to leading order in $\vec{\pi}$. From Eqs. (7.55) we find the dispersion relation for ferromagnetic spin waves

$$|p_0| \approx |J| S a_0^2 |\vec{p}|^2 \tag{7.56}$$

which is known as Bloch's law (Bloch, 1930). As expected, we find that the frequency of the low-energy excitations of a quantum ferromagnet scales as the square of the momentum.

7.5 The effective action for one-dimensional quantum antiferromagnets

We will not consider here frustrated systems. Thus, and for the sake of simplicity, we will consider the case of quantum antiferromagnets on *bipartite* lattices, such as the hypercubic lattice. We will see that, unlike in the case of the ferromagnets, the effective low-energy action is *different* for 1D systems and for higher-dimensional cases such as the square and cubic lattices. In all cases we will find a non-linear sigma model, in agreement with our previous discussion (see Chapter 3) that was based on a mean-field weak-coupling treatment of the Hubbard model. But we will get more. For the spin-chain case we will find that the action has an extra term, a topological term.

The starting point will be, once again, the real-time action of Eq. (7.39) with a nearest-neighbor *antiferromagnetic* coupling constant $J > 0$. Since we expect that *at least* the short-range order should have Néel character, it is natural to consider the *staggered* and *uniform* components of the spin field $\vec{n}$. This construction, as is, works only for two-sublattice systems close to a Néel state, although it is possible to generalize it to other cases.

Consider a spin chain with an *even* number of sites N occupied by spin-S degrees of freedom. The sites of the lattice are labeled by an integer $j = 1, \ldots, N$. The real-time action is

$$\mathcal{S}_{\mathrm{M}}[\vec{n}] = S \sum_{j=1}^{N} \mathcal{S}_{\mathrm{WZ}}[\vec{n}(j)] - \int_0^T dx_0 \sum_{j=1}^{N} J S^2 \vec{n}(j, x_0) \cdot \vec{n}(j+1, x_0) \tag{7.57}$$

where we have assumed periodic boundary conditions. Since we expect to be close to a Néel state, we will stagger the configuration

$$\vec{n}(j) \to (-1)^j \vec{n}(j) \tag{7.58}$$

On a bipartite lattice, the substitution of Eq. (7.58) into Eq. (7.57) will change the sign of the *exchange term* of the action to a *ferromagnetic* one. The Wess–Zumino terms are *odd* under the replacement of Eq. (7.58) and thus become *staggered*. Thus, it is the Wess–Zumino term, a purely quantum-mechanical effect, which will distinguish ferromagnets from antiferromagnets. After staggering the spins we get, up to an additive constant,

$$\mathcal{S}_{\mathrm{M}}[\vec{n}] = S\sum_{j=1}^{N}(-1)^j \mathcal{S}_{\mathrm{WZ}}[\vec{n}(j)] - \frac{JS^2}{2}\int_0^T dx_0 \sum_{j=1}^{N}(\vec{n}(j,x_0) - \vec{n}(j+1,x_0))^2 \tag{7.59}$$

We now split the (staggered) spin field $\vec{n}$ into a slowly varying piece $\vec{m}(j)$, the order parameter field, and a small rapidly varying part, $\vec{l}(j)$, which roughly represents the average spin (Affleck, 1990). Hence, we write

$$\vec{n}(j) = \vec{m}(j) + (-1)^j a_0 \vec{l}(j) \tag{7.60}$$

The constraint $\vec{n}^2 = 1$ and the requirement that the order-parameter field $\vec{m}$ should obey the same constraint, $\vec{m}^2 = 1$, demand that $\vec{m}$ and $\vec{l}$ be orthogonal vectors:

$$\vec{m} \cdot \vec{l} = 0 \tag{7.61}$$

The Wess–Zumino terms are rewritten as

$$S\sum_{j=1}^{N}(-1)^j \mathcal{S}_{\mathrm{WZ}}[\vec{n}(j)] = S\sum_{r=1}^{N/2}(\mathcal{S}_{\mathrm{WZ}}[\vec{n}(2r)] - \mathcal{S}_{\mathrm{WZ}}[\vec{n}(2r-1)]) \tag{7.62}$$

which, by making use of the approximation

$$\begin{aligned}\vec{n}(2r) - \vec{n}(2r-1) &= \vec{m}(2r) - \vec{m}(2r-1) + a_0(\vec{l}(2r) + \vec{l}(2r-1)) \\ &= a_0\left(\partial_1 \vec{m}(2r) + 2\vec{l}(2r)\right) + O(a_0^2)\end{aligned} \tag{7.63}$$

becomes

$$\begin{aligned}S\sum_{j=1}^{N}(-1)^j \mathcal{S}_{\mathrm{WZ}}[\vec{n}(j)] &\approx S\sum_{r=1}^{N/2}\int_0^T dx_0\, \delta\vec{n}(2r,x_0)\cdot(\vec{n}(2r,x_0)\times\partial_0\vec{n}(2r,x_0)) \\ &\approx S\sum_{r=1}^{N/2}\int_0^T dx_0 \left(a_0\,\partial_1\vec{m}(2r,x_0) + 2a_0\vec{l}(2r,x_0)\right) \\ &\quad \times (\vec{m}(2r,x_0)\times\partial_0\vec{m}(2r,x_0))\end{aligned} \tag{7.64}$$

Thus, in the continuum limit, one finds

$$\begin{aligned}\lim_{a_0\to 0} S\sum_{j=1}^{N}(-1)^j \mathcal{S}_{\mathrm{WZ}}[\vec{n}(j)] &\approx \frac{S}{2}\int d^2x\, \vec{m}\cdot(\partial_0\vec{m}\times\partial_1\vec{m}) \\ &\quad + S\int d^2x\, \vec{l}\cdot(\vec{m}\times\partial_0\vec{m})\end{aligned} \tag{7.65}$$

Similarly, the continuum limit of the potential-energy terms can also be found to be given by

$$\lim_{a_0\to 0} \frac{JS^2}{2} \sum_{j=1}^{N} \int_0^T dx_0 (\vec{n}(j,x_0) - \vec{n}(j+1,x_0))^2 \simeq \frac{a_0 JS^2}{2} \int d^2x \left((\partial_1 \vec{m})^2 + 4\vec{l}^2 \right) \tag{7.66}$$

On collecting terms we find a Lagrangian density involving both the order-parameter field $\vec{m}$ and the local spin density $\vec{l}$,

$$\mathcal{L}_{\rm M}(\vec{m},\vec{l}\,) = -2a_0 J S^2 \vec{l}^2 + s\vec{l} \cdot (\vec{m} \times \partial_0 \vec{m}) - \frac{a_0 J S^2}{2} (\partial_1 \vec{m})^2 + \frac{S}{2} \vec{m} \cdot (\partial_0 \vec{m} \times \partial_1 \vec{m}) \tag{7.67}$$

The fluctuations in the average spin density $\vec{l}$ can be integrated out. The result is the Lagrangian density of the non-linear sigma model,

$$\mathcal{L}_{\rm M}(\vec{m}) = \frac{1}{2g} \left(\frac{1}{v_{\rm s}} (\partial_0 \vec{m})^2 - v_{\rm s} (\partial_1 \vec{m})^2 \right) + \frac{\theta}{8\pi} \epsilon_{\mu\nu} \vec{m} \cdot \left(\partial_\mu \vec{m} \times \partial_\nu \vec{m} \right) \tag{7.68}$$

where g and $v_{\rm s}$ are, respectively, the coupling constant and spin-wave velocity:

$$g = \frac{2}{S} \tag{7.69}$$

$$v_{\rm s} = 2a_0 JS \tag{7.70}$$

The last term in Eq. (7.68) has topological significance. We have chosen the normalization so that the coupling constant θ is given by

$$\theta = 2\pi S \tag{7.71}$$

The tensor $\epsilon_{\mu\nu}$ is the usual Levi-Civita antisymmetric tensor in two dimensions.

Thus, apart from an anisotropy determined by the spin-wave velocity $v_{\rm s}$ and apart from the topological term, we find that the effective action for the low-frequency, long-wavelength fluctuation about a state with *short-range* Néel order is given by the non-linear sigma model. We reached the same results within the weak-coupling mean-field theory of the half-filled Hubbard model of Chapter 3. Indeed, using that approach, it is also possible to get the topological term (Wen and Zee, 1988).