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Quantum Field Theory III

Exercise sheet 2

This exercise will be discussed on 28.04.2016

2. Sine-Gordon model and BKT phase transition

It was shown in the lecture that the topological part of the partition sum $\mathcal{Z}_t$ for the XY-model with the coupling constant J can be reduced to the partition sum of a neutral 2D Coulomb plasma of $2N$ unit charges $\sigma_i \pm 1$,

$$\mathcal{Z}_t = \sum_{N=0}^{\infty} \frac{y_0^{2N}}{(N!)^2} \int \left(\prod_{i=1}^{2N} d^2\mathbf{r}_i \right) \exp\left[4\pi^2 J \sum_{i<j} \sigma_i \sigma_j C(\mathbf{r}_i - \mathbf{r}_j) \right], \tag{1}$$

where $C(\mathbf{r}) = \ln(|\mathbf{r}|/a)/(2\pi)$ is the two-dimensional Coulomb potential and a is a short-distance cut-off. In the sum above one assumes that $\sigma_i = +1$ for $i = 1, ...N$ while $\sigma_i = -1$ for $i = N+1, ..., 2N$, and y_0 denotes the fugacity of vortices (see Chapter 8.6 in [1] for more info).

It can be shown that this Coulomb gas model is equivalent to the 2D 'sine-Gordon' model defined by the action

$$S[\theta] = \frac{1}{16\pi K} \int d^2\mathbf{r} (\nabla\theta)^2 + g \int d^2\mathbf{r} \cos(\theta), \tag{2}$$

where $\theta : \mathbb{R}^2 \mapsto \mathbb{R}$ is a real scalar field.

a) Prove the equivalence of the models (1) and (2) by expanding the partition sum of the sine-Gordon model in powers of g. As the intermediate useful result establish that the multiple phase correlation function

$$V(\mathbf{r_1}, \ldots, \mathbf{r}_{2N}) = \left\langle \exp\left(i \sum_{i=1}^{2N} \epsilon_i \, \theta(\mathbf{r}_i) \right) \right\rangle_0, \quad \epsilon_i = \pm 1,$$

where the average is performed w.r.t. the free Gaussian action $S_0 = (16\pi K)^{-1} \int d^2\mathbf{r} (\nabla\theta)^2$, is non-zero in the thermodynamic limit $L \to \infty$ only under the 'charge-neutrality' condition $\sum_{i=1}^{2N} \epsilon_i = 0$.

The equivalence of the two models will hold under the identification $J = 2K/\pi$ and $y_0 = g/2$.

b) Using the standard momentum shell renormalization group derive the (perturbative) RG equation for the coupling constant g in the model (2). The result you'll get should read

$$\frac{dg}{dl} = (2 - 2K) g.$$

Compare it with the one of the RG equations for the BKT phase transition describing the renormalization of the vortex fugacity y.

Note: The 2nd RG equation describing the renormalization of the coupling constant K is somewhat harder to obtain since it requires going into the 2nd oder perturbation theory in g. Those who are interested are advised to consult Appendix E in the book [2].

References

[1] A. Altland and B. Simons, *Condensed Matter Field Theory*, 2nd Edition (Cambridge University Press, 2010).

[2] T. Giamarchi, *Quantum Physics in One Dimension* (Oxford University Press, 2003).