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Quantum Field Theory III

Exercise sheet 4

This exercise will be discussed on 26.05.2016

4. The effective action for the 1D quantum antiferromagnet

The goal of this exercise is to derive the path integral representation for the partition sum of the one-dimensional Heisenberg *antiferromagnet* with even number of sites N described by the Hamiltonian $\hat{H} = J\sum_{i=1}^{N} \hat{\mathbf{S}}_i \cdot \hat{\mathbf{S}}_{i+1}$. Here the interaction constant $J > 0$ and $\hat{\mathbf{S}}_i$ are quantum spin operators on each site with $\hat{\mathbf{S}}_i^2 = S(S+1)$. In few steps outlined below you have to show that a partition sum of this model in the continuum limit is given by a path integral,

$$\mathcal{Z} = \int\limits_{\mathbf{m}(0,x)=\mathbf{m}(\beta,x)} \mathcal{D}\mathbf{m}\, e^{-\mathcal{S}[\mathbf{m}]}, \quad \mathcal{S}[\mathbf{m}] = \int_0^L dx \int_0^\beta d\tau \mathcal{L}[\mathbf{m}], \tag{1}$$

written in terms of the unit vector on a sphere, $\mathbf{n}(\tau, x) \in S^2$, with the Lagrangian

$$\mathcal{L}[\mathbf{m}] = \frac{S}{4}\left(\frac{1}{v_s}(\partial_\tau \mathbf{m})^2 + v_s(\partial_x \mathbf{m})^2\right) + \frac{iS}{2}\,\mathbf{m}\cdot(\partial_\tau \mathbf{m} \times \partial_x \mathbf{m})\,, \tag{2}$$

where $v_s = 2aJS$ is the spin wave velocity expressed in terms of the lattice constant a.

As you know from the Exercise **3**, the imaginary time action of the magnet, when written in terms of the unit vectors $\mathbf{n}_j(\tau) \in S^2$ on each site, can be represented in the form

$$S[\mathbf{n}] = S\sum_{j=1}^{N} \mathcal{S}_{\mathrm{WZ}}[\mathbf{n}_j] + JS^2 \int_0^\beta d\tau \sum_{j=1}^{N-1} \mathbf{n}_j \cdot \mathbf{n}_{j+1}, \tag{3}$$

where $\mathcal{S}_{\mathrm{WZ}}[\mathbf{n}] \;=\; -\,i\,\int_0^\beta d\tau \dot{\phi}\cos\theta$ is the Wess-Zunino action expressed here via the spherical coordinates. Anticipating the Néel state let us introduce the stagger configuration: $\mathbf{n}_j \to (-1)^j \mathbf{n'}_j$, and then split the staggered field as $\mathbf{n'}_j = \mathbf{m}_j + (-1)^j a\mathbf{l}_j$, where $\mathbf{m}_j$ is a slow antiferromagnetic order parameter field and a small rapidly varying part $\mathbf{l}_j$ represents the average spin. If so, one should demand that they are orthogonal vectors (why?), i.e. $\mathbf{m}_j \cdot \mathbf{l}_j = 0$, hence the field $\mathbf{m}_j$ is also a unit vector, $\mathbf{m}_j^2 = 1$.

a) Check that the Wess-Zumino term is *odd* under this replacement, i.e. $\mathcal{S}_{\mathrm{WZ}}[-\mathbf{n}] = -\mathcal{S}_{\mathrm{WZ}}[\mathbf{n}]$.

b) By going to the continuum limit, expand the action (3) in the gradients $\partial_\tau \mathbf{m}(\tau, x)$, $\partial_x \mathbf{m}(\tau, x)$ and the spin fluctuations $\mathbf{l}(\tau, x)$, where $\mathbf{m}(\tau, x = 2aj) = \mathbf{m}_j(\tau)$ and etc. You should find the intermediate result for the Lagrangian density,

$$\mathcal{L}_M[\mathbf{m}, \mathbf{l}] \simeq \frac{iS}{2}\,\mathbf{m}\cdot(\partial_\tau \mathbf{m} \times \partial_x \mathbf{m}) - iS\,\mathbf{l}\cdot(\mathbf{m} \times \partial_\tau \mathbf{m}) + \frac{a}{2}J^2 S^2\Big((\partial_x \mathbf{m})^2 + 4\mathbf{l}^2\Big), \tag{4}$$

where the 1st two terms are inherited from the Wess-Zumino action while other two are due to the spin-spin interaction energy.

c) Integrate out the fluctuating spin field $\mathbf{l}(\tau, x)$ and obtain the Lagrangian (2) for the slow order parameter field $\mathbf{m}(\tau, x)$. When performing this integration one can eventually relax the constrain $\mathbf{l}(\tau, x) \cdot \mathbf{m}(\tau, x) = 0$ imposed above. Can you justify it?

References

[1] A. Altland and B. Simons, *Condensed Matter Field Theory*, 2nd Edition (Cambridge University Press, 2010).

[2] E. Fradkin, *Field Theories of Condensed Matter Physics*, 2nd Edition (Cambridge University Press, 2013).