Lecture #6 → Maj. fermions

I

- Summary of BCS superconductivity

(a) $$H = \sum_{\sigma} \int d^3x \; \psi_\sigma^+ \left(-\frac{\hbar^2}{2m}\nabla^2 - \mu\right)\psi_\sigma + H_{int}$$

$$H_{int} = -g \int d^3x \; (\psi_\uparrow^+ \psi_\downarrow^+)(\psi_\downarrow \psi_\uparrow) \to \text{attraction!}$$

(b) Mean-field approximation

(i) $$\hat{H}_{MF} = \text{Const} + \int d^3x \; (\psi_\uparrow^+, \psi_\downarrow) \begin{pmatrix} -\frac{\hbar^2}{2m}\nabla^2 - \mu & -\Delta \\ -\Delta^* & \frac{\hbar^2}{2m} + \mu \end{pmatrix} \begin{pmatrix} \psi_\uparrow \\ \psi_\downarrow^+ \end{pmatrix}$$

$\Delta \to$ "order parameter"

(ii) $$H_{MF} = \text{Const} + \sum_k (c_{k\uparrow}^+, c_{-k\downarrow}) \begin{pmatrix} \xi_k & -\Delta \\ -\Delta^* & -\xi_k \end{pmatrix} \begin{pmatrix} c_{k\uparrow} \\ c_{-k\downarrow}^+ \end{pmatrix}$$

$\xi_k \to$ electron, $-\xi_k \to$ hole

$$\xi_k = k^2/2m - \mu$$

(c) H_{MF} can be diagonalized by Bogoliubov transformation

$$\begin{pmatrix} \alpha_{k\uparrow} \\ \alpha_{-k\downarrow}^+ \end{pmatrix} = \begin{pmatrix} \cos\theta_k & +\sin\theta_k \\ \sin\theta_k & -\cos\theta_k \end{pmatrix} \begin{pmatrix} c_{k\uparrow} \\ c_{-k\downarrow}^+ \end{pmatrix}; \quad \{\alpha_{k\sigma}, \alpha_{k'\sigma'}^+\} = \delta_{kk'}\delta_{\sigma\sigma'} \; !$$

$$\Rightarrow H_{MF} = \sum_{k\sigma} \lambda_k \alpha_{k\sigma}^+ \alpha_{k\sigma}, \text{ with}$$

(i) $\lambda_k = \sqrt{\xi_k^2 + |\Delta|^2} \to$ spectrum of q.-particles

(ii) $\sin 2\theta_k = -\Delta/\lambda_k \to$ angle

(d) $$|GS\rangle = \prod_k \left(\cos\theta_k - \sin\theta_k \, c_{k\uparrow}^+ c_{-k\downarrow}^+\right) |0\rangle_{c,c^+}$$

$|0\rangle_{c,c^+} \to$ math vacuum

(e) $$\Delta \simeq 2\hbar\omega_D e^{-1/g\nu_0}$$

$\omega_D \to$ Debye frequency

$\nu_0 \to$ DoS at $\varepsilon = \varepsilon_F$.

II

p-wave sc $d=2$ (no spin!)

- Model (mean-field) Hamiltonian

$$H_x = \begin{pmatrix} \hat{h}_x & -\hat{\Delta}^* \\ \hat{\Delta} & -\hat{h}_x^T \end{pmatrix} \quad \Bigg| \quad \begin{aligned} h_x &= -\frac{\hbar^2}{2m}(\partial_x^2+\partial_y^2) - \mu \\ \hat{\Delta} &= \vartheta(\partial_x + i\partial_y) \end{aligned}$$

H_x ↓ (2x2) differ. operator!

$-h_x^T$ ↓ kin. energy of hole

$$\hat{H} = \frac{1}{2}\int \left(\psi^+(x), \psi(x)\right) H_x \begin{pmatrix} \psi(x) \\ \psi^+(x) \end{pmatrix} d^2x$$

$$= \int d^2x \left\{ \psi^+ \left(-\frac{\hbar^2}{2m} - \mu\right)\psi + \frac{1}{2}\vartheta\left[\psi(\partial_x + i\partial_y)\psi + h.c.\right]\right\}$$

(a) $\hat{\Delta}$ is dif. operator in real space

$\overset{F.T.}{\Longrightarrow}$ $\Delta_k = +i\vartheta(k_x + ik_y)$ ⇝ "p-wave"

- Symmetry of H_x:

$\sigma_1 H_x^T \sigma_1 \underset{\downarrow ex.}{=} -H_x$; $\partial_x^T \overset{!}{=} -\partial_x$; $\partial_y^T = -\partial_y$

→ p/h symmetry with sign (+)

→ BCS $\sigma_2 H_{MF}^T \sigma_2 = -H$, $\sigma_2^* = -\sigma_y$ ← sgn (−)

- Solution (real space, e.g. $\mu \to \mu(x,y)$)

Let $H_x \psi_\varepsilon = \varepsilon \psi_\varepsilon$ with $\psi_\varepsilon^{(x)} = \begin{pmatrix} u_\varepsilon^{(x)} \\ v_\varepsilon^{(x)} \end{pmatrix}$ → eigenmode

Then $H_x^+ = H_x \Rightarrow \sigma_1 H_x^* \sigma_1 = -H_x \Rightarrow$

$H_x^* \psi_\varepsilon^* = \varepsilon \psi_\varepsilon^* \Rightarrow -\sigma_1 H_x \sigma_1 \psi_\varepsilon^* = +\varepsilon \psi_\varepsilon^*$

$\Rightarrow H_x(\sigma_1 \psi_\varepsilon^*) = -\varepsilon(\sigma_1 \psi_\varepsilon^*) \Rightarrow$

3

Hence $\psi_{-\varepsilon}^{(x)} = \begin{pmatrix} u_{-\varepsilon}^{(x)} \\ v_{-\varepsilon}^{(x)} \end{pmatrix} \underset{\text{no degeneracy}}{\overset{!}{=}} \sigma_1 \psi_{\varepsilon}^{*} = \begin{pmatrix} v_{\varepsilon}^{*}(x) \\ u_{\varepsilon}^{*}(x) \end{pmatrix}$

$\Rightarrow \boxed{u_{-\varepsilon} = v_{\varepsilon}^{*} \;\&\; v_{-\varepsilon} = u_{\varepsilon}^{*}}$ (1)

By def. $\varepsilon_\alpha > 0$ if $\alpha > 0$, $\varepsilon_\alpha < 0$ if $\alpha < 0$

- Bogoliubov transformation:

$$\gamma_\alpha = \int d^2x \left(u_\alpha^{*}(x) \hat{\psi}(x) + v_\alpha^{*}(x) \hat{\psi}^{+}(x) \right)$$

(a) (b)

$\alpha \to$ quantum index of state

any quantum index of eigenstate (~energy)

(a) ~) Because of Eq. (1) $\boxed{\gamma_\alpha^{+} = \gamma_{-\alpha}}$ (2)

Indeed $\gamma_{-\alpha} = \int d^2x \left(v_\alpha(x) \hat{\psi}(x) + u_\alpha(x) \hat{\psi}^{+}(x) \right) \overset{!}{=} \gamma_\alpha^{+}$

(b)$^+$ (a)$^+$

(b) If $\{\hat{\psi}(x), \hat{\psi}^{+}(x')\} = \delta(x-x')$, then $\{\gamma_\alpha, \gamma_\beta^{+}\} = \delta_{\alpha\beta}$

This follows from orth. relations $\langle \psi_\alpha | \psi_\beta \rangle = \delta_{\alpha\beta}$

(i) $\int d^2x \left(u_\alpha^{*}(x) u_\beta(x) + v_\alpha^{*}(x) v_\beta(x) \right) = \delta_{\alpha\beta}$

(ii) $\alpha \to -\alpha$ $\int d^2x \left(v_\alpha(x) u_\beta(x) + u_\alpha(x) v_\beta(x) \right) = 0$

Note: in (i) & (ii) both $\alpha > 0$ & $\beta > 0$!

- Canonical (or eigen-) form of $\hat{H}$:

(a) $\hat{H} = \frac{1}{2} \sum_\alpha' \left(\varepsilon_\alpha \gamma_\alpha^{+} \gamma_\alpha - \varepsilon_\alpha \gamma_{-\alpha}^{+} \gamma_{-\alpha} \right) \overset{\text{use eq. (2)}}{=}$

! ← ex. $= \text{Const} + \sum_{\alpha>0}' \varepsilon_\alpha \hat{\gamma}_\alpha^{+} \hat{\gamma}_\alpha$

(b) $\gamma_\alpha | GS \rangle = 0$ for $\forall\, \alpha > 0$. (as usual)

- Zero mode $\varepsilon = 0$ (when present)

(3) $u_0 = v_0^*$ and $v_0 = u_0^*$ (the same)

$$\Rightarrow \quad \gamma_0 = \int d^2x \left(u_0^*(x)\, \hat{\psi}(x) + \underbrace{u_0(x)}_{= v_0^*} \hat{\psi}^\dagger(x) \right)$$

Hence: $\boxed{\gamma_0^\dagger = \gamma_0}$ $\quad \{\gamma_0^\dagger, \gamma_0\} = 1 \Rightarrow \gamma_0^2 = 1/2$

↓ "Majorana fermion" → its own anti-particle

(III) "Majorana chain" → 1D p-wave SC

- Model Hamiltonian (experiment → see J. Alicea's review)

$x-1 \quad x \quad x+1 \longrightarrow \mathbb{Z}$ → 1D lattice with spinless fermions

| $a = 1$ → lattice constant | $\{c_x, c_{x'}^\dagger\} = \delta_{xx'}$, $\quad x \in \mathbb{Z}$

$$H = -\mu \sum_x c_x^\dagger c_x - \frac{1}{2} \sum_x \underbrace{w}_{\text{hopping}} \left(c_x^\dagger c_{x+1} + h.c. \right)$$
$$+ \frac{1}{2} \sum_x \left(\nu\, c_x c_{x+1} + h.c. \right)$$

- k-space $\quad c_x = \frac{1}{L^{1/2}} \sum_k c_k e^{ikx}$ | $k_j = \frac{2\pi}{L} j$

$\quad \hookrightarrow \{c_k, c_{k'}^\dagger\} = \delta_{kk'}$

$$\hat{H} = \frac{1}{2} \sum_{k>0} \left(c_k^\dagger, c_{-k} \right) \begin{pmatrix} \xi_k & \Delta_k \\ \Delta_k^* & -\xi_k \end{pmatrix} \begin{pmatrix} c_k \\ c_{-k}^\dagger \end{pmatrix} + \text{Const}$$

Where $\quad \xi_k = -(\mu + w \cos k)$; $\quad \Delta_k = +i\nu \sin k$

Remarks: (i) $\frac{1}{2} \nu \sum_k \hat{c}_{-k} \hat{c}_k e^{-ik} = -i\nu \sum_{k>0} \hat{c}_{-k} \hat{c}_k \sin k$

$\quad + (k \to -k)$

(ii) $\quad \sigma_1 H_{-k}^T \sigma_1 = -H_k$ → p/h symmetry in k-space

(5)

- Linearization $k \ll 1$ or $x \gg 1$

(a) $\xi_k \simeq -\mu - w\left(1 - \frac{k^2}{2}\right) = -\tilde{\mu} + \frac{1}{2} w k^2$; $\overset{\text{mass}}{\uparrow}{m} = \frac{1}{w}$

(b) $\Delta_k \simeq -i v k = -v \partial_x$

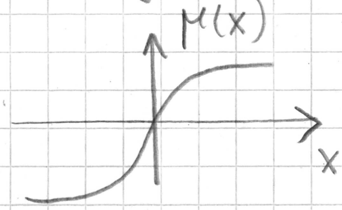
$\Rightarrow$ real space long-wave approx reads $\hat{c}_x \to \hat{\psi}(x)$

$$\hat{H} = \frac{1}{2} \int dx \, (\psi^+, \psi) \begin{pmatrix} -\tilde{\mu} - \frac{\hbar^2}{2m}\partial_x^2 & -v\partial_x \\ -v\partial_x & \tilde{\mu} + \frac{\hbar^2}{2m}\partial_x^2 \end{pmatrix} \begin{pmatrix} \psi \\ \psi^+ \end{pmatrix}$$

N.B. $\left| \begin{array}{l} \partial_x^* = \partial_x \to \text{real operator} \\ \partial_x^T = -\partial_x \to \text{anti-symmetric} \end{array} \right.$

$\rightsquigarrow$ we recognize p-wave SC with y-direction omitted

- Majorana mode : $m \to \infty$



$$\begin{pmatrix} -\tilde{\mu}(x) & -v\partial_x \\ -v\partial_x & -\tilde{\mu}(x) \end{pmatrix} \begin{pmatrix} u_0(x) \\ v_0(x) \end{pmatrix} = 0$$

Check eq. (3)

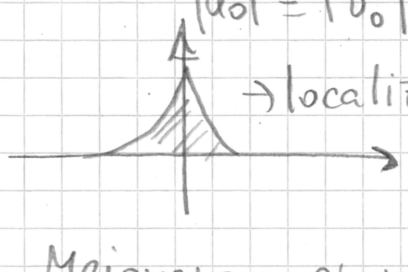
Let $u_0(x) \in \mathbb{R} \Rightarrow v_0(x) = u_0(x) \Rightarrow$

$$\left(\tilde{\mu}(x) + v\partial_x\right) u_0(x) = 0 \Rightarrow$$ solution to differ. equation

$$\| \; u_0(x) = \text{Const} \times \exp\left[-\int_0^x \frac{\tilde{\mu}(y)}{v} dy \right] \; \|$$

Example : $\tilde{\mu}(x) = \mu_0 \, \text{sgn}(x)$

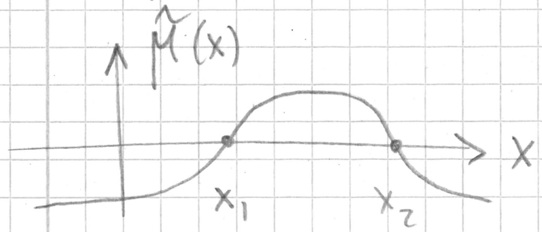
$$\Rightarrow u_0(x) \propto e^{-\frac{\mu_0}{v}|x|}$$



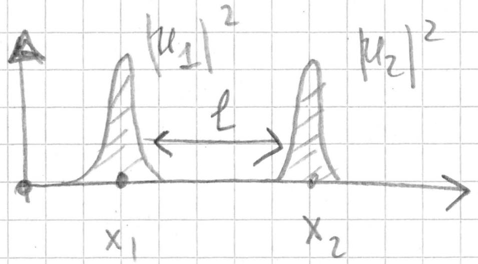
$$\hat{\gamma}_0 = \int dx \; u_0(x) \left(\psi(x) + \psi^+(x) \right) \to \text{Majorana operator}$$

6

- Hybridization of two M. modes



$\Rightarrow$



$$\begin{cases} \gamma_1 = \sqrt{2}\int dx \left(u_1^*(x)\,\psi(x) + u_1(x)\,\psi^+(x)\right) \\ \gamma_2 = \sqrt{2}\int dx \left(u_2^*(x)\,\psi(x) + u_2(x)\,\psi^+(x)\right) \end{cases}$$

if $\ell \gg v/\mu_0 = \xi$ (loc. length of M.), then

(4) $\{\gamma_i, \gamma_j\} = 2\delta_{ij}$ & $\gamma_1^+ = \gamma_1$; $\gamma_2^+ = \gamma_2$

(a) Introduce fermion: $\begin{cases} c = \frac{1}{2}(\gamma_1 + i\gamma_2) \\ c^+ = \frac{1}{2}(\gamma_1 - i\gamma_2) \end{cases}$

from (4) $\Rightarrow$ $\{c, c^+\} = 1$ as should be

(b) Introduce "parity" operator

$P = 2c^+c - 1 \equiv i\gamma_1\gamma_2$

$\gamma \otimes \gamma = \mathbb{1} + c$
$\Downarrow$
fusion of two M.

(c) Minimal Hamiltonian of hybridization is

$H_{12} = i t\, \gamma_1\gamma_2 = t\,(2c^+c - 1)$ | $t \sim e^{-\ell/\xi}$

$\Downarrow$ it has two Bogoliubov levels $\varepsilon_\pm = \pm t$

$|gs\rangle \to$ mathematical vacuum for c, c^+ $\Rightarrow$

(i) $H_{12}|gs\rangle = -t|gs\rangle$ $\to$ empty

(ii) $H_{12}\left(c^+|gs\rangle\right) = +t\left(c^+|gs\rangle\right)$ $\to$ filled