Solution to problem # 2

It is worth to begin by evaluating the free correlation function $G(\mathbf{r}) = \langle\theta(\mathbf{r})\theta(0)\rangle$. For that it is useful to regularize the Gaussian action by adding the mass term,

$$S_0 = \frac{1}{16\pi K}\int d^2\mathbf{r}\left((\nabla\theta)^2 + \xi^{-2}\theta^2\right). \tag{1}$$

Then the correlation function is well defined and well known,

$$\langle\theta(\mathbf{r})\theta(0)\rangle = G(\mathbf{r}) = 8\pi K\int\frac{d\mathbf{q}}{(2\pi)^2}\frac{e^{i\mathbf{qr}}}{q^2+\xi^{-2}} = (4K)K_0(|\mathbf{r}|/\xi). \tag{2}$$

In the limit $\xi\to\infty$ one may cut the divergence of the Bessel function ($K_0(z)\sim-\ln(z)$ at $z\to 0$) by the system size L and obtain

$$G(\mathbf{r}) = 4K\ln(L/|\mathbf{r}|). \tag{3}$$

(a)

In order to evaluate the vertex correlation function $V(\mathbf{r}_1,\dots,\mathbf{r}_n)$ it is instructive to start from the simplest cases $N=1,2$. Let us introduce the abbreviation $\theta(\mathbf{r}_i)=\theta_i$. Then at $N=1$ one gets $\langle e^{i\theta_1}\rangle = e^{-\langle\theta_1^2\rangle/2} = e^{-G(0)/2}$. Regularizing $G(0)=G(a)$, where a is the short-distance cut-off one obtains $\langle e^{i\theta_1}\rangle = (L/a)^{2K}\to 0$ in the thermodynamic limit $L\to\infty$. For $N=2$ we have

$$V(\mathbf{r}_1,\mathbf{r}_2) = \langle e^{i\epsilon_1\theta_1}e^{i\epsilon_2\theta_2}\rangle = \exp\left(-\frac{1}{2}\langle\theta_1^2\rangle - \frac{1}{2}\langle\theta_2^2\rangle - \epsilon_1\epsilon_2\langle\theta_1\theta_2\rangle\right) \tag{4}$$

This identity holds since we average over the Gaussian action. Introducing the correlation function

$$\bar{G}(\mathbf{r}) = G(\mathbf{r}) - G(0) \overset{0\to a}{=} -4K\ln(|\mathbf{r}|/a), \tag{5}$$

which is L-independent, we see that in the case $\epsilon_1=-\epsilon_2$ the correlator $V(\mathbf{r}_1,\mathbf{r}_2)=\exp(\bar{G}(\mathbf{r}))$ is finite. On the other hand, for $\epsilon_1=\epsilon_2$ the correlator vanishes:

$$V(\mathbf{r}_1,\mathbf{r}_2) = \exp(-\bar{G}(\mathbf{r}) - 2G(a)) = \exp\left(-\bar{G}(\mathbf{r})\right)\times(a/L)^{8K} \overset{L\to\infty}{\longrightarrow} 0. \tag{6}$$

One may now generalize these considerations to the n-th vertex correlation function. Using the Gaussian nature of the free action one can write

$$V(\mathbf{r}_1,\dots,\mathbf{r}_n) = \left\langle\exp(i\sum_{i=1}^{n}\epsilon_i\theta_i)\right\rangle = \exp\left(-\frac{1}{2}\sum_i\langle\theta_i^2\rangle - \frac{1}{2}\sum_{i\neq j}\epsilon_i\epsilon_j\langle\theta_i\theta_j\rangle\right) \tag{7}$$

Let now one has n_1 positive 'charges' with $\epsilon_i=+1$ and n_2 negative ones with $\epsilon_i=-1$, where $n_1+n_2=n$. Then one observes that the sum

$$S_a = \sum_i\langle\theta_i^2\rangle \tag{8}$$

has n terms. The two sums

$$S_b = \sum_{i\neq j}\langle\theta_i\theta_j\rangle, \quad \epsilon_i=\epsilon_j=+1 \tag{9}$$

and

$$S_c = \sum_{i\neq j}\langle\theta_i\theta_j\rangle, \quad \epsilon_i=\epsilon_j=-1 \tag{10}$$

have $n_1(n_1-1)$ and $n_2(n_2-1)$ terms, respectively. And finally the sum

$$S_d = -\sum_{i\neq j}\langle\theta_i\theta_j\rangle, \quad \epsilon_i = -\epsilon_j \tag{11}$$

has $2n_1n_2$ terms. The total number of terms is $n + n_1(n_1-1) + n_2(n_2-1) + 2n_1n_2 = n^2$ as it should be. Now on representing

$$\langle\theta_i\theta_j\rangle = \bar{G}(\mathbf{r}_i - \mathbf{r}_j) + G(a), \tag{12}$$

one concludes that

$$-\frac{1}{2}\sum_i\langle\theta_i^2\rangle - \frac{1}{2}\sum_{i\neq j}\epsilon_i\epsilon_j\langle\theta_i\theta_j\rangle = -\frac{1}{2}\sum_{i,j}\epsilon_i\epsilon_j\bar{G}(\mathbf{r}_i-\mathbf{r}_j) - \frac{1}{2}(n_1-n_2)^2G(a), \tag{13}$$

where the last terms originates from $n + n_1(n_1-1) + n_2(n_2-1) = (n_1^2+n_2^2)$ ('charged' terms with $\epsilon_i = \epsilon_j$) – $2n_1n_2$ ('neutral' ones, $\epsilon_i = -\epsilon_j$) $\to (n_1-n_2)^2$. As the result, the correlation function becomes

$$V(\mathbf{r}_1,\ldots,\mathbf{r}_n) = \exp\left(-\frac{1}{2}\sum_{i\neq j}\epsilon_i\epsilon_j\bar{G}(\mathbf{r}_i-\mathbf{r}_j)\right) \times \left(\frac{a}{L}\right)^{2K(n_1-n_2)^2}. \tag{14}$$

Note that the terms with $i = j$ do not contribute to the sum, since $\bar{G}(0) \overset{\text{reg.}}{\to} \bar{G}(a) = 0$. We see that V is non-zero in the limit $L\to\infty$ iff $n_1 = n_2$, meaning that $n = 2N$ is even and the total charge $\sum_i \epsilon_i = 0$ is zero.

We are now in position to establish the equivalence with the Coulomb gas model. On representing $g\cos(\theta) = (g/2)(e^{i\theta} + e^{-i\theta})$, the partition sum of the sine-Gordon model becomes

$$Z = \sum_{n=0}^{\infty}\frac{(-g/2)^n}{n!}\int\left(\prod_{i=1}^{n}d^2\mathbf{r}_i\right)\sum_{\{\epsilon_i\}}V(\mathbf{r_1},\ldots\mathbf{r}_n) = \sum_{N=0}^{\infty}\frac{(g/2)^{2N}}{(2N)!}\int\left(\prod_{i=1}^{2N}d^2\mathbf{r}_i\right)\sum_{\{\epsilon_i|\text{neutral}\}}\exp\left[-\frac{1}{2}\sum_{i\neq j}\epsilon_i\epsilon_j\bar{G}(\mathbf{r}_i-\mathbf{r}_j)\right], \tag{15}$$

where the charge neutrality condition $\sum_i\epsilon_i = 0$ is assumed in the last expression. For fixed N the number of such neutral configurations is

$$\begin{pmatrix} 2N \\ N \end{pmatrix} = \frac{2N!}{(N!)^2}. \tag{16}$$

By permuting integration variables $\mathbf{r}_i$ one can reduce any 'charge configuration' to the fixed one where $\epsilon_i = +1$ for $i = 1,...,N$ and $\epsilon_i = -1$ for other indices. Taking into account that $\bar{G}(\mathbf{r}_i-\mathbf{r}_j) = -8\pi K C(\mathbf{r}_i-\mathbf{r}_j)$, where $C(\mathbf{r})$ is the Coulomb potential in 2D, one arrives at

$$Z = \sum_{N=0}^{\infty}\frac{(g/2)^{2N}}{(N!)^2}\int\left(\prod_{i=1}^{2N}d^2\mathbf{r}_i\right)\exp\left[8\pi K\sum_{i<j}\sigma_i\sigma_jC(\mathbf{r}_i-\mathbf{r}_j)\right], \tag{17}$$

from where one reads of $y_0 = g/2$ and $J = 2K/\pi$.

(b)

Let $\Lambda\sim 1/a$ is an initial UV momentum cut-off and $\Lambda' = \Lambda/b$ with $b > 1$ is the new one. We split the field θ into slow ($\theta^<$) and fast ($\theta^>$) parts,

$$\theta(\mathbf{r}) = \theta^<(\mathbf{r}) + \theta^>(\mathbf{r}) = \frac{1}{L}\sum_{|\mathbf{q}|<\Lambda'}e^{i\mathbf{qr}}\theta_{\mathbf{q}} + \frac{1}{L}\sum_{\Lambda'<|\mathbf{q}|<\Lambda}e^{i\mathbf{qr}}\theta_{\mathbf{q}}. \tag{18}$$

Then the quadratic part of the free action (with $g = 0$) naturally splits into slow and fast terms, $S_0 = S_0^< + S_0^>$, while its partition function $Z_0 = Z_0^< Z_0^>$ factorizes. As to the partition sum of the sine-Gordon model, it reads

$$\frac{Z}{Z_0} = \frac{1}{Z_0} \int \mathcal{D}\theta e^{-S_0^< + S_0^>} \exp\left[-g \int d^2\mathbf{r} \cos(\theta^< + \theta^<)\right]. \tag{19}$$

Using the perturbation theory in g and the 1st order cumulant expansion, one may proceed as

$$\frac{Z}{Z_0} \simeq \frac{1}{Z_0^<} \int \mathcal{D}\theta^< e^{-S_0^<} \exp\left[-g \int d^2\mathbf{r} \left\langle \cos(\theta^< + \theta^<)\right\rangle_>\right], \tag{20}$$

where

$$\langle (\dots) \rangle_> = \frac{1}{Z_0^>} \int \mathcal{D}\theta^> (\dots) e^{-S_0^>}$$

means the average over the fast Gaussian action. By evaluating an average from the cosine term, one obtains

$$\begin{aligned} \left\langle \cos(\theta^<(\mathbf{r}) + \theta^<(\mathbf{r}))\right\rangle_> &= \cos\theta^<(\mathbf{r}) \times \exp\left[-\frac{1}{2}\langle \theta^>(\mathbf{r})^2\rangle_>\right] = \cos\theta^<(\mathbf{r}) \times \exp\left[-(4\pi K) \int_{\Lambda' < |q| < \Lambda} \frac{d\mathbf{q}}{(2\pi)^2} \frac{1}{\mathbf{q}^2}\right] \\ &= \cos\theta^<(\mathbf{r}) \times \exp\Big[-2K \ln(\Lambda/\Lambda')\Big] = b^{-2K} \cos\theta^<(\mathbf{r}). \end{aligned} \tag{21}$$

After that we perform the rescaling of momenta ($\mathbf{q}' = b\,\mathbf{q}$) and coordinates ($\mathbf{r}' = \mathbf{r}/b$) to bring the upper cut-off Λ' back to the original one (Λ). In combination with Eq. (21) it gives the renormalized action

$$S_{\text{ren}}[\theta^<] = S_0^< + g'(b) \int d^2\mathbf{r}' \cos(\theta^<(\mathbf{r}')) \tag{22}$$

with the new coupling constant

$$g'(b) = b^{2-2K} g,$$

which is equivalent to the differential RG equation

$$\frac{dg}{d\ln b} = (2 - 2K)g.$$

One can say that the coupling g has the 'scaling dimension' $\Delta_g = (2 - 2K)$, with $\Delta_g^0 = 2$ being its engineering (or bare) dimension (g has the physical dimension $[g] = [\Lambda]^2$, i.e. momentum ^{2}). Exactly the same RG equation one obtains for the fugacity y when studying the XY-model ($K \to \pi J/2$ and $g \to 2y$).