- Useful relations from Lecture # 2

(a) $\phi_0 = \frac{2\pi\hbar c}{e}$ – flux quantum ; $l_B = \left(\frac{\hbar c}{eB}\right)^{1/2}$

(b) $\nu = 2\pi n l_B^2 \rightarrow$ filling fraction ; $n \rightarrow$ concentration

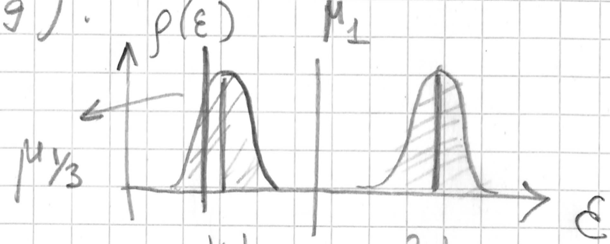
(c) $\nu = N/N_\phi$, where $N \rightarrow$ # of e ; $N_\phi = \phi/\phi_0$

FQHE

- The Laughlin wave-function at $\nu = 1/m$.

$\nu = 1/3, 1/5, 1/7, (1/9)$? $\quad m \in 2\mathbb{Z}_+ + 1$

$n = \frac{\nu}{2\pi l_B^2}$



1) at $\mu = \mu_1 \Rightarrow n_0 \simeq \frac{1}{2\pi l_B^2} \rightarrow$ IQHE $\quad \frac{1}{2}\hbar\omega_c \quad \frac{3}{2}\hbar\omega_c$

2) at $\mu = \mu_{1/3} \Rightarrow n = \frac{1}{3} n_0 \rightarrow$ partially filled 1^{st} LL

Reminder: $n = \int_{-\infty}^{\mu} \underbrace{\rho(\varepsilon)}_{DoS} f_F(\varepsilon) d\varepsilon \rightarrow$ basic formula

- No e-e $\rightarrow$ highly degenerate state $\rightarrow$ unstable to e-e inter.

- $H = \frac{1}{2m} \sum_i' \left(\vec{p}_i - \frac{e}{c}\vec{A}\right)^2 + \underbrace{\sum_{i<j} V(\vec{r}_i - \vec{r}_j)}_{\text{interaction}}$

a) $V(\vec{r}_i - \vec{r}_j) = \frac{e^2}{\epsilon|\vec{r}_i - \vec{r}_j|} \rightarrow$ physical

b) $V(\vec{r}_i - \vec{r}_j) = V_0 \nabla_i^2 \underbrace{\delta^2(\vec{r}_i - \vec{r}_j)}_{\text{2D } \delta\text{-function}} \rightarrow$ model

c) $\Psi_L(z) = \prod_{i<j}^{N} (z_i - z_j)^m \, e^{-\sum_{i=1}^{N} |z_i|^2/4l_B^2}$

(2)

d) $\Psi_L(z)$ is $|GS\rangle$ of $\hat{H}$ with $\nabla^2 \delta_{12}$ interaction

$$\hat{H}\,\Psi_m = E_N \Psi_m \,, \qquad E_N = \tfrac{1}{2} N \hbar\omega_c$$

this can be rigorously shown

- Quasi-holes

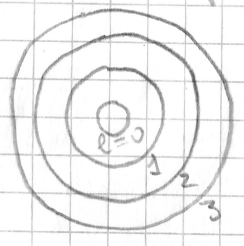
(I) Let $\nu = 1$; $\Rightarrow$ $\Psi_h(z) = \Big(\prod_{i=1}^{N} z_i\Big) \prod_{i<j}^{N} (z_i - z_j)\, e^{-\sum_{i=1}^{N} |z_i|^2/4\ell_B^2}$

$\ell = 0$ $\qquad \ell = N-1$

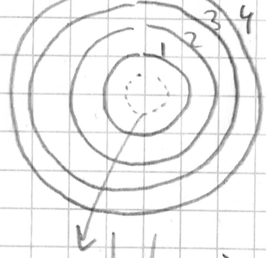
$$\Psi_h(z) \overset{!}{=} \exp\Big(-\sum_{i=1}^{N} |z_i|^2/4\ell_B^2\Big) \times \begin{array}{c} 1 \\ 2 \\ \vdots \\ N \end{array} \begin{vmatrix} z_1 & z_1^2 & \dots & z_1^N \\ z_2 & z_2^2 & \dots & z_2^N \\ \vdots & & & \\ z_N & z_N^2 & \dots & z_N^N \end{vmatrix}$$

! $\ell = m$ (of 2nd lecture)

$N = 4$



$$z_i^{\ell} \to z_i^{\ell+1} \quad \Rightarrow$$



$\Psi_{GS}(z)$

$\ell = \{0, 1, 2, 3\}$

hole $\to$ $\ell = 0$ is empty

$\ell = \{1, 2, 3, 4\}$

(II) $\nu = 1/m$ $\Rightarrow$ $\Psi_h(z) = \Big(\prod_{i=1}^{N} z_i\Big)\, \Psi_L(z)$

ground state

Idea is the same!

By expanding $\Psi_m(z)$ one obtains | Coherent superp. of states with dif. ℓ's

$$\Psi_h(z) = \Big(\prod_{i=1}^{N} z_i\Big) \sum_{\{k_1, \dots k_N\}}' C_{k_1 \dots k_N}\, z_1^{k_1} z_2^{k_2} \dots z_N^{k_N} \times \exp[\dots]$$

where $\sum_{j=1}^{N} k_j = \tfrac{1}{2} m\, N(N-1)$

$\Rightarrow$ for $\Psi_h(z)$ all momenta are shifted: $z_j^{k_j} \to z_j^{k_j+1}$

III Angular momentum $\qquad z_j = r_j e^{i\varphi_j}$

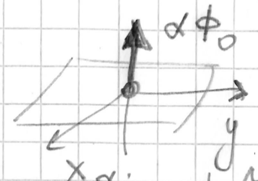
$$\hat L_z = \sum_{j=1}^{N} L_z^{(j)} = -i\sum_{j=1}^{N}\frac{\partial}{\partial\varphi_j} \Rightarrow \quad \Big| \; M_L = \tfrac{1}{2} m N(N-1)$$

- $\hat L_z \Psi_L(z) = \frac{1}{2} m\, N(N-1)\, \Psi_m(z)$; M_L
- $L_z \Psi_h(z) = \left[\frac{1}{2} m\, N(N-1) + N\right] \Psi_h(z)$; $M_L + N$

IV • Adiabatic insertion of a singular unit ϕ_0.

$$\vec a = \frac{\alpha\phi_0}{2\pi}\left(-y/r^2,\; x/r^2\right) \quad \text{with} \quad \alpha: 0 \to 1$$

- Flux: $\oint_C a\,dl \overset{!}{=} \alpha\phi_0$; $\Rightarrow \nabla\times\vec a = \alpha\phi_0\,\hat z\,\delta^2(\vec r)$

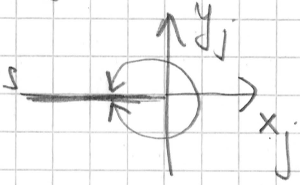


- In $\hat H$ we shift $\vec A \to \vec A + \vec a$

$\Psi(x,y) \to e^{i\alpha\,\mathrm{arctg}(y/x)}\,\Psi(x,y)$

- $\vec a$ can be eliminated by gauge transformation

$\Rightarrow$ branch cut for $\Psi(z)$ appears for each particle j



$$\Psi(x_j + i0)\Big|_{x_j<0} = e^{2\pi i\alpha}\,\Psi(x_j - i0)\Big|_{x_j<0}$$

- For 0^{th} LL single particle state is $z_j^{\,l+\alpha}\, e^{-|z_j|^2/4l_B^2}$
- As shown by Laughlin & others for $\Psi_L(z)$ state $C_{\{k\}}$ almost do not change (!)

$\Rightarrow$ (a) $\Psi_L(z) \xmapsto{\alpha} \Psi_\alpha(z) = \prod_{j=1}^{N}(z_j)^{\alpha}\,\Psi_L(z)$

adiab. flux insertion

(b) $\Psi_{\alpha=1}(z) \equiv \Psi_h(z) \to$ quasi-hole state!

$\underline{\text{V}}$: $E_h = E_N + \varepsilon_N$; $E_0 \propto N$; $\lim_{N\to\infty} \varepsilon_N \to \varepsilon_0$

E_N — energy of GS; ε_0 — finite!

- quasi-hole state is separated by the gap ε_0 from the ground state.

$\underline{\text{VI}}$ General (many) quasi-hole state with η_j; $j=1,\dots,M$

- $$\Psi_h(\eta, z) = \prod_{j=1}^{M}\prod_{i=1}^{N}(z_i - \eta_j) \times \prod_{i<j}^{N}(z_i - z_j)^m \exp(\dots)$$

Next goals:

1) Evaluate $n \overset{!}{=} \frac{1}{2\pi m \ell_B^2} \Rightarrow \nu = 1/m$

2) Charge of q.-hole $q^* = -e/m$

3) Braiding of q.-holes $\Rightarrow \alpha = \pi/m \to$ anyons!

Plasma analogy

- Introduce normalized GS: $\langle\Psi|\Psi\rangle = 1$

$$\langle z|\Psi\rangle = \frac{1}{\mathcal{Z}^{1/2}}\Psi_L(z), \quad \mathcal{Z} \to \text{normal. factor}$$

$$\mathcal{Z} = \int \prod_{i=1}^{N} d^2 z_i \; P(z), \text{ where } \underbrace{P(z)}_{|\Psi_L(z)|^2 \,=} = e^{-\beta U(z)}$$

with $$\Psi_L(z) = \prod_{i<j}^{N}\left(\frac{z_i - z_j}{\ell_B}\right)^m \exp\left[-\sum_{i=1}^{N}|z_i|^2/4\ell_B^2\right]$$

$$\beta U(z) \overset{\text{def}}{=} -2m\sum_{i<j}^{N}\ln\frac{|z_i - z_j|}{\ell_B} + \frac{2}{\ell_B^2}\sum_{i=1}^{N}|z_i|^2$$

Info: $|z|^{2m} = e^{2m\ln|z|}$

- Let us set $\beta \equiv \frac{2}{m}$ → ficticios inverse temperature

$$\Rightarrow \mathcal{U}(z) = -m^2 \sum_{i<j}^{N} \ln \frac{|z_i - z_j|}{\ell_B} + \frac{m}{4\ell_B^2} \sum_{i=1}^{N} |z_i|^2$$

- Remainder from statistical physics

$$Z = \operatorname{tr} e^{-\beta H(\hat{p},\hat{q})} \underset{\substack{\downarrow \\ \text{classical limit}}}{\overset{\hbar \to 0}{=}} \frac{1}{N!} \int \frac{d^N p \, d^N q}{(2\pi)^N} e^{-\beta H(\vec{p},\vec{q})}$$

$\vec{p} = (p_1, \dots, p_N)$ and $\vec{q} = (q_1, \dots, q_N)$

! if $H(p,q) = \underbrace{K(\vec{p})}_{\text{kinetic}} + \underbrace{V(\vec{q})}_{\text{potential}} \Rightarrow Z \underset{\text{factorizes!}}{=} Z_K \cdot Z_V$

- $\begin{cases} \Rightarrow \ln\text{-like interaction between fict. charges } Q_i \equiv m \\ \Rightarrow 2^{nd}\text{-term} \to \text{needs interpretation} \end{cases}$

3D versus 2D electrostatics

3D	2D
1) Interaction $\mathcal{U}_{12} = \int \frac{\rho_1(r_1)\rho_2(r_2)}{\vert r_1 - r_2\vert} d^3r_1 d^3r_2$	$\mathcal{U}_{12} = -\int \ln\left(\frac{r_1 - r_2}{\ell}\right) \rho(r_1)\rho(r_2)\, d^2r_1 d^2r_2$
2) Potential: $\mathcal{U}_{12} = \int \rho_1(r)\, \varphi_2(r)\, d^D r$	
$\varphi_2(r) = \int \frac{\rho_2(r')}{\vert r - r'\vert} d^3r'$	$\varphi_2(r) = -\int \ln\left\vert\frac{r-r'}{\ell}\right\vert \rho_2(r') d^2r'$
3) Green's function $\nabla^2 \frac{1}{\vert r_1 - r_2\vert} = -4\pi\delta^3(r_1 - r_2)$	$-\nabla^2 \ln\left\vert\frac{r_1 - r_2}{\ell}\right\vert = -2\pi\delta^2(r_1 - r_2)$
Poisson eq.	
4) $\nabla^2 \varphi_2(\vec{r}) = -4\pi\rho_2(\vec{r})$	$\nabla^2 \varphi_2(\vec{r}) = -2\pi\rho_2(\vec{r})$

⑥

- Let's intepret $\frac{m}{4\ell_B^2}\sum_{i=1}^{N}|z_i|^2 \stackrel{?}{=}$

$$= \sum_{i=1}^{N}\int \underbrace{(-m)\delta^2(z_i - z)}_{\text{density of point charges } Q_i = -m}\cdot\left(\frac{-|z|^2}{4\ell_B^2}\right) d^2z$$

$$\Rightarrow \quad \varphi(z) = -\frac{|z|^2}{4\ell_B}$$

$\downarrow$ potential of background chare

$$\nabla^2 = \partial_x^2 + \partial_y^2 \stackrel{!}{=} 4\partial_z\partial_{\bar z} \Rightarrow \quad \left(|z|^2 = z\bar z\right)$$

$$\nabla^2\varphi(z) = -\frac{1}{\ell_B^2} = -2\pi\rho_0 \Rightarrow \boxed{\rho_0 = \frac{1}{2\pi\ell_B^2}}$$

$\downarrow$ background charge density

Evaluation of $\nu \stackrel{?}{=} 1/m$

- Concentration : $\hat n(z) = \sum_{i=1}^{N}\delta(z - z_i)$

$$\langle n\rangle = \langle\psi|n|\psi\rangle = \frac{1}{Z}\int\prod_{i=1}^{N}d^2z_i\ \delta(z - z_i)\,|\psi_L(z)|^2, \text{or}$$

$$n = \langle n\rangle = \frac{1}{Z}\int\prod_{i=1}^{N}d^2z_i\ \delta(z - z_i)\,P(z) \quad \text{with } P(z) = e^{-\beta U(z)}$$

- "Charge neutrality" $\Rightarrow$ $\left[\begin{array}{l} +Qn + \rho_0 \stackrel{!}{=} 0 \Rightarrow \\ Q = -m \end{array}\right.$

ρ_0 $\searrow$ positive backgr

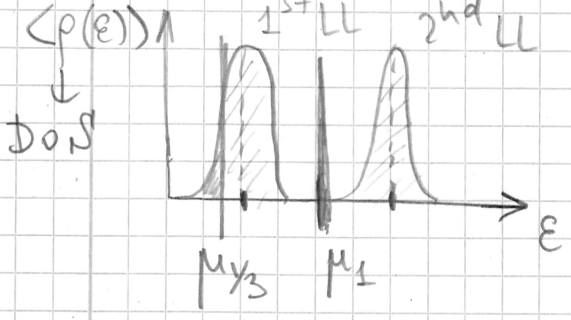
Q $\downarrow$ neg. fic. charges

$$n = \frac{1}{2\pi m\ell_B^2} \Rightarrow \underline{\nu = 1/m}$$

Anyons in FQHE → Lecture #4

- Laughlin wave function at $\nu = 1/m$

$m = 3, 5, 7, \ldots, \to$ odd integer



$\mu_{1/3}$ corresponds to

$$n = \frac{1}{2\pi l_B^2 m} \to \text{concentr.}$$

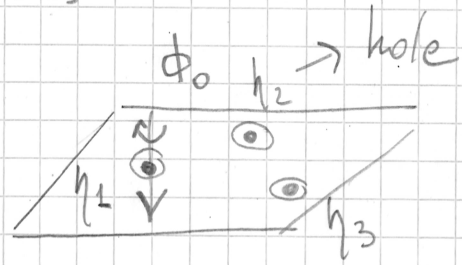
$$l_B = \left(\hbar c / eB\right)^{1/2} \to \text{magnetic length}$$

$$\Psi_L(z) = \prod_{i<j}^{N} \left(\frac{z_i - z_j}{l_B}\right)^m \exp\left[-\sum_{i=1}^{N} |z_i|^2 / 4 l_B^2\right]$$

$z_j = x_j + i y_j \to$ complex coordinates

- Quasi-hole wave function

$$\Psi_h(\eta; z) = \prod_{j=1}^{M} \prod_{i=1}^{N} \left(\frac{z_i - \eta_j}{l_B}\right) \times \Psi_L(z)$$



M holes

Ψ_h is obtained from $\Psi_L(z)$ by flux attachment at each point $\eta_k = x_k + i y_k$

$$\vec{a}(x,y) = -\frac{\alpha \Phi_0}{2\pi}\left(-\frac{y - y_0}{|\vec{r} - \vec{r}_0|^2}, \frac{x - x_0}{|\vec{r} - \vec{r}_0|^2}\right) \to \text{singular flux at point } \vec{r}_0$$

$\alpha \in [0, 1]$

Plasma analogy

- Introduce normalized GS: $\langle \Psi | \Psi \rangle = 1$

$$\langle z | \Psi \rangle = \frac{1}{Z^{1/2}} \Psi_L(z), \quad \text{where } Z \to \text{norm. factor}$$

$$Z = \int \prod_{i=1}^{N} d^2 z_i \, |\Psi_L(z)|^2 \quad \text{and} \quad |\Psi_L(z)|^2 \stackrel{\text{def}}{=} P(z)$$

$P(z)$ → a probability density

Let's set $P(z) = e^{-\beta U(z)}$

with $\beta = 2/m \to$ auxiliarly temperature

(2)

- $P(z)$ is Boltzmann (or Gibbs) weight with potential

$$\mathcal{U}(z) = -m^2 \sum_{i<j}^{N} \ln\left(\frac{z_i - z_j}{\ell_B}\right) + \frac{m}{4\ell_B^2} \sum_{i=1}^{N} |z_i|^2$$

- $\ln$-like interaction betwee two point-charges $Q = m$

3D versus 2D electrostatic

(a) Poisson equation

3D | 2D

- $\nabla^2 \varphi(\vec{r}) = -4\pi \rho(r)$ | $\nabla^2 \varphi(\vec{r}) = -2\pi \rho(\vec{r})$

(b) Green's function

- $\nabla^2 \left(\frac{1}{\vec{r}_1 - \vec{r}_2}\right) = -4\pi \delta^3(\vec{r}_1 - \vec{r}_2)$ | $-\nabla^2 \ln\left(\frac{\vec{r}_1 - \vec{r}_2}{\ell}\right) = -2\pi \delta^2(\vec{r}_1 - \vec{r}_2)$

(c) Interaction

$\mathcal{U}_{12} = \int \frac{\rho_1(\vec{r}_1)\rho_2(\vec{r}_2)}{|\vec{r}_1 - \vec{r}_2|} d^3r_1 d^3r_2$ | $\mathcal{U}_{12} = -\int \ln\left(\frac{\vec{r}_1 - \vec{r}_2}{\ell}\right) \rho(r_1)\rho(r_2)\, d^2r_1 d^2r_2$

(c) Potential: $\mathcal{U}_{12} = \int \rho(r_1) \varphi_2(r_1)\, d^D r_1$

$\varphi_2(r_1) = \int \frac{\rho_2(r_2)}{|\vec{r}_1 - \vec{r}_2|} d^3\vec{r}_2$ | $\varphi_2(\vec{r}_1) = -\int \rho_2(\vec{r}_2) \ln\left|\frac{r_1 - r_2}{\ell}\right| d^2r_2$

- Let's interpret | $\frac{m}{4\ell_B^2} \sum_i |z_i|^2$ -?

$\varphi(z) = \frac{|z|^2}{4\ell_B^2}$ $\Leftarrow$ | $= \int \underbrace{\sum_{i=1}^{N} m\, \delta(z - z_i)}_{\text{density of point charges } Q_i = m} \frac{|z|^2}{4\ell_B^2}\, d^2z$

$\varphi(z)$: potential of background charge

$$\nabla^2 = \partial_x^2 + \partial_y^2 = 4\partial_z \partial_{\bar{z}} \Rightarrow$$

$$\nabla^2 \varphi(z) = 1/\ell_B^2 = -2\pi\rho_0 \Rightarrow \rho_0 = \frac{1}{2\pi\ell_B^2}$$

ρ_0: background charge density

(3)

- "Charge neutrality" of plasma

Consider $\hat{n}(z) = \sum_{j=1}^{N} \delta(z - z_j)$ → concentration of e

$$\langle n \rangle = \langle \psi | \hat{n}(z) | \psi \rangle = \frac{1}{\mathcal{Z}} \int \prod_{i=1}^{N} d^2 z_i \sum_{j=1}^{N} \delta(z - z_j)\, P(Z)$$

$Q\langle n \rangle = m \langle n \rangle$ – charge density of free charges in plasma

$\left| \, Q\langle n \rangle + \rho_0 = 0 \, \right|$ → charge neutrality ⇒

$$n = \frac{1}{2\pi m \ell_B^2} \rightarrow \text{concentration} \Rightarrow \nu = 1/m \,!$$

Quasi-hole wave function & partition sum

- $\langle z, \bar{z} | \eta_1, \ldots, \eta_M \rangle = \prod_{j=1}^{M} \prod_{i=1}^{N} \left(\frac{z_i - \eta_j}{\ell_B} \right) \times \psi_L(z) = \psi_h(z; \eta)$

  $\langle z, \bar{z} | \eta_1, \ldots, \eta_M \rangle$ ↓ unnormalized; $\psi_L(z)$ → GS; ↓ unnormalized

- $|\psi\rangle_{\text{hole } \leftarrow h} = \frac{1}{\mathcal{Z}} |\eta_1, \ldots, \eta_M\rangle$ – normalized state

$$\Rightarrow \mathcal{Z} = \int \prod_{i=1}^{N} d^2 z_i \, |\psi_h(z;\eta)|^2 = \int \prod_{i=1}^{N} d^2 z_i \, e^{-\beta U(z;\eta)}$$

⇝ Applying again the idea of plasma analogy ⇒

$$U(z;\eta) = -m^2 \sum_{k<\ell}' \ln \left| \frac{z_k - z_\ell}{\ell_B} \right| - m \sum_{k,i} \ln \frac{|z_k - \eta_i|}{\ell_B}$$

(+m) at z_k ~~~ (+m) at z_ℓ ; (+m) at z_k ~~~ (+1) at η_i → impurity

$$+ \frac{m}{4\ell_B^2} \sum_{k=1}^{N} |z_k|^2$$

↓ inter. with background → $\rho_0 = -\frac{1}{2\pi \ell_B^2}$ → back. gr. density

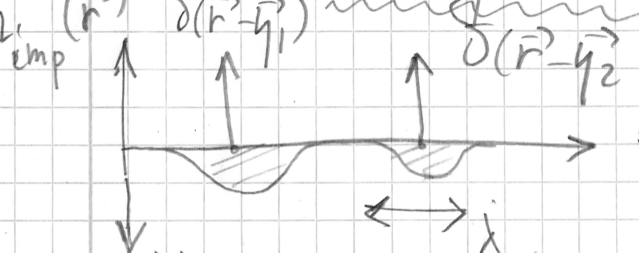
(4):

- The energy $\mathcal{U}(z;\eta)$ is not yet the full inter. energy of plasma & impurities

$$(*)\quad \mathcal{U}_{full}(z;\eta) = \mathcal{U}(z,\eta) - \sum_{i<j} \ln \frac{|\eta_i - \eta_j|}{l_B} + \underbrace{\frac{1}{4l_B^2}\sum_{i=1}^{M} |\eta_i|^2}_{\text{interaction with backgr.}}$$

η_i (+1) ~~~ (+1) η_j

Screening in plasma



$m\,n(\vec{r}) \to$ mean concentration

$\underbrace{m\,n(\vec{r})}$ induced charge(!) density

$$n_{imp}(\vec{r}) = \sum_{i=1}^{M} \delta(\vec{r}-\vec{\eta}_i)$$

impurity density

$$\hat{n}(\vec{r}) = \sum_{i=1}^{N} \delta(\vec{r}-\vec{z}_i)$$

$$n(\vec{r}) = {}_{\eta}\langle \psi | \hat{n}(\vec{r}) | \psi \rangle_{\eta} = \frac{1}{\mathcal{Z}} \langle \eta | \hat{n} | \eta \rangle$$

$$m \int n(\vec{r})\, d^2\vec{r} \overset{!}{=} -1 \quad \Rightarrow \quad q^* = -e/m \to \text{quasi-hole charge}$$

$|\vec{r}-\vec{r}_i| \lesssim \lambda_D \to$ Debye screening radius, $\sim l_B$

Partition sum of quasi-holes

- If $|\eta_i - \eta_j| \gg \lambda_D \sim l_B$ for all $i \neq j$, then

$$\mathcal{Z}_{full} = \int \prod_{i=1}^{N} d^2 z_i \exp\left[-\beta\, \mathcal{U}_{full}(z;\eta)\right] = \text{Const}$$

$\sim$ energy doesn't change if positions η_i vary.

- Using (*) we can say that $\quad (\beta = 2/m)$

$$\mathcal{Z}_{full} = \mathcal{Z} \exp\left[+\frac{2}{m} \sum_{i<j} \ln \frac{|\eta_i - \eta_j|}{l_B} - \frac{1}{2m l_B^2} \sum_{i=1}^{M} |\eta_i|^2 \right]$$

(5) $\Rightarrow$

- $$\mathcal{Z}[\eta,\bar\eta] = \exp\left[-\frac{2}{m}\sum_{i<j} \ln\frac{|\eta_i-\eta_j|}{l_B} + \frac{1}{2ml_B^2}\sum_{i=1}^{M}|\eta_i|^2\right]$$

$\downarrow$ norm. const for $|\psi\rangle = \frac{1}{\mathcal{Z}^{1/2}}|\eta\rangle \rightarrow$ quasi-hole wave function

Berry connection

$\eta_1, \eta_2, \ldots \rightarrow \lambda \rightarrow$ parameters

$$a_\eta = i\langle\psi|\frac{\partial}{\partial\eta}|\psi\rangle$$, where $\eta \rightarrow$ any of η_k

- Using, that (i) $\mathcal{Z} = \langle\eta|\eta\rangle$ and (ii) $|\eta\rangle$ – holomorphic function of η $\Rightarrow$

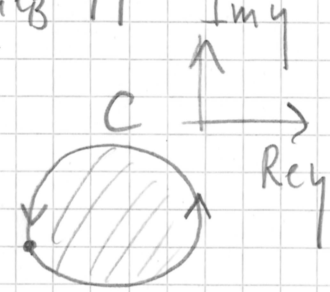
$$a_\eta = \frac{i}{2}\frac{\partial}{\partial\eta}\ln\mathcal{Z} \quad \text{and} \quad a_{\bar\eta} = -i\frac{\partial}{\partial\bar\eta}\ln\mathcal{Z}$$

$\uparrow$ exercise!

Quasi-hole charge via AB-effect

- Single quasi-hole η $\Rightarrow$ $\mathcal{Z}[\eta] = \frac{1}{2ml_B^2}\bar\eta\eta$ $\Rightarrow$

$$a_\eta = \frac{i}{4ml_B^2}\bar\eta \quad \text{and} \quad a_{\bar\eta} = -\frac{i\eta}{4ml_B^2}$$



$$e^{i\gamma} = \exp\left[i\oint_C (a_\eta d\eta + a_{\bar\eta} d\bar\eta)\right]$$

$\downarrow$ Berry phase

trajectory $\eta(t)$

if $\eta = x+iy$, then $\bar\eta\, d\eta + \eta\, d\bar\eta = 2i(x\,dy - y\,dx)$

$$\Rightarrow \gamma = -\oint_C \frac{1}{2ml_B^2}(-y\,dx + x\,dy);$$

$1/l_B^2 = eB/\hbar c$ and $\vec{A} = \frac{B}{2}(-y, x)$ $\Rightarrow$ (other side $\rightarrow$)

$\swarrow$ radial gauge

5.a

$$\gamma = -\left(\frac{e}{m}\right) \frac{1}{\hbar c} \oint_C \left(A_x\, dx + A_y\, dy\right) \Rightarrow$$

$$e^{i\gamma} = \exp\left[\frac{iq^*}{\hbar c} \oint_C A\, dl\right] \quad \text{with} \quad q^* = -e/m \;!$$

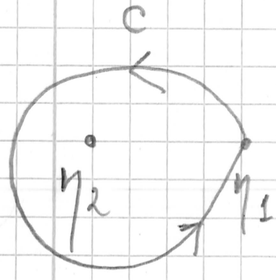
quasi-hole charge

Fractional statistics of q.-holes

- $\Psi_h(\eta_2,\eta_1;z) = e^{i\pi/m}\,\Psi_h(\eta_1,\eta_2;z) \quad \Big| \quad e^{i\pi/m} = e^{i\alpha}$

  $e^{i\pi/m}$ → statistical phase

- $\ln Z[\eta_1,\eta_2] = -\frac{1}{m}\ln\left|\frac{\eta_1-\eta_2}{\ell_B}\right|^2 + \frac{1}{2m\ell_B^2}\left(|\eta_1|^2+|\eta_2|^2\right)$



$\eta_2 \to$ fixed; $\quad \eta_1(t) \to$ moves (adiabatic)

$$a_{\eta_1} = \frac{i}{2}\frac{\partial}{\partial \eta_1}\ln Z = -\frac{i}{2m}\frac{1}{(\eta_1-\eta_2)}$$

$$a_{\bar\eta_1} = -\frac{i}{2}\frac{\partial}{\partial \bar\eta_1}\ln Z = \frac{i}{2m}\frac{1}{(\bar\eta_1-\bar\eta_2)}$$

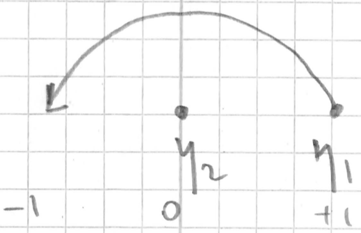
- Berry phase along C:

$$\gamma = \oint a_{\eta_1}\,d\eta_1 + \oint a_{\bar\eta_1}\,d\bar\eta_1 = \quad \to \text{counterclockwise}$$

$$= -\frac{i}{2m}\oint\frac{d\eta_1}{(\eta_1-\eta_2)} + \frac{i}{2m}\oint\frac{d\bar\eta_1}{(\bar\eta_1-\bar\eta_2)}$$

↙ simple pole, clockwise(!)

$$= 2\times\left(-\frac{i}{2m}\right)\cdot 2\pi i = \frac{2\pi}{m} \leftarrow \text{Berry phase}$$

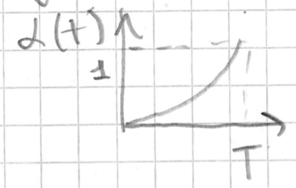


→ Right move → η_1 at 0, η_2 at +1 → initial state

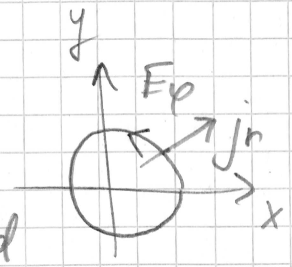
$\Rightarrow \alpha = \pi/m \to$ statistical phase

Hall resistance

• Singular flux $\vec{a} = -\frac{\alpha\Phi_0}{2\pi}\left(-y/r^2, x/r^2\right)$



→ adiabatic insertion



• $\vec{E}(t) = -\frac{1}{c}\frac{\partial \vec{a}}{\partial t}$ ⇒ electric field

$(\Phi_0 = 2\pi\hbar c/e)$ $\qquad E_\varphi(t) = \frac{\hbar\dot{\alpha}(t)}{e}\,\frac{1}{r}$ ⇒

• Current (in radial direction)

$$j_r(r,t) = \sigma_H E_\varphi = \frac{\hbar}{e}\sigma_H\,\dot{\alpha}(t)\,\frac{1}{r}$$

$$I(t) = 2\pi r\, j_r(r,t) = \frac{2\pi\hbar}{e}\sigma_H\,\dot{\alpha}(t)$$

• Charge $\quad q = \int_0^T I(t)\,dt = \frac{2\pi\hbar}{e}\sigma_H \stackrel{!}{=} e/m$

• ⇒ $\sigma_H = \frac{e^2}{2\pi\hbar}\cdot\frac{1}{m}$ ⇒ $\rho_H = R_Q m = R_Q\frac{1}{\nu}$

with $\nu = 1/m$ → filling fraction