Path integral for spin

The goal of this chapter is to derive the path integral representation of the partition sum $\mathcal{Z} = \mathrm{tr}\, e^{-\beta \hat{H}}$ for a particle of spin S in magnetic field, $\hat{H} = \vec{B}\cdot\hat{\vec{S}}$.

- Particle with spin $\hat{S}^2 = S(S+1)$ is related to $(2S+1)$-dimensional representation of the group $SU(2)$.

- Any element $g \in SU(2)$ affords parametrization in terms of Euler angles

$$SU(2) = \{ g(\varphi,\theta,\psi) = e^{-i\varphi \hat{S}_3} e^{-i\theta \hat{S}_2} e^{-i\psi \hat{S}_3} \mid \varphi, \psi \in [0, 2\pi] \ \& \ \theta \in [0,\pi] \}.$$

- Let $|\uparrow\rangle$ is maximal weight state, i.e. $\hat{S}_3 |\uparrow\rangle = |\uparrow\rangle \times S$.

Then $|g\rangle \equiv g|\uparrow\rangle$ is spin coherent state.

- States $|g\rangle$ form the complete basis,

$$\mathbb{1} = c \int dg \, |g\rangle\langle g| \rightarrow \text{resolution of identity}$$

(i) $c \rightarrow$ some constant

(ii) $dg \rightarrow$ Haar measure, $\forall\, h \in SU(2)$ we have

$$\int dg\, f(gh) = \int dg\, f(hg) = \int dg\, f(g).$$

$$dg = \frac{1}{\pi} \sin^2\psi \, \sin\theta \, d\theta \, d\psi \, d\varphi \rightarrow \text{for Euler parametrization}$$

- Partition sum; $N \gg 1$, $\varepsilon = \beta/N \rightarrow 0$.

$$\mathcal{Z} = \lim_{N\to\infty} \mathrm{tr}\left(e^{-\beta/N\, \hat{H}}\right)^N = \lim_{N\to\infty} \int_{g_N = g_0} \prod_{i=0}^{N-1} \langle g_{i+1} | e^{-\varepsilon \hat{H}} | g_i\rangle \prod_{i=0}^{N-1} dg_i$$

- Matrix element reads, ($\varepsilon \ll 1$) ; $\langle g_i | g_i \rangle \equiv 1$.

$$\langle g_{i+1} | e^{-\varepsilon \hat{H}} | g_i \rangle \simeq \langle g_{i+1} | g_i \rangle - \varepsilon \langle g_{i+1} | \hat{H} | g_i \rangle$$

$$= 1 + \{ \langle g_{i+1} | g_i \rangle - \langle g_i | g_i \rangle \} - \varepsilon \langle g_{i+1} | \hat{H} | g_i \rangle \simeq$$

(the bracket $\{ \} \ll 1$)

$$\simeq \exp \{ \langle g_{i+1} | g_i \rangle - \langle g_i | g_i \rangle - \varepsilon \langle g_{i+1} | \hat{H} | g_i \rangle \}.$$

- The partition sum then takes the form

$$Z = \lim_{N \to \infty} \int_{g_N = g_0} \left(\prod_{i=1}^{N-1} dg_i \right) \times \exp \left\{ -\varepsilon \left(\sum_{i=0}^{N-1} \frac{\langle g_i | g_i \rangle - \langle g_{i+1} | g_i \rangle}{\varepsilon} + \langle g_{i+1} | \vec{B} \cdot \hat{S} | g_i \rangle \right) \right\}.$$

In the continuum limit we can write

$$Z = \int Dg \exp \left(-\int_0^\beta d\tau \left(-\langle \partial_\tau g | g \rangle + \langle g | \vec{B} \cdot \hat{S} | g \rangle \right) \right).$$

Let us further consider two terms of this action

A. - Magnetic term: $\langle g | \vec{B} \cdot \hat{S} | g \rangle$ - ?

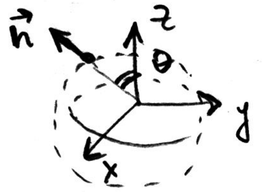
Let us define $n_i \equiv \langle g | \hat{S}_i | g \rangle$. We then write

$$g(\theta, \varphi, \psi) = \tilde{g}(\theta, \varphi) e^{-i\psi \hat{S}_3} \Rightarrow |g\rangle = \tilde{g} |\uparrow\rangle e^{-i\psi S}$$

and we see that $n_i = n_i(\theta, \varphi)$ and does not depend on ψ!

Lemma: $\vec{n} = (n_1, n_2, n_3) = (\sin\theta \cos\varphi, \sin\theta \sin\varphi, \cos\theta) \cdot S$

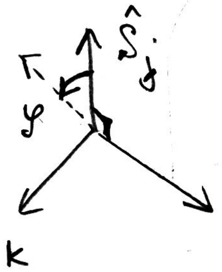
S.3| We see that $\vec{n}$ is unit vector, parametrized in terms of spherical coordinates.



Proof: one can use identity for $i \neq j$

$$(*) \quad e^{-i\varphi \hat{S}_i} \hat{S}_j e^{i\varphi \hat{S}_i} = \hat{S}_j \cos\varphi + \varepsilon_{ijk} \hat{S}_k \sin\varphi$$

Geom. meaning.



See p. S.4a for the proof.

One uses $(*)$ to evaluate $n_i = \langle \hat{g} | \hat{S}_i | \hat{g} \rangle$ with $|\hat{g}\rangle = e^{-i\varphi \hat{S}_3} e^{-i\theta \hat{S}_2} |\uparrow\rangle$ and $\langle \hat{g}| = \langle \uparrow | e^{i\theta \hat{S}_2} e^{i\varphi \hat{S}_3}$

Then magnetic term reads

$$\sum_{k=1}^{3} \langle g | B_k \hat{S}_k | g \rangle = \sum_k B_k n_k = (\vec{B}\cdot\vec{n}) = S \cdot B \cdot \cos\theta.$$

$\uparrow$ for $\vec{B} = B\vec{e}_z$

or

$$S_B[\theta,\varphi] = \int_0^\beta S \cdot B \cdot \cos\theta \, d\tau$$

B. Topological term, $-\int_0^\beta \langle \partial_\tau g | g \rangle d\tau$ - ?

1) Let $g = \hat{g} e^{-i\psi \hat{S}_3}$, then

$$\int_0^\beta d\tau \langle \partial_\tau g | g \rangle = + \int_0^\beta d\tau \langle \uparrow | \partial_\tau (e^{i\psi \hat{S}_3} \tilde{g}^{-1}) \tilde{g} e^{i\psi \hat{S}_3} | \uparrow \rangle$$

$$= \int_0^\beta d\tau \langle \uparrow | i(\partial_\tau \psi) \hat{S}_3 | \uparrow \rangle + \int_0^\beta d\tau \langle \partial_\tau \hat{g} | \hat{g} \rangle$$

$$= \underbrace{i S \psi \Big|_0^\beta}_{=0} + \int_0^\beta d\tau \langle \partial_\tau \tilde{g} | \hat{g} \rangle.$$

Result: since $\psi(\beta) = \psi(0)$ (periodic bound. conditions for tr)

we get $\int_0^\beta d\tau \langle \partial_\tau g | g \rangle = \int_0^\beta \langle \partial_\tau \tilde{g} | \hat{g} \rangle . d\tau$.

2) Direct evaluation, using identity (*) on p S.3 gives

$$S_{top} = -\int_0^\beta d\tau \langle \partial_\tau g | g \rangle = -i S \int_0^\beta d\tau \, (\partial_\tau \varphi) \cos\theta .$$

- Final result (action of a spin S in magnetic field B_z)

$$S[\theta,\varphi] = S_B[\theta,\varphi] + S_{top}[\theta,\varphi]$$

$$\underset{\text{action}}{S[\theta,\varphi]} = \underset{\text{spin}}{S} \int_0^\beta d\tau \left(B \cdot \cos\theta - i \dot{\varphi} \cos\theta \right)$$

- Minimization of action leads to Bloch equation for spin

$$\boxed{ i \partial_\tau \vec{n} = [\vec{B} \times \vec{n}] } \qquad (**)$$

$\vec{n} \in S^2$, $\vec{n} = \vec{n}(\theta,\varphi)$ - unit vector defined by angles θ & φ.

Eq (**) defines saddle point trajectory of the action $S[\theta,\varphi]$ in the limit of large ($S \gg 1$), i.e. classical spin.

Supplementary

A. We prove here that $e^{-i\varphi \hat{S}_i}\, \hat{S}_j\, e^{i\varphi S_i} = S_j \cos\varphi + \varepsilon_{ijk} \hat{S}_k \sin\varphi$, (*)

if $i \neq j$.

- Let us define $\hat{S}_j(\varphi) \equiv e^{-i\varphi \hat{S}_i}\, \hat{S}_j\, e^{i\varphi S_i}$, then $\frac{d}{d\varphi} \hat{S}_j(\varphi) = i\,[\hat{S}_j(\varphi), \hat{S}_i]$. We can now check that (*) satisfies to this relation, indeed:

- L.h.s. $= \frac{d}{d\varphi} \hat{S}_j(\varphi) = -\hat{S}_j \sin\varphi + \varepsilon_{ijk} \hat{S}_k \cos\varphi$
- R.h.s. $= i\,[\hat{S}_j, \hat{S}_i] \cos\varphi + i\varepsilon_{ijk} [\hat{S}_k, \hat{S}_i] \sin\varphi$

$$= \varepsilon_{ijk} \hat{S}_k \cos\varphi - \hat{S}_j \sin\varphi . \quad ◪$$

B. Let us evaluate h_z:

$$h_z = \langle g | \hat{S}_3 | g \rangle = \langle \tilde{g} | \hat{S}_3 | \tilde{g} \rangle =$$

$$= \langle \uparrow | e^{i\theta S_2} e^{i\varphi S_3} S_3 e^{-i\varphi S_3} e^{-i\theta S_2} | \uparrow \rangle =$$

$$= \langle \uparrow | e^{i\theta S_2} S_3 e^{-i\theta S_2} | \uparrow \rangle = \langle \uparrow | \hat{S}_3 \cos\theta - \hat{S}_1 \sin\theta | \uparrow \rangle$$

$$= \cos\theta$$

Other components (h_x & h_y) can be evaluated in the same way.

Geometrical picture of topological term.

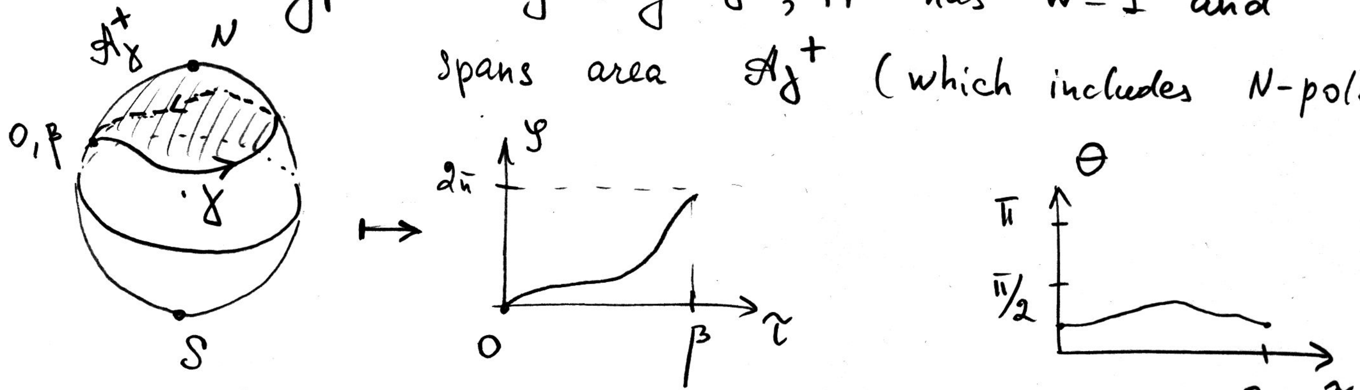
- Let us write $S_{top}[\theta,\varphi]$ in the form

$$S_{top}[\theta,\varphi] = S i \int_0^\beta \dot\varphi \, d\tau + S i \int_0^\beta (1-\cos\theta)\dot\varphi \, d\tau$$

Setting $\varphi(\beta) = \varphi(0) + 2\pi W$ $(W \in \mathbb{Z})$ we get

$$S_{top} = 2\pi i (S \cdot W) + \tilde{S}_{top}; \quad \tilde{S}_{top} = i S \int_0^\beta (1-\cos\theta)\dot\varphi \, d\tau$$

- Consider typical trajectory γ, it has $W = 1$ and spans area A_γ^+ (which includes N-pole).



$$\boxed{S_{top} = (2\pi i + i A_\gamma^+) \cdot S}$$

Topological action of a spin is geometric phase.
(Velocity along γ, defined by $\dot\varphi$ & $\dot\theta$ does not play any role).

Proof:

ⓐ We represent $\tilde{S}_{top} = i S \int_0^\beta (\dot{\vec{n}} \cdot \vec{A}) \, d\tau$, where $\vec{A}$ is properly chosen artificial vector potential (see below)

S.61 (b) $\vec{n} = (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta)$, hence

$$\dot{\vec{n}} = \frac{d}{d\tau}\vec{n} = \dot\theta\,\vec{e_\theta} + \dot\varphi\sin\theta\,\vec{e_\varphi}, \text{ with}$$

$$\vec{e_\theta} = (\cos\theta\cos\varphi, +\cos\theta\sin\varphi, -\sin\theta) \text{ and}$$
$$\vec{e_\varphi} = \sin\theta(-\sin\varphi, \cos\varphi, 0)$$

$\vec{n}, \vec{e_\theta}$ & $\vec{e_\varphi}$ form local orthogonal basis

(c) Let us take $\boxed{\vec{A} = \frac{1-\cos\theta}{\sin\theta}\vec{e_\varphi}}$, then

$\vec{A}$ is singular along the string $\theta = \pi$!

$$\left(\frac{d\vec{n}}{d\tau}\cdot\vec{A}\right) = (1-\cos\theta)\frac{d\varphi}{d\tau} \quad \text{and} \quad \tilde{S}_{top} = Si\int_0^\beta (1-\cos\theta)\dot\varphi\, d\tau \quad \text{as}$$

it should be

(d) let us use Stokes' theorem to write

$$\tilde{S}_{top} = iS\int_0^\beta \left(\vec{A}\cdot\frac{d\vec{n}}{d\tau}\right)d\tau = iS\oint \vec{A}\cdot d\vec{\ell} = iS\int_{A_\gamma^+} dS(\vec{\nabla}\times\vec{A})$$

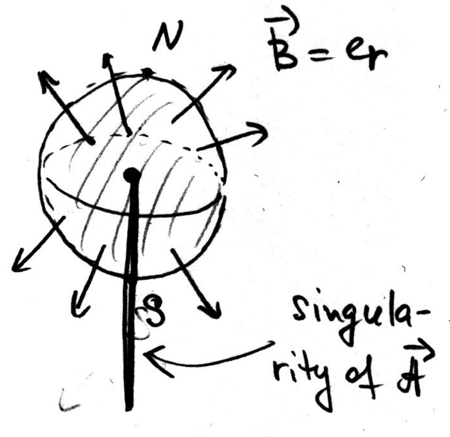
(e) For the vector field $\vec{f} = f_r\vec{e_r} + f_\theta\vec{e_\theta} + f_\varphi\vec{e_\varphi}$

$$(\vec{\nabla}\times\vec{f}) = \begin{vmatrix} \vec{e_r} & \vec{e_\theta} & \vec{e_\varphi} \\ \partial_r & \partial_\theta & \partial_\varphi \\ f_r & rf_\theta & r\sin\theta f_\varphi \end{vmatrix} \frac{1}{r^2\sin\theta}.$$

Using this formula (for $r \equiv 1$) we get $\vec{B} = \vec{\nabla}\times A =$

$$= \frac{\vec{e_r}}{\sin\theta}\,\partial_\theta(1-\cos\theta) = \vec{e_r}$$

(f) $$\tilde{S}_{top} = iS\int_{A_\gamma^+} dS = iS\,A_\gamma^+ .$$ ∎



Dirac monopole