(1)

Lecture #7: Braiding of Maj. in $D=2$

- Scope: 2D spinless p-wave $\to$ half-flux vortexes $\to$ host Maj. zero modes $\to$ non-trivial braiding statistics of Maj (non-Abelian anyons)

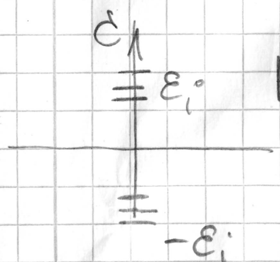
(I) p-wave SC in $D=2$

$$\hat{H} = \int d^2x \, (\psi^+, \psi) \begin{pmatrix} \frac{p_x^2+p_y^2}{2m} - \mu & -i\Delta^* e^{-i\varphi}(p_x - ip_y) \\ i\Delta e^{i\varphi}(p_x + ip_y) & -\frac{p_x^2+p_y^2}{2m} + \mu \end{pmatrix} \begin{pmatrix} \psi \\ \psi^+ \end{pmatrix}$$

$\varphi \to$ arbitrary (const.) phase

single-particle

- Spectrum of H_{BdG}: $\quad H\psi_\varepsilon = \varepsilon \psi_\varepsilon \Rightarrow \psi_{-\varepsilon} = \sigma_1 \psi_\varepsilon^*$



p/h symmetry $\quad \psi_\varepsilon = \begin{pmatrix} u_\varepsilon \\ v_\varepsilon \end{pmatrix} \Rightarrow \begin{matrix} u_{-\varepsilon} = v_\varepsilon^* \\ v_{-\varepsilon} = u_\varepsilon^* \end{matrix}$

- Many-body picture:

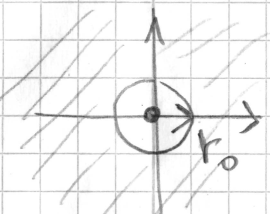
$$H = \frac{1}{2}\sum_{\varepsilon_i > 0} \left(\varepsilon_i \gamma^+_{+\varepsilon_i} \hat{\gamma}_{\varepsilon_i} - \varepsilon_i \hat{\gamma}^+_{-\varepsilon_i} \hat{\gamma}_{\varepsilon_i} \right) + \text{Const}$$

$$= \underbrace{E_g}_{\text{ground state energy}} + \underbrace{\sum_{\varepsilon_i > 0} \varepsilon_i \hat{\gamma}^+_{\varepsilon_i} \hat{\gamma}_{\varepsilon_i}}_{\text{excitations}}$$

$$\hat{\gamma}_\varepsilon = \int d^2x \left(u^*(x)\psi(x) + v^*(x)\psi^+(x) \right)$$

and $\gamma^+_{-\varepsilon} \overset{!}{=} \gamma_\varepsilon$

(II) Vortexes.



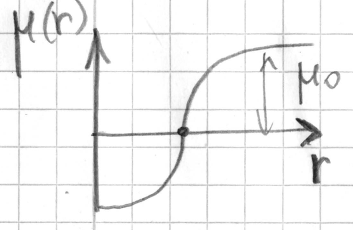
(a) Add singular flux at origin:

$$\vec{a} = \frac{\phi_0}{4\pi}\left(-y/r^2, \, x/r^2 \right) \; ; \; \phi_0 = \frac{2\pi\hbar c}{e}$$

(i) $\oint \vec{a} \cdot dl = \phi_0/2 \to$ half-quantum

(ii) $\frac{e}{\hbar c} \oint \vec{a}\, dl = \pi \to$ "π-flux"

the same



(b) How H_{BdG} changes?

(i) $p_i \to p_i - e/c \, a_i$ $\quad$ (ii) $\Delta \to |\Delta| e^{-i\theta}$, where polar coordinate (r, θ) will be used

② Appearence of phase θ is due to self-consistent solution for $\Delta \sim \langle \psi^+ \psi^+ \rangle \to$ 2 AB phase.

ⓒ Zero-mode in the vortex core

(i) $1/m \to 0$ (ii) $p_i \to p_i - \frac{e}{c} a_i$ in $\hat{\Delta}_x \to$ not important, since $\hat{\psi}\, a_i\, \hat{\psi} = a_i\, \hat{\psi}^2 = 0$

(iii) $i(\hat{p}_x \pm i\hat{p}_y) = e^{\pm i\theta} (\partial_r \pm \frac{i}{r} \partial_\theta)$

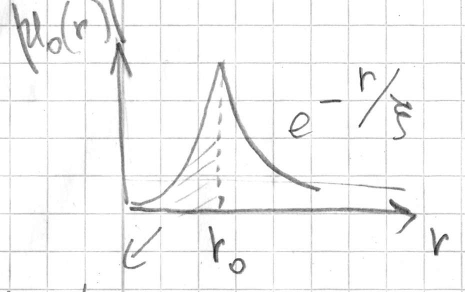
- Hence
$$H_{BdG} = \begin{pmatrix} -\mu(r) & -|\Delta| e^{-i\varphi} (\partial_r - \frac{i}{r} \partial_\theta) \\ |\Delta| e^{i\varphi} (\partial_r + \frac{i}{r} \partial_\theta) & +\mu(r) \end{pmatrix}$$
$e^{\pm i\theta}$ goes away

- Quest: $H_{BdG} \psi_0 = 0$, ansatz $\psi_0 = \begin{pmatrix} u_0(r) \\ u_0^*(r) \end{pmatrix} \Rightarrow$

$$\begin{cases} 1) & -\mu u_0 - |\Delta| e^{-i\varphi} \partial_r u_0^* = 0 \\ 2) & |\Delta| e^{i\varphi} \partial_r u_0 + \mu u_0^* = 0 \end{cases} \Rightarrow u_0(r) = e^{-i\varphi/2} f(r)$$

with $\mu(r) f - |\Delta| \partial_r f = 0$ (aka 1D!)

$$\Rightarrow f(r) = \text{Const} \times \exp\left[- \int_{r_0}^{r} \frac{\mu(r')}{|\Delta|} dr' \right]$$



$e^{-r/\xi}$ with $\xi \sim \mu_0/|\Delta| \to$ localiz. lenth

Vortex core

- (*) $\hat{\gamma}_0 = \sqrt{2} \int d^2x\, f(r) \left(e^{i\varphi/2} \hat{\psi}(x) + e^{-i\varphi/2} \psi^+(x) \right)$

Maj. mode. Note: $\varphi \to \varphi + 2\pi \Rightarrow \gamma_0 \to -\gamma_0$

sign change!

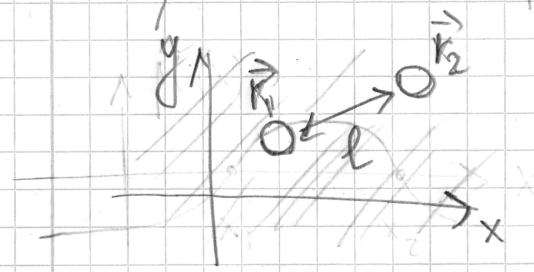
- Excited states in the vortex

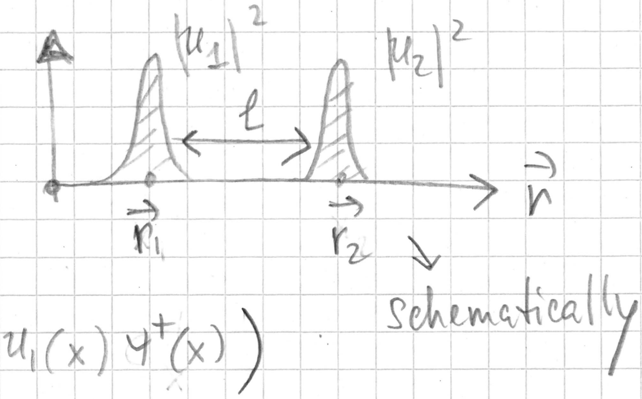
- $\varepsilon_n \sim n|\Delta|/r_0$; $n \in \mathbb{Z}$

(3)

III Fusion

- Hybridization of two M. modes





$$\begin{cases} \gamma_1 = \sqrt{2}\int d^2x \left(u_1^*(x)\,\psi(x) + u_1(x)\,\psi^\dagger(x) \right) \\ \gamma_2 = \sqrt{2}\int d^2x \left(u_2^*(x)\,\psi(x) + u_2(x)\,\psi^\dagger(x) \right) \end{cases}$$

if $\ell \gg |\Delta|/\mu_0 = \xi$ (loc. length of M.), then

(4) $\{\gamma_i, \gamma_j\} = 2\delta_{ij}$ & $\gamma_1^\dagger = \gamma_1$; $\gamma_2^\dagger = \gamma_2$

(a) Introduce fermion: $\begin{cases} c = \frac{1}{2}(\gamma_1 + i\gamma_2) \\ c^\dagger = \frac{1}{2}(\gamma_1 - i\gamma_2) \end{cases}$

from (4) $\Rightarrow$ $\{c, c^\dagger\} = 1$ as should be

(b) Introduce "parity" operator

$$P_{12} = -2c^\dagger c + 1 \equiv -i\gamma_1\gamma_2$$

$\gamma \otimes \gamma = \mathbb{1} + c$
$\Downarrow$
fusion of two M.

(c) Minimal Hamiltonian of hybridization is

$$-P_{12} \propto H_{12} = +it\,\gamma_1\gamma_2 = +t(2c^\dagger c - 1) \quad \Big| \quad t \sim e^{-\ell/\xi}$$

$\Downarrow$ it has two Bogoliubov levels $\varepsilon_\pm = \pm t$

$|gs\rangle \rightarrow$ mathematical vacuum for $c, c^\dagger \Rightarrow$

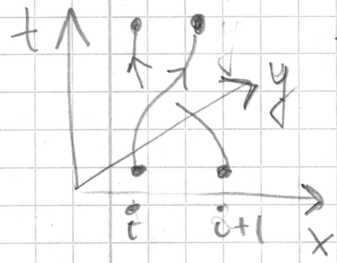
(i) $H_{12}|gs\rangle = -t|gs\rangle \rightarrow$ empty

(ii) $H_{12}(c^\dagger|gs\rangle) = +t(c^\dagger|gs\rangle) \rightarrow$ filled

N.B. $P_{12}|0\rangle = |0\rangle$ – even # of electrons | $P_{12}(c^\dagger|0\rangle) = -c^\dagger|0\rangle$ $\rightarrow$ odd!

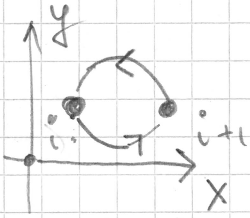
④ IV **Braiding of Majoranas**

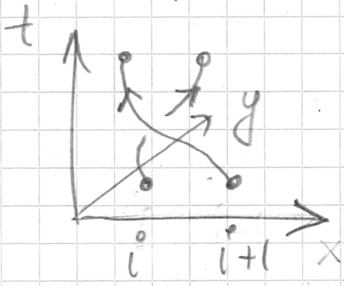
- Braiding group: consider $2n$ vortexes/particles;



→ world lines

$B_{i,i+1}$





$B_{i+1,i}$

anti-clockwise

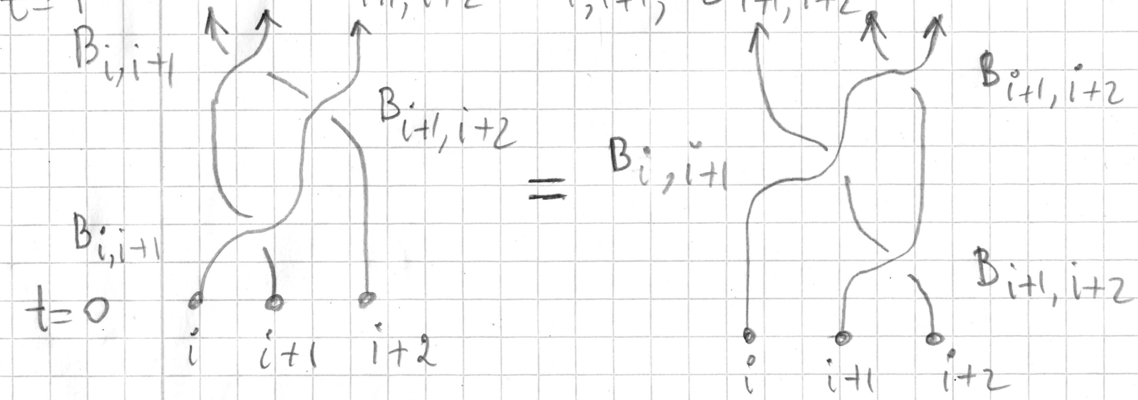
$B_{i+1,i} = B^{-1}_{i,i+1}$

clockwise

Defining properties: ⓐ $B_{i,i+1} B_{j,j+1} = B_{j,j+1} B_{i,i+1}$ if $|i-j| > 1$

ⓑ $B_{i,i+1} B_{i+1,i+2} B_{i,i+1}$

$= B_{i+1,i+2} B_{i,i+1} B_{i+1,i+2}$

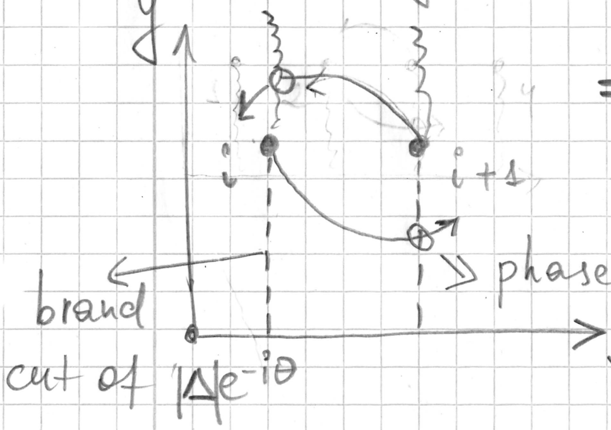


- Braiding group induces projective representation on the Hilbert space

$$B_{i,i+1} \mapsto |\Psi(t=T)\rangle = \hat{U}_{i,i+1} |\Psi(t=0)\rangle$$

! U's are defined upto global phase

- Braiding of Majoranas



$\Rightarrow$ $B_{i,i+1}$: $\begin{cases} \gamma_i \to +\gamma_{i+1} \\ \gamma_{i+1} \to -\gamma_i \quad ! \\ \gamma_j \to \gamma_j \text{ if } j \neq i, i+1 \end{cases}$

Wavy cuts:

$\gg$ phase slip ($\varphi = 2\pi$) in Δ for $\hat{\gamma}_i$, see eq. (*), page 2.

V. Representation of braiding group for Majoranas

- Take $i=1$:
$$\begin{cases} \gamma_2(t=T) = U_{12}\,\gamma_2\,U_{12}^{-1} \overset{!}{=} -\gamma_1 & (1) \\ \gamma_1(t=T) = U_{12}\,\gamma_1\,U_{12}^{-1} = \gamma_2 & (2) \end{cases}$$

Heisenberg evolution of operators

$$P_{12} = -i\gamma_1\gamma_2$$

Q: Which U_{12} makes the job?

A: $$U_{12} = e^{-i\frac{\pi}{4}P_{12}} = e^{\frac{\pi}{4}\gamma_2\gamma_1} \overset{*}{=} \frac{1}{\sqrt{2}}(1+\gamma_2\gamma_1)$$

$$\left\{ \begin{array}{l} e^{i\alpha\hat{\sigma}} \overset{*}{=} \cos\alpha + i\sin\alpha\,\hat{\sigma} \\ \text{if } \hat{\sigma}^2 = 1 \end{array} \right\} \qquad \underbrace{P_{12}^2 = 1}_{\text{parity}}$$

Check:

(1) $e^{\frac{\pi}{4}\gamma_2\gamma_1}\,\gamma_2\,e^{-\frac{\pi}{4}\gamma_2\gamma_1} = \gamma_2\,e^{+i\frac{\pi}{2}P_{12}} = \gamma_2\gamma_1\gamma_2 = -\gamma_1$

(2) $e^{\frac{\pi}{4}\gamma_2\gamma_1}\,\gamma_1\,e^{-\frac{\pi}{4}\gamma_2\gamma_1} = \gamma_1\,e^{+i\frac{\pi}{2}P_{12}} = \gamma_1\gamma_1\gamma_2 = \gamma_2$

- In general $$U_{i,i+1} = \exp\left[-i\frac{\pi}{4}P_{i,i+1}\right] = \frac{1}{\sqrt{2}}(1+\gamma_{i+1}\gamma_i)$$

- Consider 4 Majoranas → two fermions

(a) $c_1 = \frac{1}{2}(\gamma_1 + i\gamma_2)$; $c_2 = \frac{1}{2}(\gamma_3 + i\gamma_4)$

(b) 4 states: $|0\rangle,\ c_1^\dagger|0\rangle,\ c_2^\dagger|0\rangle,\ c_1^\dagger c_2^\dagger|0\rangle$

(c) Braiding matrices: (when acting on Hilbert space)

$$\left.\begin{array}{l} U_{12} = \mathrm{diag}\left(e^{-i\frac{\pi}{4}}, e^{i\frac{\pi}{4}}, e^{-i\frac{\pi}{4}}, e^{i\frac{\pi}{4}}\right) \\ U_{34} = \mathrm{diag}\left(e^{-i\frac{\pi}{4}}, e^{-i\frac{\pi}{4}}, e^{i\frac{\pi}{4}}, e^{i\frac{\pi}{4}}\right) \end{array}\right\} \to \text{exercise}$$

$$U_{23} = \frac{1}{\sqrt{2}}\left(\begin{array}{ccc|c} 1 & 0 & 0 & -i \\ \hline 0 & 1 & -i & 0 \\ 0 & -i & 1 & 0 \\ \hline -i & 0 & 0 & 1 \end{array}\right) \to \text{preserves total parity}$$

$$P = P_{12}P_{34}$$

$$\hat{P}\,U_{23} = U_{23}\,\hat{P} \to \text{hence block structure of } U_{23}$$

⑥

- Check:

$$\gamma_1 = c_1 + c_1^\dagger \,, \qquad \gamma_2 = i\,(c_1^\dagger - c_1)$$
$$\gamma_3 = c_2 + c_2^\dagger \,; \qquad \gamma_4 = i\,(c_2^\dagger - c_2)$$

Hence
$$U_{23} = \frac{1}{\sqrt{2}}\left(1 + \gamma_3\gamma_2\right) = \frac{1}{\sqrt{2}}\left(1 + i\,(c_2 + c_2^\dagger)(c_1^\dagger - c_1)\right)$$

ⓐ 1st column: let $|\psi(0)\rangle = |0\rangle \Rightarrow$

$$|\psi(t=T)\rangle = U_{23}|0\rangle =$$
$$= \frac{1}{\sqrt{2}}\left(1 + i c_2 \cancel{c_1^\dagger} - i c_2 \cancel{c_1} + i c_2^\dagger c_1^\dagger - i c_2^\dagger \cancel{c_1}\right)|0\rangle$$
$$= \frac{1}{\sqrt{2}}\left(|0\rangle - i c_1^\dagger c_2^\dagger |0\rangle\right) \quad \blacksquare$$

ⓑ 2nd column: let $|\psi(0)\rangle = c_1^\dagger|0\rangle \Rightarrow$

$$|\psi(t=T)\rangle = \frac{1}{\sqrt{2}}\left(1 + i c_2 \cancel{c_1^\dagger} - i c_2 c_1 + i c_2^\dagger \cancel{c_1^\dagger} - i c_2^\dagger c_1\right) c_1^\dagger |0\rangle$$
$$= \frac{1}{\sqrt{2}}\left(c_1^\dagger|0\rangle - i c_2^\dagger |0\rangle\right) \quad \blacksquare$$

Summary: (i) Majoranas are non-Abelian anyons!

(ii) $\hat{U}_{i,i+1}$ generally do not commute

→ resource to create entanglement!