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Quantum Field Theory III

Exercise sheet 3

This exercise will be discussed on 12.05.2016

3. Path integral for spin

The goal of this exercise is to derive the path integral representation of the partition sum $\mathcal{Z} = \operatorname{tr} e^{-\beta \hat{H}}$ of a spin S in a magnetic field subject to the Hamiltonian $\hat{H} = \mathbf{B} \cdot \hat{\mathbf{S}}$.

As you know, in quantum mechanics the spin $\hat{\mathbf{S}}^2 = S(S+1)$ is related to the $(2S+1)$–dimensional representation of the group SU(2). Any element of this group, $g \in \mathrm{SU}(2)$, affords a parametrization in terms of three Euler angles,

$$\mathrm{SU}(2) = \left\{ g(\phi, \theta, \psi) = e^{-i\phi \hat{S}_3} e^{-i\theta \hat{S}_2} e^{-i\psi \hat{S}_3} \middle| \phi, \psi \in [0, 2\pi),\, \theta \in [0, \pi] \right\}$$

and for any representation there exists a *maximal weight state* $|\uparrow\rangle \equiv |S, m = S\rangle$ such that $\hat{S}_3|\uparrow\rangle = |\uparrow\rangle S$. To construct the path integral one considers the spin *coherent states* $|g\rangle$ which are defined by the action of a group element g on this maximal weight state, i.e. by definition

$$|g\rangle = g|\uparrow\rangle.$$

One can show that states $|g\rangle$ form the (over)-complete basis in the $2S+1$-dimensional representation space, and

$$\mathbb{1} = (2S+1) \int dg |g\rangle\langle g|, \quad dg = \frac{1}{\pi} \sin^2 \psi \sin \theta d\psi d\theta d\phi \tag{1}$$

is 'the resolution of identity', where dg is the so-called Haar measure on the group SU(2) (in the Euler parametrization).

a) Using the resolution of identity (1) prove, that

$$\mathcal{Z} = \lim_{N \to \infty} \operatorname{tr} \left(e^{-\epsilon \hat{H}} \right)^N = \int dg \exp \left[- \int_0^\beta d\tau \left(-\langle \partial_\tau g | g \rangle + \langle g | \mathbf{B} \cdot \hat{\mathbf{S}} | g \rangle \right) \right],$$

where $\epsilon = \beta / N \to 0$.

In order to write the above path integral in terms of Euler angles one should compute the expectation values $n_i = \langle g(\psi, \theta, \phi) | \hat{S}_i | g(\psi, \theta, \phi) \rangle$ of the spin operators.

b) Check that n_i's do not depend on ψ and

$$\mathbf{n}(\theta, \phi) \equiv (n_1, n_2, n_3) = S \times (\sin\theta \cos\phi, \sin\theta \sin\phi, \cos\theta),$$

is proportional to the unit vector on a sphere S^2 parametrized by other two angles θ and ϕ.

c) Using the above result evaluate the action $\mathcal{S}[\theta, \phi]$ for the spin in Euler coordinates assuming $\mathbf{B} = B\mathbf{e}_z$ is pointing along z-axis. You should obtain the result

$$\mathcal{Z} = \int dg e^{-\mathcal{S}[\theta, \phi]}, \qquad \mathcal{S}[\theta, \phi] = \int_0^\beta d\tau (-i\dot{\phi} \cos\theta + B \cos\theta), \tag{2}$$

which was 'conjectured' on the lecture.

d) Apply the variational principle to the action $\mathcal{S}[\theta, \phi]$ and verify that the corresponding Euler-Lagrange equations agree with the classical Bloch equation for the spin,

$$i\partial_\tau \mathbf{n} = [\mathbf{B} \times \mathbf{n}],$$

written in imaginary time.

One can further perceive the geometric meaning of the topological term

$$\mathcal{S}_{\rm top}[\theta, \phi] = iS \int_0^\beta d\tau (1 - \cos\theta)\dot{\phi}, \tag{3}$$

which up to the boundary term $2i\pi WS$ is the 1st term in the action (2) provided the periodic boundary condition, $\phi(\beta) = \phi(0) + 2\pi W$, is assumed and $W \in \mathbb{Z}$ is known as the winding number. Let $\mathbf{n}, \mathbf{e}_\theta$ and $\mathbf{e}_\phi$ form the orthonormal(!) basis in the spherical coordinates.

e) Prove that the topological action is the integral along the curve γ,

$$\mathcal{S}_{\rm top}[\theta, \phi] = iS \oint_\gamma \mathbf{A} \cdot d\mathbf{n}, \qquad \mathbf{A} = \frac{1 - \cos\theta}{\sin\theta} \mathbf{e}_\theta, \tag{4}$$

spanned by the spin, see Fig.1. Note, that the effective vector potential $\mathbf{A}$ is singular here along the negative z-axis, also known as the "Dirac's string".

f) With the help of the Stoke's theorem (in the spherical coordinates) verify that

$$\mathcal{S}_{\rm top}[\theta, \phi] = iSA_{\gamma,n},$$

where $A_{\gamma,n}$ is the domain on the sphere S^2 which has the curve γ as its boundary and may contain the north (but not the south) pole.

References

[1] A. Altland and B. Simons, *Condensed Matter Field Theory*, 2nd Edition (Cambridge University Press, 2010).

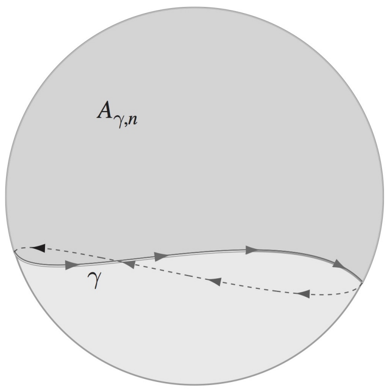

Figure 1: