\equiv$ Controlled X-gate, $\hat{U} \equiv \hat{X}$

$$\hat{U}_{CNOT} = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \left(\begin{array}{cc|cc} 1 & & & \\ & 1 & & \\ \hline & & & 1 \\ & & 1 & \end{array}\right) \end{array}$$

$\to$ matrix representation of U_{CNOT}

- Universal quantum computing in the quantum circuit model.

(a) N-qubits, $\dim(H) = 2^N$, $U(D) \to$ group of all possible unitary operations

(b) Assume, q. comp. can implement p different elementary operations $U_n \in U(D)$ with $n = 1, \dots, p$ ($\sim$ gates)

(c) $U = U_{i_t} \cdot \ldots \cdot U_{i_1} \to$ unitary after t steps "discrete time"

(d) $U \sim e^{i\varphi} U$ with $\varphi \in U(1)$

$\uparrow$ equivalent operations

e) For comp. purposes it is sufficient to approximate any desired $\mathcal{U}$ by sequence c) with any desired accuracy. Sets $\{\mathcal{U}_n\}$, $n = 1, \dots, p$ of such gates which makes it possible are known as universal.

f) Define distance between two unitaries

$$\mathrm{dist}(\mathcal{U}; V) = \sqrt{1 - \frac{1}{D} \mathrm{tr}(\mathcal{U}^\dagger V)}$$

g) More precise statement about universality:
Set $\{\mathcal{U}_n\}$ is universal if for $\forall\ \varepsilon > 0$ & $\forall\ \mathcal{U}_{taget} \in \mathcal{U}(D)$ $\exists$ sequence $(i_t, \dots, i_1)$ such that $\mathrm{dist}(\mathcal{U}_{taget}; \mathcal{U}_{i_t} \dots \mathcal{U}_{i_1}) < \varepsilon$

h) Q: How long a sequence should be
A: Kitaev & Solovay theorem, $t \sim O(\log \frac{1}{\varepsilon})$

→ Raugh argument: for t steps we get $\sim p^t$ possible gates. If they cover $\mathcal{U}(D)$ uniformly we get typical dist $\sim p^{-t}$ of our "grid" in $\mathcal{U}(D)$ $\Rightarrow$ $\varepsilon \sim p^{-t} \Rightarrow t \sim O(\ln \frac{1}{\varepsilon})$