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Quantum Field Theory III

Exercise sheet 1

This exercise will be discussed on 21.04.2016

1. Basics of differential geometry

The aim of this exercise is to get the familiarity with some basic definitions from the theory of differentiable manifolds.

Let the manifold M be a unit sphere $M = S^2$ which has a natural embedding into $\mathbb{R}^3$: $S^2 = \{\mathbf{x} \in \mathbb{R}^3 | \mathbf{x}^2 = 1\}$.

a) Consider the spherical coordinates on S^2 related to the Cartesian coordinates (x^1, x^2, x^3) in $\mathbb{R}^3$ by the well known formulae

$$x^1 = \sin\theta\cos\phi, \quad x^2 = \sin\theta\sin\phi, \quad x^3 = \cos\theta,$$

and two vector fields $\mathbf{e}_\phi \equiv \partial_\phi$, $\mathbf{e}_\theta \equiv \partial_\theta$ which form a natural basis set (induced by the chosen spherical coordinates) in the tangent bundle TS^2. Compute the Cartesian coordinates $(\mathbf{e}_\phi)^i$ and $(\mathbf{e}_\theta)^i$ (where $i = 1, 2, 3$) of such basis.

b) Another coordinate system on S^2 is the Riemann one $(z, \bar{z})$ which is defined by

$$x^1 = \frac{z + \bar{z}}{1 + z\bar{z}}, \quad x^2 = \frac{i(\bar{z} - z)}{1 + z\bar{z}}, \quad x^3 = \frac{-1 + z\bar{z}}{1 + z\bar{z}}. \tag{1}$$

These coordinates are also known as *stereographic projection* of the sphere S^2 on the complex plane $\mathbb{C}$, where the inverse relation to Eq. (1) is the mapping

$$z : S^2 \mapsto \mathbb{C}, \qquad z(\mathbf{x}) = (x^1 + ix^2)/(1 - x^3), \qquad \mathbf{x}^2 = 1,$$

and $\bar{z}(\mathbf{x}) \equiv (z(\mathbf{x}))^*$. What are Cartesian coordinates of vector fields $\mathbf{e}_z \equiv \partial_z$ and $\mathbf{e}_{\bar{z}} \equiv \partial_{\bar{z}}$?

We remind that given any differentiable function $f : M \mapsto \mathbb{R}$ on the manifold M, its differential df along a vector $\mathbf{v}_p \in T_pM$ at point $p \in M$ is defined as $df_p(\mathbf{v}_p) \equiv \mathbf{v}_p(f)$.

c) Compute $dz(\mathbf{e}_\phi)$, $d\bar{z}(\mathbf{e}_\phi)$ and the same for the vector field $\mathbf{e}_\theta$ for all points on S^2. Express the final result in spherical coordinates.

d) Compute $d\phi(\mathbf{e_z})$, $d\theta(\mathbf{e_z})$ and the analogous differentials for the vector field $\mathbf{e_{\bar{z}}}$. Once again express the final result in spherical coordinates and check the self-consistency with the previous task **c)**.