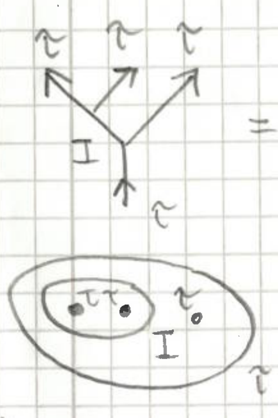
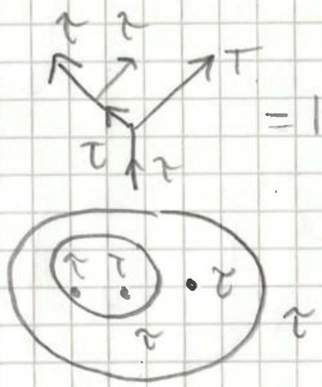
Lecture #18: "Top. quantum computing with anyons" (1)

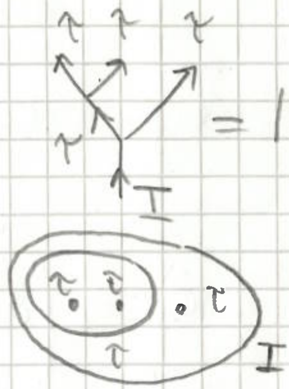
Example: qubit made out of 3 Fib. anyons



$= |0\rangle$



$= |1\rangle$



$= |N\rangle$

!!! $\{|0\rangle, |1\rangle\} \to$ basis in $V^{\tau\tau\tau}_{\tau}$ $\quad$ $|N\rangle \in V^{\tau\tau\tau}_{I}$

$\to$ form a qubit $\quad$ "non-operational state"

$$V^{\tau\tau\tau}_{\tau} \oplus V^{\tau\tau\tau}_{I} = H^{(3)}_{\tau} \quad \text{with} \quad \dim\left(H^{(3)}_{\tau}\right) = F_3 = 3$$

- Any braid of three anyons can be constructed as sequence of braids $\hat{\sigma}_1, \hat{\sigma}_1^{-1}, \hat{\sigma}_2$ & $\hat{\sigma}_2^{-1}$ which is a braid group B_3 of three strands

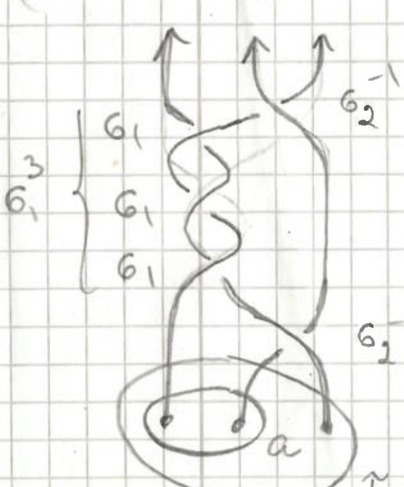
See back page!

- For Fib anyons it is known to be universal

(I) Single q-bit gate:

$$U_{target} = \hat{X} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$
$\quad\quad |0\rangle \quad |1\rangle$

1) 1^{st} approximation:



$$U_1 = \sigma_2^{-1}\sigma_1^{3}\sigma_2^{-1} \simeq e^{3i\pi/5}\begin{pmatrix} O(10^{-1}) & 0.98 \\ 0.98 & O(10^{-1}) \end{pmatrix}$$

$$\text{dist}(\hat{U}_1, \hat{X}) = 0.17$$

2) 2^{nd} approximation (better)

$$U_2 = \sigma_1^{-2}\sigma_2^{3}\sigma_1^{2}\sigma_2^{-2} \simeq e^{i\alpha}\begin{pmatrix} O(10^{-2}) & 0.99 \\ 0.99 & O(10^{-2}) \end{pmatrix}$$

$$\text{dist}(\hat{U}_2, \hat{X}) = 0.08$$

(*) Two-qubit controlled gate $\to$ next pages

1.a

Reminder: in the basis $\{|0\rangle, |1\rangle, |N\rangle\}$

$$\hat{\sigma}_1 = \mathrm{diag}\left(e^{4i\pi/5}, e^{-3i\pi/5}, e^{-3i\pi/5}\right)$$

$$\sigma_2 = \begin{array}{c} |0\rangle \\ |1\rangle \\ |N\rangle \end{array} \left(\begin{array}{cc|c} e^{-4i\pi/5}\varphi^{-1} & e^{3i\pi/5}\varphi^{-1/2} & 0 \\ e^{3i\pi/5}\varphi^{-1/2} & -\varphi^{-1} & 0 \\ \hline 0 & 0 & e^{-3i\pi/5} \end{array}\right)$$

See derivation in Lecture # 15

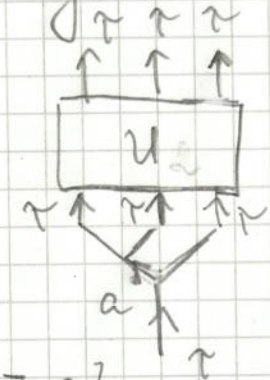
- "Distance" between two unitary matrices is defined by

$$\mathrm{dist}(U,V) = \sqrt{1 - \frac{1}{D}\left|\mathrm{tr}\left(U^{\dagger}V\right)\right|} \in [0,1]$$

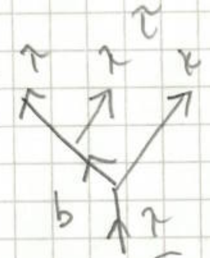
if U, V are matrices of size $D \times D$.

Another drawings: $|a\rangle =$  $\rightarrow$ initial state

$$\hat{u}\,|a\rangle = $$



$$= \sum_b$$



$$(\hat{u})^b_a$$

$\downarrow$

[matrix elements of $\hat{u}$ in basis $\{|0\rangle, |1\rangle\}$

$$\begin{cases} |I\rangle \equiv |0\rangle \\ |\tau\rangle \equiv |1\rangle \end{cases}$$

Here $a, b \in \{I, \tau\}$

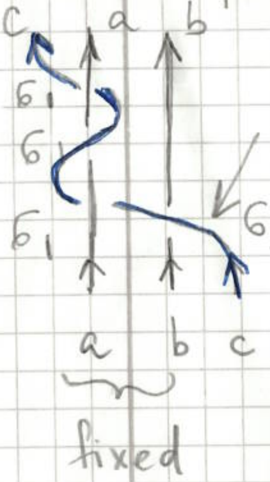
$\because$ Since $\hat{u}$ is composed from braiding operations, the overall quantum number τ doesn't change.

Info: the general task to identify a sequence of elementary gates to approx. $\hat{u}_{taget}$ is known as "quantum compiling".

II Constructing gates by "weaving".

Def. One fixes positions of $N-1$ anyons and only moves the N^{th} one around all other stationary anyons.

1) Simplest example:



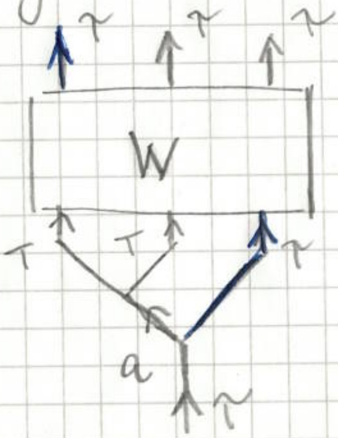
c-anyon is special

$$\hat{W} = \sigma_1^3 \sigma_2^{-1}$$

$\Downarrow$

weave with 3^d (or c) anyon.

Diagram in terms of trees

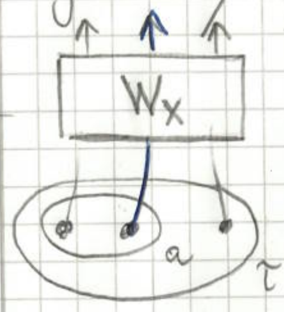


(Omit on lecture)

2) If system of N identical anyons forms the univ. quantum computation by braiding, it is also capabable of univ. q. comp. by weaving!

3) X-gate by weaving

Omit!



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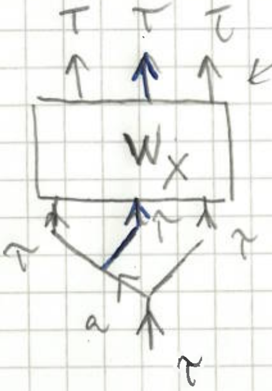


shortend drawing

$\mathrm{dist}(\hat{W}_X, \hat{X}) = 0.18$

→ after long figure!

$W_X = \sigma_2^4\, \sigma_1^{-2}\, \sigma_2^2\, \sigma_1^{-2}$ (Fig 11.7 from Simon's book)

→ 2nd anyon is a control one

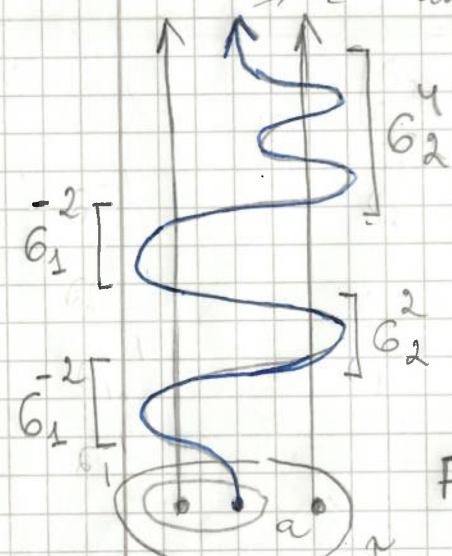


Fig 11.7

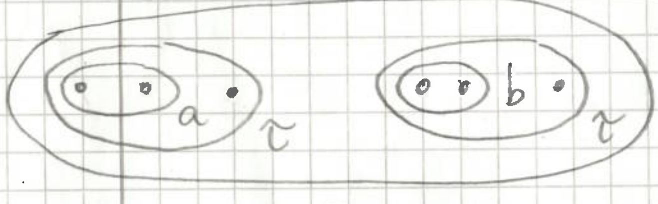
Note: Clockwise or anti-clockwise type of braidings can be read off from the formula for W_X

anti-clockwise clockwise

III Two-qubit gates with Fib. anyons

- Computational space $\mathcal{H}_c \subset \mathcal{H}_I^{(6)}$

$\dim \mathcal{H}_I^{(6)} = F_4 = 5 \mid \dim \mathcal{H}_c = 4$



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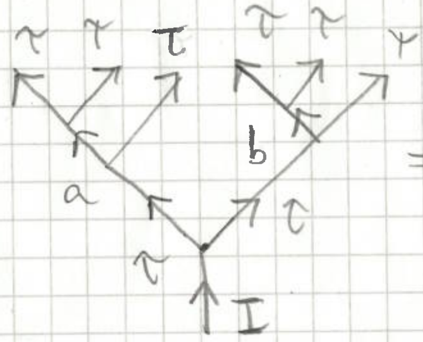
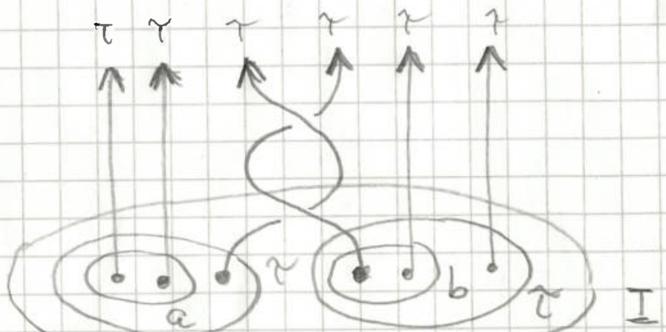


$= |a, b\rangle$

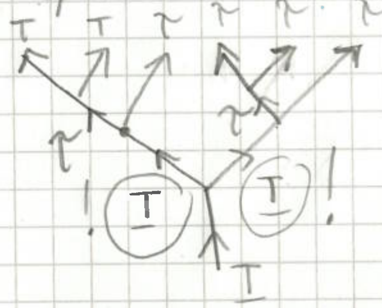
$a, b \in \{I, \tau\}$

- Braid between two qubits entangles them, but may lead to "leakage error"



- Non-computational space

$|N, N\rangle =$



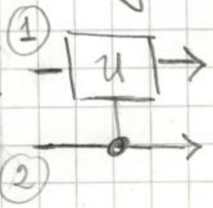
→ the fusion channel of 3 anyons in each qubit can be swaped to I

⤳ (i) the problem is generic for all anyon theorie with universal braiding, it is not specific to Fibonacci

(ii) one should try to design 2-qubit gate so that to make leakage error sufficiently small.

IV Controlled U-gate ; $C(U)$ → another notation

- Reminder



initial state ← $|\psi_i\rangle$ → $|\psi_f\rangle$ | final state $|\psi_f\rangle = C(U)|\psi_i\rangle$

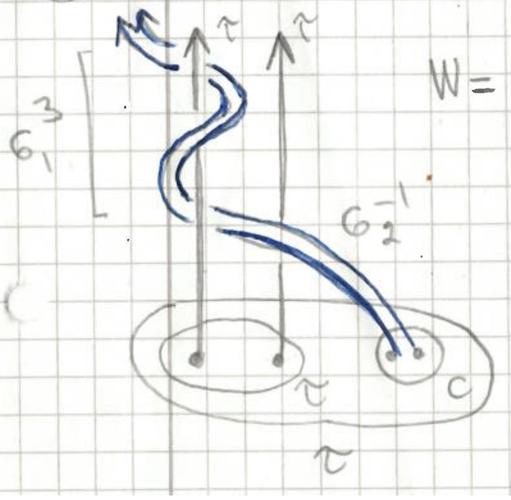
$H^{(1)} \otimes H^{(2)}$ → Hilbert space of 2 qubits

$$C(U): \begin{cases} |0\rangle \otimes |0\rangle \to |0\rangle \otimes |0\rangle \\ |1\rangle \otimes |0\rangle \to |1\rangle \otimes |0\rangle \\ |0\rangle \otimes |1\rangle \to (U|0\rangle) \otimes |1\rangle \\ |1\rangle \otimes |1\rangle \to (U|1\rangle) \otimes |1\rangle \end{cases}$$

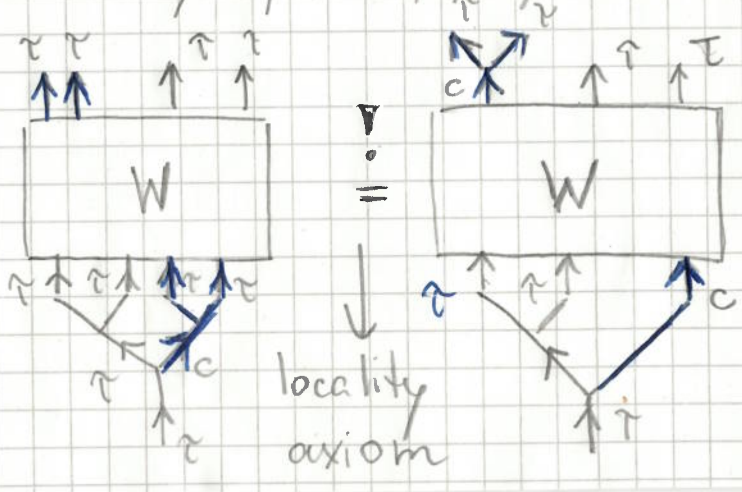
→ 2^{nd} qubit controlls the 1^{st}.

- Controlled $\hat{\sigma}_2^2$-gate (realized with weave)

(A) Consider weave controlled by pair of anyons:



$W = \sigma_1^3 \sigma_2^{-1}$



$\stackrel{?}{=}$

↓

locality axiom

▼ Pair of τ-anyons split from $c \in \{I, \tau\}$ behaves under weave as the composed particle with quantum number being c.

(B) Let's design weave to approximate $\hat{\sigma}_1^2$-gate.

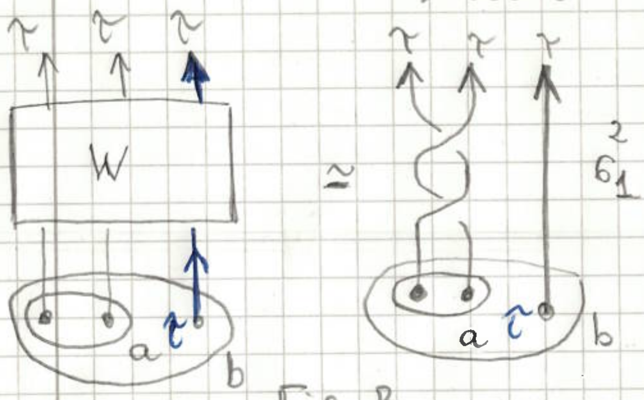


Fig. B

- To long to draw ⇒

$$W = \sigma_2^3 \sigma_1^{-4} \sigma_2^{-2} \sigma_1^4 \sigma_2^{-4} \sigma_1^{-4} \sigma_2^{-1}$$

↓

weave controlled by 3d anyon

$$\mathrm{dist}(W, \hat{\sigma}_1^2) \simeq 0.12$$

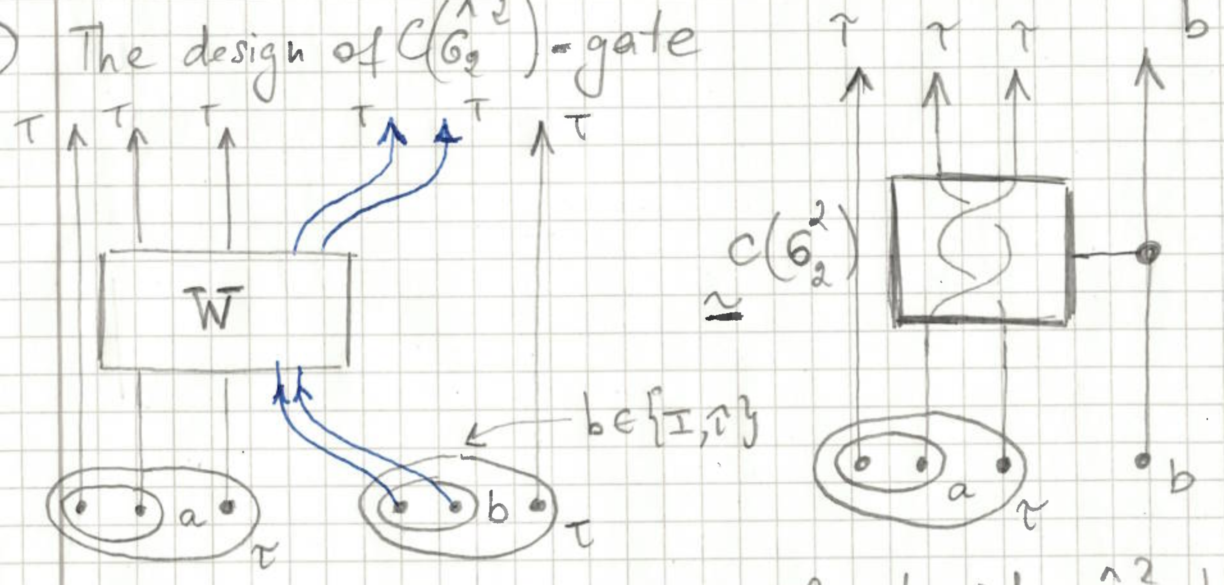
- Equivalence holds for all possible a & b $\in \{I, \tau\}$, i.e. in 3-dim. Hilbert space spanned by $\{|0\rangle, |1\rangle, |N\rangle\}$

$$W \simeq \begin{array}{c|c} & \begin{array}{ccc} |0\rangle & |1\rangle & |N\rangle \end{array} \\ \begin{array}{c} |0\rangle \\ |1\rangle \\ |N\rangle \end{array} & \left(\begin{array}{cc|c} \multicolumn{2}{c|}{\text{////}} & \emptyset \\ \hline \multicolumn{2}{c|}{\emptyset} & e^{-6i\pi/5} \end{array}\right) \end{array} \cdot e^{i\alpha}$$

$e^{i\alpha}$ → irrelevant phase

(See Mathematica file for check)

(C) The design of $C(\hat{\sigma}_2^2)$-gate



1^{st} qbit $\quad$ 2^{nd} qbit

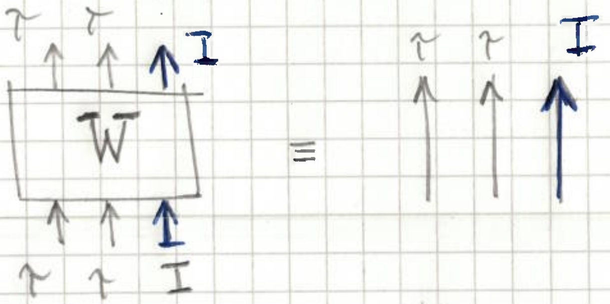
(with overall fusion channel I which is not shown)

- If braid $\hat{\sigma}_2^2$ braid is performed or not is controlled by quantum number $\underline{\underline{b}}$ (the state of 2^{nd} qbit

- Verification:

(i) If $b = \tau$ then the diagram is reduced to Fig B $\Rightarrow$ approx. braid $\hat{\sigma}_2^2$ is performed

(ii) If $b = I$, then braiding of vacuum particle around pair of τ-anyons has no effect on Hilbert space



$\Rightarrow$ the state $|a\rangle$ of 1st qbit doesn't change.

(D) The controlled CNOT gate ($C(\hat{X})$) can be implemented by using the same ideas of weaving (see the book of S. Simon for more details)
In combination with any desired single qbit gates it is sufficient for universal quantum computation.