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Quantum Field Theory III

Exercise sheet 10

This exercise will be discussed on 21.07.2016

11. The Hilbert space of $\mathbb{Z}_2$ lattice gauge theory

Consider the two-dimensional square lattice with cites labeled by $\mathbf{i}$. For every nearest-neighbor link we assign a link variable $s_{\mathbf{ij}}$ which takes values ± 1 . To obtain the Hilbert space of the $\mathbb{Z}_2$ gauge theory we require that gauge equivalent configurations $\{s_{\mathbf{ij}}\}$ and $\{\tilde{s}_{\mathbf{ij}}\}$ label the same quantum state, where by definition $\{s_{\mathbf{ij}}\}$ and $\{\tilde{s}_{\mathbf{ij}}\}$ are gauge equivalent if they are related by the $\mathbb{Z}_2$ gauge transformation

$$\tilde{s}_{\mathbf{ij}} = W_{\mathbf{i}} s_{\mathbf{ij}} W_{\mathbf{j}}, \tag{1}$$

where $W_{\mathbf{i}}$ is arbitrary function with values ± 1. We group all such $\{\tilde{s}_{\mathbf{ij}}\}$ into a class and name it the gauge-equivalent class.

a) To begin with consider one square with 4 cites and 4 links. What is the number of different gauge-equivalent classes (= dimension of the Hilbert space)? **Hint**: The answer is 2 states.

b) Take further the lattice with the periodic boundary conditions in both direction (i.e. the lattice forms a torus) which has N cites and $2N$ nearest-neighbor links. What is the number of gauge-equivalent classes (or quantum states) in this case? **Hint**: 2×2^N states.

To find the way to label the states in such Hilbert space one considers the gauge-invariant quantities. One of them is the Wegner-Wilson loop variable defined for any loop C as

$$U(C) = s_{\mathbf{ij}} s_{\mathbf{jk}} \dots s_{\mathbf{li}}, \tag{2}$$

where $\mathbf{i}, \mathbf{j}, \mathbf{k}$ and $\mathbf{l}$ are cites of the loop. One calls $U(C)$ the $\mathbb{Z}_2$ flux through the loop. In particular, if $\mathbf{x}$ and $\mathbf{y}$ are unit vectors in the directions x and y one defines the $\mathbb{Z}_2$ flux through the plaquette,

$$F_i = s_{\mathbf{i},\mathbf{i}+\mathbf{x}} s_{\mathbf{i}+\mathbf{x},\mathbf{i}+\mathbf{x}+\mathbf{y}} s_{\mathbf{i}+\mathbf{x}+\mathbf{y},\mathbf{i}+\mathbf{y}} s_{\mathbf{i}+\mathbf{y},\mathbf{i}}. \tag{3}$$

c) Prove that on a torus

$$\prod_i F_i = 1. \tag{4}$$

Since we have N plaquettes with $F_i = \pm 1$, and because of (4) the set $\{F_i\}$ provide only $2^N/2$ labels for quantum states. As you have to show below each configuration $\{F_i\}$ corresponds to four different states, thereby one gets 2×2^N as the total number of states, see **b)**.

Let $s^0_{\mathbf{ij}}$ be an initial configuration and four other configurations are obtained from it as

$$s^{m,n}_{\mathbf{ij}} = f^m_x(\mathbf{ij}) f^n_y(\mathbf{ij}) s^0_{\mathbf{ij}}, \qquad m, n = 0, 1 \tag{5}$$

where functions $f_{x,y}(\mathbf{ij})$ take the values ± 1. Explicitly, $f_x(\mathbf{ij}) = -1$ if the link $\mathbf{ij}$ crosses the line x (see Fig. 1) and $f_x(\mathbf{ij}) = 1$ otherwise. Similarly, $f_y(\mathbf{ij}) = -1$ if the link $\mathbf{ij}$ crosses the line y, and $f_y(\mathbf{ij}) = 1$ otherwise.

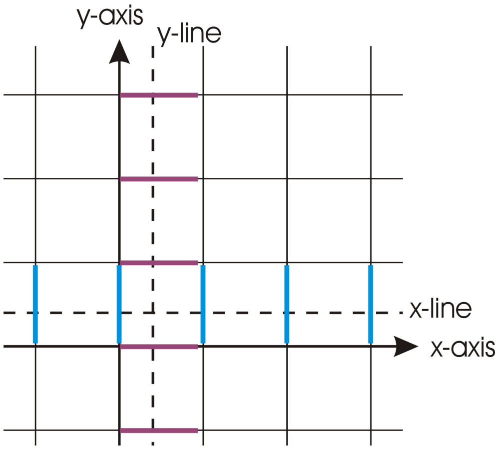

Figure 1: The variables $s^0_{\mathbf{ij}}$ on blue (red) links are changing signs under the transformation (5) if m=1 (n=1), the variables $s^0_{\mathbf{ij}}$ on all others links are unaffected.

d) Check that four such configurations corresponds to the same flux F_i through each plaquette.

e) Check that four configurations $s^{m,n}_{\mathbf{ij}}$ are *not* gauge equivalent. **Hint:** Consider $U(C)$ with a non-contractible loop C going all the way around the torus.