Solution to problem # 1

(a)

Observe that $\mathbf{e}_\phi$ and $\mathbf{e}_\theta$ from TS^2 are vector fields along the directions ϕ and θ, resp. To find their coordinates in $\mathbb{R}^3$, one should use Eq. (2.29) from the Altand's book, since we have the mapping $S^2 \mapsto \mathbb{R}^3$. Hence

$$\begin{aligned}\mathbf{e}_\phi &= \partial_\phi(x^1, x^2, x^3) = (-\sin\theta\sin\phi, \sin\theta\cos\phi, 0),\\ \mathbf{e}_\theta &= \partial_\theta(x^1, x^2, x^3) = (\cos\theta\cos\phi, \cos\theta\sin\phi, -\sin\theta).\end{aligned} \tag{1}$$

(b)

In the same fashion as above,

$$\begin{aligned}\mathbf{e}_z &= \partial_z(x^1, x^2, x^3) = \left(\frac{1-\bar{z}^2}{(1+z\bar{z})^2}, -i\frac{1+\bar{z}^2}{(1+z\bar{z})^2}, \frac{2\bar{z}}{(1+z\bar{z})^2}\right)\\ \mathbf{e}_{\bar{z}} &= \partial_{\bar{z}}(x^1, x^2, x^3) = \left(\frac{1-z^2}{(1+z\bar{z})^2}, i\frac{1+z^2}{(1+z\bar{z})^2}, \frac{2z}{(1+z\bar{z})^2}\right)\end{aligned} \tag{2}$$

(c) & (d)

One may start by relating spherical and stereographic coordinates using the mapping $z(\mathbf{x}) = (x^1 + ix^2)/(1 - x^3)$ to obtain

$$z(\theta,\phi) = \frac{\sin\theta e^{i\phi}}{1-\cos\theta} = \cot(\theta/2)e^{i\phi}, \qquad \bar{z}(\theta,\phi) = \cot(\theta/2)e^{-i\phi}. \tag{3}$$

The inverse mapping $\mathbb{C} \to S^2$ is then

$$\phi(z,\bar{z}) = -\frac{i}{2}\ln\left(z/\bar{z}\right), \qquad \theta(z,\bar{z}) = 2\,\mathrm{arccot}(z\bar{z}). \tag{4}$$

Next, by definition, $dz(\mathbf{e}_\phi) = \mathbf{e}_\phi(z) = \partial_\phi z$, and etc. To be more formal, I'm using the material from page 539 of the Altland's book. Hence we may organize the matrix of transformation from stereographic to sperical basis,

$$\begin{pmatrix} \partial_\phi z & \partial_\theta z \\ \partial_\phi \bar{z} & \partial_\theta \bar{z} \end{pmatrix} = \begin{pmatrix} ie^{i\varphi}\cot\left(\frac{\theta}{2}\right) & \frac{e^{i\varphi}}{\cos(\theta)-1} \\ -ie^{-i\varphi}\cot\left(\frac{\theta}{2}\right) & \frac{e^{-i\varphi}}{\cos(\theta)-1} \end{pmatrix} \tag{5}$$

For the inverse mapping one gets

$$\begin{pmatrix} \partial_z \phi & \partial_{\bar{z}} \phi \\ \partial_z \theta & \partial_{\bar{z}} \theta \end{pmatrix} = \begin{pmatrix} -\frac{1}{2}ie^{-i\varphi}\tan\left(\frac{\theta}{2}\right) & \frac{1}{2}ie^{i\varphi}\tan\left(\frac{\theta}{2}\right) \\ \frac{1}{2}e^{-i\varphi}(\cos(\theta)-1) & \frac{1}{2}e^{i\varphi}(\cos(\theta)-1) \end{pmatrix} \tag{6}$$

Finally, one checks that matrices (5) and (6) are inverse to each other, as it should be.